Perturbation Analysis and Optimization of Stochastic Flow Networks

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Abstract— **We consider a Stochastic Fluid Model (SFM) of a network consisting of several single-class nodes in tandem and perform perturbation analysis for the node queue contents and associated event times with respect to a threshold parameter at the first node. We then derive Infinitesimal Perturbation Analysis (IPA) derivative estimators for loss and buffer occupancy performance metrics with respect to this parameter and show that these estimators are unbiased. We also show that the estimators depend only on data directly observable from a sample path of the actual underlying discrete event system, without any knowledge of the stochastic characteristics of the random processes involved. This renders them computable in on-line environments and easily implementable for network management and optimization. This is illustrated by combining the IPA estimators with standard gradient based stochastic optimization methods and providing simulation examples.**

Keywords— **Infinitesimal Perturbation Analysis, Stochastic Fluid Models, Non-linear Optimization.**

I. Introduction

Stochastic Fluid Models (SFM) have recently been adopted as an alternative modeling paradigm to queueing networks for telecommunication applications, as well as other complex discrete event systems. Introduced in [1] and then in [2] for the purpose of analysis, fluid models have also been considered for simulation and control [3],[4],[5],[6],[7],[8],[9]. Using this modeling framework, a new approach for network congestion management has been proposed, based on Infinitesimal Perturbation Analysis (IPA) [10],[11],[12],[13]. The cornerstone of this approach is the on-line estimation of gradients (sensitivities) of certain congestion-related performance measures (e.g., loss rates, average buffer levels) as functions of various controllable parameters. These gradient estimates are used in conjunction with standard stochastic approximation algorithms to optimize the parameter settings. As operating conditions change, the gradient estimates change, therefore, this approach aims at continuously seeking to optimize a generally time-varying performance metric. All work to date has been limited to a single node SFM. In this paper, we extend the approach to networks of nodes connected in tandem and, in the process, study how a buffer level perturbation in one node in a network can propagate to other nodes and how local congestion control may affect the rest of a network.

To date, many implementations of network control mechanisms have relied on adjusting traffic parameters (e.g., inflow rates) by monitoring and measuring certain performance measures (e.g., average buffer levels, delay jitter, and loss rates). Arguably, control algorithms that rely on both performance measures and their gradients with respect to controllable parameters will perform better. In fact, some derivative-based congestion control algorithms have been proposed in [14],[15]. Our approach is centered around the *on-line estimation* of such derivatives and it relies on the use of IPA. IPA has been developed in the general setting of Discrete Event Dynamic Systems (DEDS), and queueing models in particular. However, in the setting of queueing networks, IPA cannot usually provide unbiased gradient estimators outside the realm of simple models with a single customer class, infinite buffers, and state-independent routing [16],[17]. These limitations exclude many telecommunication application features such as differentiated services, packet loss due to buffer capacity limitations, and virtual-path routing. However, in the context of SFMs, as opposed to queueing systems, recent work [10] has shown that IPA gradient estimators for important performance metrics are endowed with the following crucial properties: (i) They are unbiased, (ii) They are nonparametric, i.e., they are computable by expressions that are independent of the probability laws of the underlying traffic processes, and (iii) They are extremely simple and easy to implement. The first property implies that the IPA gradient estimators can be trusted in performance prediction; the second implies that the IPA estimators can be computed from field measurements instead of merely simulation environments; and the third property points to the possibility of real-time computation.

The use of IPA in single-node SFMs has been studied in [18],[10],[11],[12]. In [10], a SFM was adopted for a single traffic class network node in which threshold-based buffer control is exercised. For the problem of determining a threshold that minimizes a weighted sum of loss volume and buffer content, it was shown that IPA yields remarkably simple nonparametric sensitivity estimators for this performance metric with respect to a threshold parameter, which, in addition, are unbiased under very weak structural assumptions on the defining traffic processes. More-

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over, a solution of the performance optimization problem based on the IPA-based approach outlined above recovers or gives close approximations to the solution of the associated queueing model. Extensions of the results derived in [10],[11] to general networks have had to proceed in two directions: the incorporation of multiple traffic classes and the analysis of general topology networks. The former direction has been pursued in [12],[19], where results analogous to those in [10] were obtained. The latter direction is pursued in the present paper, whose primary focus is on tandem networks; some early results for the two-node case may be found in [20]. The main contributions are as follows. First, we consider a SFM consisting of $M \geq 2$ single-class nodes in tandem and develop IPA derivative estimators of loss and buffer occupancy performance metrics with respect to a threshold parameter at the first node. We show that these estimators are unbiased and discuss their applicability to general-topology networks. Despite the inevitable burdensome notation necessary to derive and analyze the estimators, their implementation is simple and rests on two intuitively appealing "perturbation propagation rules": *(i)* A queue content perturbation at node m propagates downstream whenever the buffer at m becomes empty, and *(ii)* A perturbation at node m is eliminated after its buffer becomes either empty or full. Finally, we demonstrate the use of the IPA estimators for network performance optimization purposes through simulation experiments.

The rest of the paper is organized as follows. Section 2 presents the stochastic flow modeling framework for a network of nodes in tandem. In Section 3 we carry out perturbation analysis of the network with respect to a threshold parameter at node 1 and derive explicit IPA estimators for loss and queue content metrics. We also prove the unbiasedness of these estimators. Section 4 presents some simulation results illustrating the use of the estimators in network performance optimization. Section 5 concludes the paper and outlines related ongoing work.

II. Tandem Network SFM and Preliminary Results

Consider a tandem network viewed as a Stochastic Fluid Model (SFM) as shown in Fig. 1 with M nodes indexed by $m = 1, \ldots, M$. The outflow of node m is the inflow to node $m+1$, and we assume there is no feedback in the system. In the context of communication network applications, this implies that we limit ourselves here to network settings operating with protocols such as the User Datagram Protocol (UDP), but not the Transmission Control Protocol (TCP); the inclusion of feedback information that affects the incoming flow is a separate problem we address elsewhere (see [21]) and it has not yet been incorporated in this multinode analysis. Let b_m denote the buffer size of node m, $m = 1, \ldots, M$, where $b_m > 0$. At the first node, we consider the buffer size as a controllable parameter; equivalently, we view it as a *threshold* denoted by $\theta = b_1$ which is adjustable for the purpose of congestion control. We will assume that the real-valued parameter θ is confined to a closed and bounded (compact) interval Θ. The inflow rate of each node $m = 2, \ldots, M$ is denoted by $\alpha_m(\theta;t)$, to indicate the fact that it generally depends on θ, whereas $\alpha_1(t)$ is an external process independent of θ. The processing rate of node $m = 1, \ldots, M$ at time t is denoted by $\beta_m(t)$ and is independent of θ. The buffer level is denoted by $x_m(\theta;t)$, the outflow rate is denoted by $\delta_m(\theta;t)$ and the overflow rate is denoted by $\gamma_m(\theta;t)$. The external processes $\{\alpha_1(t)\}$ and $\{\beta_m(t)\}$, $m = 1, \ldots, M$, which are independent of θ, can have a very general form for the purpose of our analysis; in particular, they need not be statistically independent. We are interested in studying sample paths of this SFM over a time interval $[0, T]$ for a given fixed $0 < T < \infty$.

The dynamics of the buffer level $x_m(\theta;t)$, $m = 1, \ldots, M$, are described by the following one-sided differential equation:

$$\frac{dx_m(\theta;t)}{dt^+} = \begin{cases} 0, & \text{if } \mathrm{x_m}(\theta;\mathrm{t}) = 0 \text{ and} \\ & \quad \alpha_m(\theta;t) - \beta_m(t) \leq 0, \\ 0, & \text{if } \mathrm{x_m}(\theta;\mathrm{t}) = \mathrm{b_m} \text{ and} \\ & \quad \alpha_m(\theta;t) - \beta_m(t) \geq 0, \\ \alpha_m(\theta;t) - \beta_m(t), & \text{otherwise.} \end{cases} \tag{1}$$

where, to maintain uniformity in the notation, it is understood that $\alpha_1(\theta;t) = \alpha_1(t)$. With this convention in mind, the outflow rate from node $m = 1, \ldots, M-1$ is the inflow rate to the downstream node $m+1$, so that for all $m = 2, \ldots, M$ we have

$$\alpha_m(\theta;t) = \begin{cases} \beta_{m-1}(t), & \text{if } \mathrm{x_{m-1}}(\theta;\mathrm{t}) > 0 \\ \alpha_{m-1}(\theta;t), & \text{if } \mathrm{x_{m-1}}(\theta;\mathrm{t}) = 0 \end{cases}. \tag{2}$$

Finally, the overflow rate $\gamma_m(\theta;t)$ at node m due to a full buffer is defined by

$$\gamma_m(\theta;t) = \begin{cases} \alpha_m(\theta;t) - \beta_m(t), & \text{if } \mathrm{x_m}(\theta;\mathrm{t}) = \mathrm{b_m} \text{ and} \\ & \quad \alpha_m(\theta;t) - \beta_m(t) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For convenience, we define

$$A_m(\theta;t) := \alpha_m(\theta;t) - \beta_m(t). \tag{4}$$

We stress again that in this SFM the flow rates $\{\alpha_1(t)\}$ and $\{\beta_m(t)\}$, $m = 1, ..., M$, are treated as *stochastic* processes representing the random instantaneous rates of the arriving traffic and of the node processing rates. This is why in considering a typical sample path of the SFM (as in Fig. 2) the buffer content is shown not as piecewise linear (which corresponds to fixed flow rates over specific intervals), but only as piecewise analytic.

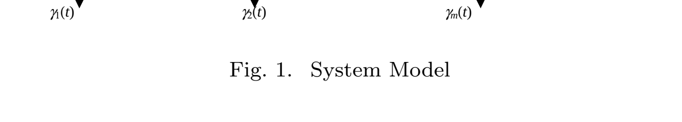

Fig. 1. System Model

Let us now take a closer look at (2) which describes the only connection between node m and its upstream nodes. The value of $\alpha_m(\theta;t)$, $m>1$, is given by either $\beta_{m-1}(t)$, which is independent of θ, or by $\alpha_{m-1}(\theta;t)$. In turn, the value of $\alpha_{m-1}(\theta;t)$ is given by either $\beta_{m-2}(t)$ or by $\alpha_{m-2}(\theta;t)$. Proceeding recursively, we find that the value of $\alpha_2(\theta;t)$ is either $\beta_1(t)$ or $\alpha_1(t)$ which are both independent of θ. Thus, the value of $\alpha_m(\theta;t)$ is ultimately given by one of the processes $\{\alpha_1(t)\}$ and $\{\beta_i(t),\ i=1,\ldots,m\}$ which are all independent of θ; the way in which $\alpha_m(\theta;t)$ switches among them depends on θ through the states $x_i(\theta;t),\ i=1,\ldots,m-1$ and the points in time when this switching occurs defines the "switchover points" discussed in the sequel.

Focusing on node m, the inflow process $\{\alpha_m(\theta;t)\}$ and the service process $\{\beta_m(t)\}$ are referred as *defining processes* of node m, since they define the local dynamics at that node. The buffer level $\{x_m(\theta;t)\}$, outflow process $\{\delta_m(\theta;t)\}$ and overflow process $\{\gamma_m(\theta;t)\}$ are referred as *derived processes*, since they can be derived from the defining processes via (1)-(3).

Viewing the network as a discrete event system, the SFM dynamics are dependent on a number of events. For the purpose of our analysis, we define an *event* of node $m=1,\ldots,M$ to be one of the following:

e_1 - A jump (discontinuity) in either $\alpha_m(\theta;t)$ or $\beta_m(t)$.

e_2 - A time instant when $A_m(\theta;t)$ becomes 0 with no discontinuity in $A_m(\theta;t)$ at t.

e_3 - A time instant when the buffer level $x_m(\theta;t)$ becomes full or empty.

Two types of sample performance metrics will be considered throughout this paper, both over the time interval $[0,T]$. The *loss volume* at node $m=1,\ldots,M$, denoted by $L_m(\theta;T)$, is defined by

$$L_m(\theta;T)=\int_0^T \gamma_m(\theta;t)dt, \tag{5}$$

and the *work* at node $m=1,\ldots,M$, denoted by $Q_m(\theta;T)$, is defined by

$$Q_m(\theta;T)=\int_0^T x_m(\theta;t)dt. \tag{6}$$

IPA provides the derivatives (gradient) of the sample performance functions with respect to various control parameters. In our case, we concentrate on the derivatives $L_m^{'}(\theta;T)$ and $Q_m^{'}(\theta;T)$, where we shall use the "prime" notation to denote a derivative with respect to θ throughout the rest of the paper.

Considering a typical sample path of the buffer level $x_m(\theta;t)$ in this SFM, as shown in Fig. 2, we observe that it can be decomposed into *Boundary Periods* (BP) and *Non-Boundary Periods* (NBP). A BP is one during which $x_m(\theta;t)=0$ or $x_m(\theta;t)=b_m$, whereas a NBP is one during which $0<x_m(\theta;t)<b_m$. A BP is further categorized as either an *Empty Period* (EP) during which $x_m(\theta;t)=0$ or as a *Full Period* (FP) during which $x_m(\theta;t)=b_m$. Since the function $x_m(\theta;t)$ is generally continuous in t for a fixed θ, we will consider EPs and FPs to be closed intervals and NBPs to be open intervals in the relative topology induced by $[0,T]$. Let

$$B_{m,n}=[\tau_{m,n}(\theta),\sigma_{m,n}(\theta)]$$

denote the nth BP, $n=1,\ldots,N_m$, where N_m is the total (random) number of BPs in $[0,T]$. Note that the start of $B_{m,n}$, $\tau_{m,n}(\theta)$, is an e_3 event of node m. For notational economy, we will omit θ in $\tau_{m,n}(\theta)$ and $\sigma_{m,n}(\theta)$ in what follows, but will keep in mind that $\tau_{m,n}$ and $\sigma_{m,n}$ are generally functions of θ. Next, observe that NBPs and BPs appear alternately throughout $[0,T]$ and let

$$\overline{B}_{m,n}=(\sigma_{m,n-1},\tau_{m,n})$$

denote the NBP that precedes $B_{m,n}$. For convenience, we shall set $\sigma_{m,0}=0$ and $\sigma_{m,N_m}=T$.

Depending on the value of $x_m(\theta;t)$ at the starting and ending points of a NBP $\overline{B}_{m,n}=(\sigma_{m,n-1},\tau_{m,n})$, we can define four types of NBPs ('E' stands for 'Empty' and 'F' stands for 'Full'):

1. (E,E): $x_m(\theta;\sigma_{m,n-1})=0$ and $x_m(\theta;\tau_{m,n})=0$.
2. (E,F): $x_m(\theta;\sigma_{m,n-1})=0$ and $x_m(\theta;\tau_{m,n})=b_m$.
3. (F,E): $x_m(\theta;\sigma_{m,n-1})=b_m$ and $x_m(\theta;\tau_{m,n})=0$.
4. (F,F): $x_m(\theta;\sigma_{m,n-1})=b_m$ and $x_m(\theta;\tau_{m,n})=b_m$.

In the example shown in Fig. 2, the BPs $B_{m,n-1}=[\tau_{m,n-1},\sigma_{m,n-1}]$, and $B_{m,n}=[\tau_{m,n},\sigma_{m,n}]$ are both FPs, whereas $B_{m,n+1}=[\tau_{m,n+1},\sigma_{m,n+1}]$ is an EP. The NBP $\overline{B}_{m,n-1}=(\sigma_{m,n-2},\tau_{m,n-1})$ is of type (E,F), $\overline{B}_{m,n}=(\sigma_{m,n-1},\tau_{m,n})$ is of type (F,F), $\overline{B}_{m,n+1}=(\sigma_{m,n},\tau_{m,n+1})$ is of type (F,E), and $\overline{B}_{m,n+2}=(\sigma_{m,n+1},\tau_{m,n+2})$ is of type (E,E).

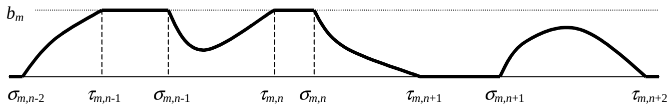

Fig. 2. Typical Sample Path of Node m

The switchover points of $\alpha_m(\theta;t)$ for $m>1$, as seen in (2), occur as follows: (i) Just before an EP of node $m-1$ starts, we have $\alpha_m(\theta;t)=\beta_{m-1}(t)$. When the EP starts, the output of $m-1$ switches from $\beta_{m-1}(t)$ to $\alpha_{m-1}(\theta;t)$. (ii) When the EP of node $m-1$ ends, the output of $m-1$ switches once again from $\alpha_{m-1}(\theta;t)$ to $\beta_{m-1}(t)$. (iii) The third instance is less obvious. During the EP at node $m-1$, it is possible that an EP at node $m-2$ starts, in which case $\alpha_{m-1}(\theta;t)$ switches from $\beta_{m-2}(t)$ to $\alpha_{m-2}(\theta;t)$. When this happens, the output of $m-1$ switches from $\alpha_{m-1}(t)$ to $\alpha_{m-2}(\theta;t)$, therefore, $\alpha_m(\theta;t)=\alpha_{m-1}(t)=\alpha_{m-2}(t)$. Clearly, it is possible that a sequence of j such events occurs so that $\alpha_m(\theta;t)=\alpha_{m-1}(t)=\ldots=\alpha_{m-j}(t)$, where $j=1,\ldots,m-1$. In this case, all nodes $m-j,\ldots,m-1$ are empty and m inherits all switchovers experienced by these upstream nodes as each one starts an EP.

For switchover points of $\alpha_m(\theta;t)$ under case (ii) above, we next prove that they are locally independent of θ.

Lemma II.1: Let σ_{m-1}, $m>1$, be a switchover point of $\alpha_m(\theta;t)$ with $\alpha_m(\theta;\sigma_{m-1}^-)=\alpha_{m-1}(\theta;\sigma_{m-1}^-)$ and $\alpha_m(\theta;\sigma_{m-1}^+)=\beta_{m-1}(\sigma_{m-1}^+)$. Then, σ_{m-1} is locally independent of θ.

Proof. See Appendix I.

It immediately follows from Lemma II.1 that the end of an EP is independent of θ. Moreover, for $m>2$, during an EP of node $m-1$ we can see in (2) that $\alpha_m(\theta;t)=\alpha_{m-1}(\theta;t)$, which implies that if a switchover occurs at $\alpha_{m-1}(\theta;t)$, this switchover will be inherited by $\alpha_m(\theta;t)$, as well as the θ-dependence of it.

This discussion motivates our definition of an *active* switchover point, which is generally a function of θ and is denoted by $s_{m,i}(\theta)$, $m>2$, $i=1,2,\ldots$:

Definition 1. A switchover point of $\alpha_m(\theta;t)$ is termed *active*, if:
1. $s_{m,i}(\theta)$ is the time when an EP at node $m-1$ starts; or
2. $s_{m,i}(\theta)$ is the time when $\alpha_{m-1}(\theta;t)$ experiences an active switchover within an EP of node $m-1$.

In Fig. 2, assuming $m>2$, the points $\tau_{m,n+1}$ and $\tau_{m,n+2}$ both start EPs and are, therefore, active switchover points of $\alpha_{m+1}(\theta;t)$. In addition, any point in $[\tau_{m,n+1},\sigma_{m,n+1}]$ is potentially an active switchover point of $\alpha_{m+1}(\theta;t)$ if it happens to be an active switchover point of $\alpha_m(\theta;t)$.

An active switchover point $s_{m,i}(\theta)$ at node m may belong to a BP $B_{m,n}$ or to a NBP $\overline{B}_{m,n}$. We define the following index sets that will help differentiating between different types of active switchover points depending on the type of interval they belong to:

$$\Psi_{m,n}:=\{i:s_{m,i}\in B_{m,n}\} \tag{7}$$
$$\Psi^o_{m,n}:=\{i:s_{m,i}\in(\tau_{m,n},\sigma_{m,n})\} \tag{8}$$
$$\overline{\Psi}_{m,n}:=\{i:s_{m,i}\in\overline{B}_{m,n}\} \tag{9}$$

Note that $B_{m,n}=[\tau_{m,n},\sigma_{m,n}]$, so we differentiate between open and closed intervals that define BPs in defining the sets $\Psi_{m,n}$ and $\Psi^o_{m,n}$. As we will see, of particular interest are active switchover points that coincide with the end of a FP, so we define the set of all BP indices that include such a point, Φ_m, as well as $\Gamma_m\subseteq\Phi_m$, a subset that includes those FPs that are followed by a NBP of type (F,E):

$$\Phi_m:=\{n:\sigma_{m,n}\text{ is an active switchover point},\ n=1,\ldots,N_m\} \tag{10}$$
$$\Gamma_m:=\{n:n\in\Phi_m\text{ and }\overline{B}_{m,n+1}\text{ is of type }(F,E)\}. \tag{11}$$

III. Infinitesimal Perturbation Analysis (IPA)

Our objective is to estimate the derivatives of the performance metrics $E[L_m(\theta;t)]$ and $E[Q_m(\theta;t)]$, where $L_m(\theta;t)$ and $Q_m(\theta;t)$ were defined in (5) and (6), through the sample derivative $L_m^{'}(\theta;T)$ and $Q_m^{'}(\theta;T)$, which is commonly referred to as the Infinitesimal Perturbation Analysis (IPA) estimators; comprehensive discussions of IPA and its applications can be found in [16],[17]. The IPA derivative-estimation technique computes the sample derivative $\mathcal{L}_{\mathcal{T}}^{'}(\theta)$ of some performance metric $\mathcal{L}_{\mathcal{T}}(\theta)$ along an observed sample path ω. An IPA-based estimate $\mathcal{L}_{\mathcal{T}}^{'}(\theta)$ of a performance metric derivative $dE[\mathcal{L}_{\mathcal{T}}(\theta)]/\lceil\theta$ is *unbiased* if $dE[\mathcal{L}_{\mathcal{T}}(\theta)]/\lceil\theta=\mathcal{E}[\mathcal{L}_{\mathcal{T}}^{'}(\theta)]$. Unbiasedness is the principal condition for making the application of IPA useful in practice, since it enables the use of the sample (IPA) derivative in control and optimization methods that employ stochastic gradient-based techniques.

The case of a single node where we are interested in $L_1'(\theta;T)$ and $Q_1'(\theta;T)$ has been studied in [10], so here we address the inter-node effects and study the resulting IPA estimators $L_m^{'}(\theta;T)$ and $Q_m^{'}(\theta;T)$ for $m>1$. Due to the tandem topology and the absence of feedback between nodes, the inter-node effects have only one direction: from upstream to downstream. Therefore, our analysis is based on the impact of the threshold parameter at the first node on performance metrics at the downstream nodes.

Since we are concerned with the sample derivatives $L_m^{'}(\theta;T)$ and $Q_m^{'}(\theta;T)$ we have to identify conditions under which they exist. As we will see, these derivatives depend on the derivatives of the active switchover points, i.e., specific event times, with respect to θ. Excluding the possibility of the simultaneous occurrence of two events (e_1, e_2, or e_3 as defined earlier), the only situation preventing the existence of these derivatives involves some t such that $A_m(\theta;t)=\alpha_m(\theta;t)-\beta_m(t)=0$; in such cases, the one-sided derivatives exist and can be obtained through a finite difference analysis (as in [10]). However, to keep the analysis simple, we focus only on the differentiable case by proceeding under the following technical conditions:

Assumption 1.

a. W.p.1, the functions $\alpha_1(t)$, and $\beta_m(t)$, $m=1,\ldots,M$ are piecewise analytic in the interval $[0,T]$.
b. For every $\theta\in\Theta$, w.p.1 no two events of a certain node m occur at the same time.
c. W.p.1, no two processes $\{\alpha_1(t)\}$, $\{\beta_m(t),m=1,\ldots,M\}$ have identical values during any open subinterval of $[0,T]$.

All three parts of **Assumption 1** are mild technical conditions. Regarding part *c*, note that $\alpha_m(\theta;t)$, through (2), ultimately depends on one or more of the processes $\{\alpha_1(t)\}$, $\{\beta_i(t)\}$, $i=1,\ldots,m$, therefore the requirement $A_m(\theta;t)\neq 0$ is reflected by the general statement under *c*.

Recall that a switchover point of $\alpha_m(\theta;t)$ is the time it switches among $\{\alpha_1(t)\}$ and $\{\beta_i(t)\}$, $i=1,\ldots,m$. It is possible that a switchover may not cause a jump (discontinuity) in $\alpha_m(\theta;t)$; for example, at $t=s$, $\alpha_m(\theta;t)$ switches from $\alpha_{m-1}(\theta;t)$ to $\beta_{m-1}(t)$ while $\alpha_{m-1}(\theta;s)=\beta_{m-1}(s)$ and such a switchover is not qualified as a node m event (e_1, e_2, or e_3 as defined earlier). The following lemma is a consequence of **Assumption 1** and shows that for an *active* switchover point, $\alpha_m(\theta;t)$ must experience a jump. Recall that an active switchover point $s_{m,i}(\theta)$ is generally a function of θ, but, for the sake of notational simplicity, we shall simply write $s_{m,i}$.

Lemma III.1: If an active switchover point of $\alpha_m(\theta;t)$ occurs at $t=s_{m,i}$, then w.p. 1 it is an e_1 event of node m.

Proof. See Appendix I.

A. Queue Content Derivatives

We shall proceed by determining the derivative $x_m^{'}(\theta;t)$ of a buffer level in the SFM with respect to the controllable parameter θ and will show that it depends exclusively on the way that θ affects the switchover points of $\alpha_m(\theta;t)$ that were termed "active" in **Definition 1**. Focusing on active switchover points $s_{m,i},\ i=1,2,\ldots$ we define the following two quantities for $m>1$ that turn out to be crucial in our analysis:

$$\psi_{m,i} \ := \ [\alpha_m(\theta;s_{m,i}^{+})-\alpha_m(\theta;s_{m,i}^{-})]s_{m,i}^{'}, \tag{12}$$

and, for $n\in\Phi_m$:

$$\phi_{m,n} := \ [\alpha_m(\theta;\sigma_{m,n}^{+})-\beta_m(\sigma_{m,n})]\sigma_{m,n}^{'}. \tag{13}$$

Let us now consider the derivative $x_m^{'}(\theta;t)$ of a buffer level in the network with respect to the controllable parameter θ. The case $m=1$ was considered in [10], so we shall focus on cases with $m>1$. The following establishes the connection between $x_m^{'}(\theta;t)$ and the two crucial quantities defined above. Note that $\mathbf{1}\left[\cdot\right]$ is the usual indicator function.

Lemma III.2: If $m=1$, for $n=1,...,N_1$

$$x_1^{'}(\theta;t)=\begin{cases}1 & \text{if } t\in B_{1,n} \text{ or } t\in\overline{B}_{1,n+1},\ x_1(\theta;\sigma_{1,n})=\theta\\ 0 & \text{otherwise}\end{cases} \tag{14}$$

If $m>1$, then for $n=1,...,N_m$

$$x_m^{'}(\theta;t)=$$
$$\begin{cases}0 & \text{if } t\in B_{m,n}\\ -\sum_{k=1}^{K_{m,n}(t)}\psi_{m,k}\ -\mathbf{1}\left[\mathbf{n}\in\mathbf{\Phi_m}\right]\cdot\phi_{\mathbf{m,n}} & \text{if } t\in\overline{B}_{m,n+1}\end{cases} \tag{15}$$

where $K_{m,n}(t)$ is the number of active switchover points in the interval $(\sigma_{m,n},t)\subset\overline{B}_{m,n+1}$.

Proof. See Appendix I.

It is now clear from (15) that $\psi_{m,k}$ and $\phi_{m,n}$ are crucial quantities associated with node m. In the next two lemmas, we show that they provide the means to connect $x_m^{'}(\theta;t)$ to $x_{m-1}^{'}(\theta;t)$ and hence shed light into the way in which buffer level perturbations propagate across nodes.

Lemma III.3: For $m>1$, let $s_{m,i}$ be an active switchover point of $\alpha_m(\theta;t)$. If it is the start of an EP at node $m-1$, then

$$\psi_{m,i} \ = -x_{m-1}^{'}(\theta;s_{m,i}^{-}) \tag{16}$$

Otherwise, if $s_{m,i}$ occurs during an EP of node $m-1$, then

$$\psi_{m,i}=\psi_{m-1,j} \tag{17}$$

for some j such that $s_{m,i}=s_{m-1,j}$.

Proof. See Appendix I.

Next for $m>1$**,** we define:

$$R_{m,n}(\theta) := \frac{\alpha_m(\theta;\sigma_{m,n}^{+})-\beta_m(\sigma_{m,n})}{\alpha_m(\theta;\sigma_{m,n}^{+})-\alpha_m(\theta;\sigma_{m,n}^{-})} \tag{18}$$

By definition, $\sigma_{m,n}$ is the end of a BP at node m. We will make use of $R_{m,n}(\theta)$ when $n\in\Phi_m$, i.e., when $\sigma_{m,n}$ happens to be an active switchover point. If this is the case, then it follows from Lemma III.1 and **Assumption 1(*b*)** that $\beta_m(t)$ is continuous at $t=\sigma_{m,n}$. Note that this quantity involves the processing rate information $\beta_m(\sigma_{m,n})$ (typically known, otherwise measurable) at $t=\sigma_{m,n}$, and the values of the incoming traffic rates before and after a BP ends at $t=\sigma_{m,n}$. Using this definition, the next lemma allows us to obtain a simple relationship between the two crucial quantities $\psi_{m,i}$ and $\phi_{m,n}$.

Lemma III.4: Let $n\in\Phi_m$ and $\sigma_{m,n}=s_{m,i}$ for some active switchover point of $\alpha_m(\theta;t)$. Then,

$$\phi_{m,n}=R_{m,n}(\theta)\cdot\psi_{m,i} \tag{19}$$

where

$$0<R_{m,n}(\theta)\leq\ 1. \tag{20}$$

Proof. See Appendix I.

Combining Lemmas III.2-III.4 we obtain the following:

Theorem III.1: For $m>1$ and $n=1,\ldots,N_m$:

$$x_m^{'}(\theta;t)=$$
$$\begin{cases}0 & \text{if } t\in B_{m,n}\\ \sum_{k=1}^{K_{m,n}(t)}x_{m-i^*}^{'}(\theta;s_{m,k}^{-})\ + & \\ \mathbf{1}\left[\mathbf{n}\in\mathbf{\Phi_m}\right]\cdot\mathbf{R_{m,n}}(\theta)\mathbf{x}_{\mathbf{m-i^*}}^{'}(\theta;\sigma_{\mathbf{m,n}}^{-}) & \text{if } t\in\overline{B}_{m,n+1}\end{cases} \tag{21}$$

where

$$i^* := \min_{j=1,\ldots,m-1}\{j: x_{m-j}(\theta;s_{m,k})>0\} \tag{22}$$

and $K_{m,n}(t)$ is the number of active switchover points in the interval $(\sigma_{m,n},t)\subset\overline{B}_{m,n+1}$.

Proof. See Appendix I.

Taking a closer look at (21) we get significant insight regarding the process through which changes in the buffer level of one node affect the buffer levels of downstream nodes. Let us view $x_m^{'}(\theta;t)$ as a perturbation in $x_m(\theta;t)$. For simplicity, let us initially ignore the case where $n\in\Phi_m$ and assume $i^*=1$. Thus, we have $x_m^{'}(\theta;t)=\sum_{k=1}^{K_{m,n}(t)}x_{m-1}^{'}(\theta;s_{m,k}^{-})$ if $t\in\overline{B}_{m,n+1}$. We can see that node $m-1$ only affects node m at time $s_{m,k}$ when an EP at node $m-1$ starts (recalling Definition 1). In simple terms: *whenever node $m-1$ becomes empty, it propagates downstream to m its current perturbation.* These perturbations accumulate at m over all $K_{m,n}(t)$ active switchover points contained in a NBP $\overline{B}_{m,n+1}$. For example, in Fig. 3, $s_{m,i+1}$ is a point where an EP ends at node $m-1$ while node m is in a NBP; at that time we get $x_m^{'}(\theta;t)=x_{m-1}^{'}(\theta;s_{m,i+1}^{-})$. Moreover, when the NBP ends at $\tau_{m,n+1}$, the value of $x_m^{'}(\theta;\tau_{m,n+1}^{-})$ is in turn propagated downstream to $m+1$, before setting $x_m^{'}(\theta;\tau_{m,n+1}^{+})=0$ at the start of the ensuing EP at m.

Any cumulative perturbation at m is *eliminated by the presence of any BP*, i.e., when $t\in B_{m,n}$ as indicated by (21). For example, in Fig. 3, $s_{m,i}$ is a point where an EP ends at node $m-1$ while node m is in a FP; therefore, it has no effect on $x_m(\theta;t)$, i.e., $x_m^{'}(\theta;t)=0$. The conclusion is that in order for a node to have a chance to propagate a

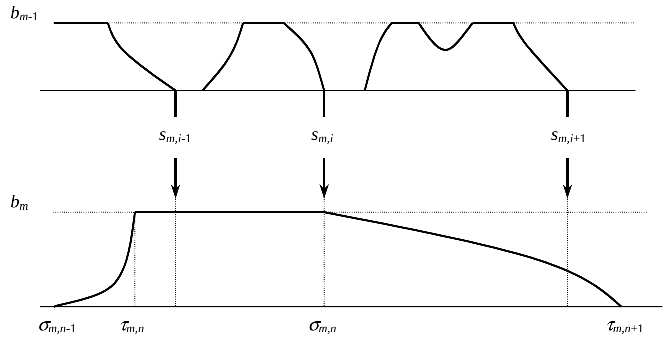

Fig. 3. A sample path example with two adjacent nodes and three active switchover points for $\alpha_m(\theta;t)$

perturbation downstream, *it must become empty before it becomes full.* In view of this fact, we can argue that control at the edge of a tandem network is generally expected to have a limited impact on nodes that are several hops away, since propagating perturbations requires the combination of several events: a perturbation to be present and to be propagated at the start of an EP before it is eliminated by a FP; moreover this has to be true for a sequence of nodes. The probability of such a joint event is likely to be small as the number of hops increases. This provides an analytical substantiation to the conjecture that congestion in a network cannot be easily regulated through control exercised several hops away, unless the intermediate nodes experience frequent EPs providing the opportunity for perturbation propagation events.

Let us now look at the two aspects that were ignored in the discussion above. First, suppose that $i^* > 1$. This means that an EP occurs not just at node $m-1$, but also nodes $m-2,\ldots,m-i^*$, all at the same time. Thus, instead of propagating a perturbation from $m-1$ to m, the propagation now takes place from $m-i^*$ to m. Second, let us consider the case where $n \in \Phi_m$ in (21). This allows an EP that starts at $m-1$ to cause the end of a FP at node m. When this occurs, only a fraction, given by $R_{m,n}(\theta)$, of the perturbation at $m-1$ is propagated to node m. For example, in Fig. 3, the point $s_{m,i}$ coincides with $\sigma_{m,n}$ and it therefore contributes another term scaled by $R_{m,n}$ as seen in (21).

Finally, note that the discussion above is independent of the way in which the controllable parameter affects the buffer content at $m=2$ and subsequently all downstream nodes through (21). In the particular case we are considering, however, we can see from (14) that the derivatives at node 1 are always given by 1. Thus, *the entire perturbation analysis process here reduces to counting EP events at all nodes that cause propagations* through (21). The only exception is for those events that end an EP at some $m-1$ and at the same time a FP at m; in this case, the derivative at node m is affected by some amount dependent on $R_{m,n}(\theta) \in (0,1]$.Up to this point, we have characterized the mechanism through which $x_m^{'}(\theta;t)$ can be evaluated recursively for all $m=1,\ldots,M$, making use of the quantities $\psi_m(s_{m,i})$ and $\phi_m(s_{m,i})$. In the next two sections, we concentrate on the sample derivatives of the two performance metrics we have identified, $L_m(\theta;T)$ and $Q_m(\theta;T)$ defined in (5) and (6). The case of $L_1(\theta;T)$ and $Q_1(\theta;T)$ was considered in [10], so we will focus on $m>1$ in what follows.

B. The IPA Derivative $L_m^{'}(\theta;T)$

Our objective here is to estimate the derivative of the expected loss volume $E[L_m(\theta;T)]$ at node $m=2,\ldots,M$ through the sample derivative $L_m^{'}(\theta;T)$. Let us define $\digamma_m$ to be the set of all indices of BPs that happen to be FPs at node m over $[0,T]$, i.e.,

$$\digamma_m := \{n : x_m(\theta;t) = b_m \text{ for all } t \in B_{m,n},\ n=1,\ldots,N_m\}.$$

Observing that only FPs at node m will experience loss, we have

$$L_m(\theta;T) = \sum_{n\in\digamma_m} \int_{\tau_{m,n}}^{\sigma_{m,n}} \gamma_m(\theta;t)dt,$$

and

$$L_m^{'}(\theta;T) = \sum_{n\in\digamma_m} \frac{d}{d\theta}\int_{\tau_{m,n}}^{\sigma_{m,n}} \gamma_m(\theta;t)dt. \tag{23}$$

By Lemma III.1 and **Assumption 1**(b), $\tau_{m,n}$ cannot be an active switchover point, since at $\tau_{m,n}$ a node m event of type e_3 must occur. Therefore, for any $n\in\digamma_m$, active switchover points can occur either in the open FP interval $(\tau_{m,n},\sigma_{m,n})$ or they may coincide with the end of the FP at time $\sigma_{m,n}$.

To establish an expression for $L_m^{'}(\theta;T)$ in terms of observable sample path data we need three preliminary results, stated below as Lemmas III.5-III.7. Since we focus on node m, we drop the subscript m for notational convenience in presenting these results.

Lemma III.5: For $n\in\digamma$,

$$\frac{d}{d\theta}\int_{\tau_n}^{\sigma_n}\gamma(\theta;t)dt = \left[A(\theta;\sigma_n^-)\sigma_n^{'} - A(\theta;\tau_n)\tau_n^{'}\right] - \sum_{k\in\Psi_n^o}\psi_k \tag{24}$$

Proof. See Appendix I.

Lemma III.6: For $n\in\digamma$,

$$A(\theta;\tau_n)\tau_n^{'}(\theta) \;=\; \sum_{k\in\overline{\Psi}_n}\psi_k + A(\theta;\sigma_{n-1}^+)\sigma_{n-1}^{'} \tag{25}$$

Proof. See Appendix I.

The next result concerns the end point σ_n of a FP.

Lemma III.7: For $n\in\digamma$,

$$[A(\theta;\sigma_n^+) - A(\theta;\sigma_n^-)]\sigma_n^{'} = \begin{cases} \psi_i, & \text{if } n\in\Phi \text{ with } \sigma_n = s_i \\ 0, & \text{if } n\notin\Phi \end{cases} \tag{26}$$

Proof. See Appendix I.

We can now obtain the IPA derivative $L_m^{'}(\theta;T)$, using once again the subscript m. We will also introduce the set

$$\Omega_{m,n} = \Psi_{m,n}\cup\overline{\Psi}_{m,n} \tag{27}$$

which, recalling (7) and (9), includes the indices i of all active switchover points in the BP $B_{m,n} = [\tau_{m,n}(\theta), \sigma_{m,n}(\theta)]$ and the NBP that precedes it $\overline{B}_{m,n} = (\sigma_{m,n-1}, \tau_{m,n})$.

Theorem III.2: The loss volume IPA derivative, $L_m^{'}(\theta;T)$, $m = 2, \ldots, M$, has the following form:

$$L_m^{'}(\theta;T) = -\sum_{n\in F_m}\sum_{i\in\Omega_{m,n}} \psi_{m,i} + \sum_{n\in\Gamma_m} \phi_{m,n} \tag{28}$$

where $\psi_{m,i}$ and $\phi_{m,n}$ are given by (16)-(17) and (19).
Proof. See Appendix I.

In simple terms, to obtain $L_m^{'}(\theta;T)$ we accumulate terms $-\psi_{m,i}$ over all active switchover points $s_{m,i}$ for each interval $(\sigma_{m,n-1}, \sigma_{m,n}]$, $n = 1, 2, \ldots$ However, the result contributes to $L_m^{'}(\theta;T)$ only if $\sigma_{m,n}$ ends a FP. The second term of (28) modifies the accumulation process as follows: Occasionally, $\sigma_{m,n}$ is followed by a NBP $(\sigma_{m,n}, \tau_{m,n+1})$ of type (F,E), i.e., the buffer at node m becomes empty. When this event takes place, the contribution $-\psi_{m,i}$ for $s_{m,i} = \sigma_{m,n}$ is modified by adding $\phi_{m,n}$ to it. In the example shown in Fig. 3, there are two active switchover points in the interval $(\sigma_{m,n-1}, \sigma_{m,n}]$ at $s_{m,i-1}$ and at $s_{m,i}$. These contribute terms $-\psi_{m,i-1}$ and $-\psi_{m,i}$ to $L_m^{'}(\theta;T)$ since the BP that ends at $\sigma_{m,n}$ is a FP. The second one happens to coincide with the end of the FP, i.e., $s_{m,i} = \sigma_{m,n}$. Since the next NBP is of type (F,E), we have $n \in \Gamma_m$ and a term $\phi_{m,n}$ is contributed to $L_m^{'}(\theta;T)$. In addition, the active switchover point at $s_{m,i+1}$ does not contribute to $L_m^{'}(\theta;T)$.

The terms $\psi_{m,i}$ and $\phi_{m,n}$ are given in Lemmas III.3 and III.4, where we can see that they depend on the derivatives $x_{m-1}^{'}(\theta;s_{m,i}^{-})$ propagated from the upstream node $m-1$ through every EP event that occurs at $m-1$. These derivatives are in turn provided by (21) in Theorem III.1. We emphasize the fact that, as in earlier work for a single node SFM [10], the IPA estimator *does not involve any knowledge of the stochastic processes characterizing arriving traffic or node processing* and allows for the possibility of correlations. The only information involved is the one required to calculate $R_{m,n}$ in (21), which, incidentally, occurs only when the end of a FP happens to be an active switchover point; one can argue that under certain loading conditions such contributions (recall also that $0 < R_{m,n} \leq 1$) are minimal and could be ignored for the benefit of obtaining computationally efficient approximations; in this case, (28) becomes a simple counter, since the values of $\psi_{m,i}$ are originally given by -1 at node 1, as seen in (14). This is further discussed in Section 4.

Theorem III.3: The IPA derivative, $L_m^{'}(\theta;T)$, $m = 2, \ldots, M$, is unbiased, i.e.,

$$E\left[L_m^{'}(\theta;T)\right] = \frac{dE[L_m(\theta;T)]}{d\theta}$$

Proof: See Appendix II. ■

C. The IPA Derivative $Q_m^{'}(\theta;T)$

Recall the definition of $Q_m(\theta;T)$ in (6). By partioning $[0,T]$ into NBPs and BPs and recalling that N_m was defined as the total number of BPs in $[0,T]$, we have

$$Q_m(\theta;T) = \sum_{n=1}^{N_m}\left[\int_{\sigma_{m,n-1}}^{\tau_{m,n}} x_m(\theta;t)dt + \int_{\tau_{m,n}}^{\sigma_{m,n}} x_m(\theta;t)dt\right]$$

Upon taking derivatives with respect to θ and in view of the fact that $x_m(\theta;t)$ is continuous in t, we obtain

$$\begin{aligned} Q_m^{'}(\theta;T) &= \sum_{n=1}^{N_m}\int_{\sigma_{m,n-1}}^{\tau_{m,n}} x_m^{'}(\theta;t)dt \\ &+ \sum_{n=1}^{N_m}\left\{x_m(\theta;\tau_{m,n})\tau_{m,n}^{'} - x_m(\theta;\sigma_{m,n-1})\sigma_{m,n-1}^{'}\right\} \\ &+ \sum_{n=1}^{N_m}\int_{\tau_{m,n}}^{\sigma_{m,n}} x_m^{'}(\theta;t)dt \\ &+ \sum_{n=1}^{N_m}\left\{x_m(\theta;\sigma_{m,n})\sigma_{m,n}^{'} - x_m(\theta;\tau_{m,n})\tau_{m,n}^{'}\right\} \end{aligned}$$

After taking into account the cancellation of several terms and in view of the fact that $\sigma'_{m,0} = \sigma'_{m,N_m} = 0$, this reduces to

$$Q_m^{'}(\theta;T) = \sum_{n=1}^{N_m}\left[\int_{\sigma_{m,n-1}}^{\tau_{m,n}} x_m^{'}(\theta;t)dt + \int_{\tau_{m,n}}^{\sigma_{m,n}} x_m^{'}(\theta;t)dt\right]. \tag{29}$$

We can now make use of the expression for $x_m^{'}(\theta;t)$ derived in Lemma III.2 and Theorem III.1 to obtain the IPA estimator $Q_m^{'}(\theta;T)$ for $m = 2, \ldots, M$.

Theorem III.4: The workload IPA derivative, $Q_m^{'}(\theta;T)$, $m = 2, \ldots, M$, has the following form:

$$\begin{aligned} Q_m^{'}(\theta;T) &= -\sum_{n=1}^{N_m}\sum_{i\in\overline{\Psi}_{m,n}} [\tau_{m,n} - s_{m,i}]\psi_{m,i} \\ &\quad - \sum_{n\in\Phi_m} [\tau_{m,n+1} - \sigma_{m,n}]\phi_{m,n} \end{aligned} \tag{30}$$

where $\psi_{m,i}$ and $\phi_{m,n}$ are given by (16)-(17) and (19).
Proof. See Appendix I.

For a simple interpretation of the IPA estimator (30), note that, similar to the IPA estimator in (28), it involves accumulating terms $-\psi_{m,i}$ over active switchover points $s_{m,i}$. In this case, however, we are only interested in $s_{m,i}$ contained in NBPs $(\sigma_{m,n-1}, \tau_{m,n})$, $n = 1, \ldots, N_m$. The accumulation is done at $\tau_{m,n}$ with each such term scaled by $[\tau_{m,n} - s_{m,i}]$ measuring the time elapsed since the switchover point took place. The second term in (30) adds similar contributions made at the end of a NBP of type (F,E) due to active switchover points that coincide with the end of a FP at some time $\sigma_{m,n}$.

Theorem III.5: The IPA derivative, $Q_m^{'}(\theta;T)$, $m = 2, \ldots, M$, is unbiased, i.e.,

$$E\left[Q_m^{'}(\theta;T)\right] = \frac{dE[Q_m(\theta;T)]}{d\theta}$$

Proof. See Appendix II.

IV. Experimental Network Optimization Results

This section presents results of simulation experiments in which we optimized a weighted sum of loss and workload in the two-queue tandem system shown in Fig. 4, as a function of the buffer limits (buffer sizes) at the two queues. All of the experiments were performed using the *Georgia Tech Network Simulator* (*GTNetS*) [22], modified to include the requisite IPA derivative calculations.

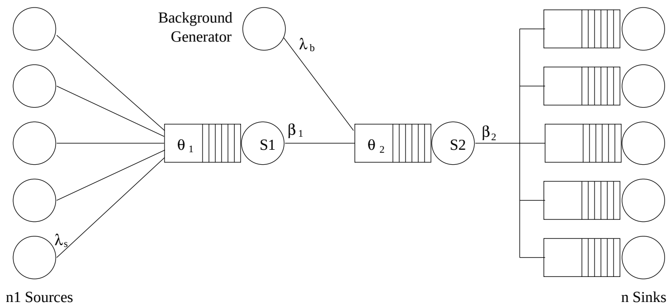

Fig. 4. Topology: Two-Stage Simulations

The approach we have taken here is to purposefully adopt a very practical engineering point of view in trying to integrate the analytical results of the previous section with a stochastic optimization methodology. We have made several simplifications, our goal being to test the "practical" value of using IPA estimates to dynamically improve (in an acceptable time scale) network performance within an optimization framework. First of all, because of the simple form of the IPA estimators of the derivatives of loss (28) and work (30) for the SFM, all data required for their evaluation can be directly obtained from a sample path of the actual queueing system, as was also done and explained in detail in our earlier work [10]. In other words, the *form* of the IPA estimators is obtained by analyzing the system as a SFM, but the associated *values* are based on real data. This provides a good approximation of the performance derivative estimates of the queueing system (which, if obtained directly from the queueing system, would be biased). Secondly, we implemented a standard stochastic approximation technique (e.g., see [23]) in conjunction with the IPA derivatives obtained in Section 3, but included some simple heuristics that are empirically known to accelerate convergence, at the expense of staying within the bounds of the usual technical conditions required to guarantee convergence. In addition, although all our analysis is based on the assumption that all observed sample paths start with all queues at the empty state, we have nonetheless applied the IPA estimates at the nth iteration of the optimization algorithm using the ending state of the $(n-1)$th iteration.

The final simplifying step we have taken concerns the contribution of the term involving $\phi_{m,n}$ in the IPA estimators (28) and (30). As already argued, based on (14), (16)-(17), and (19)-(20), each instance of this term is bounded by $[0,1]$. Moreover, the term is nonzero only when an active switchover point coincides with the end of a FP at node 2, i.e., an EP starts at node 1 causing a FP to end at node 2. This is likely to occur only when the buffer limits are largely imbalanced (that of node 2 is too small), in which case the performance sensitivity with respect to the buffer limit of node 2 is expected to be large (hence, the buffer limit of node 2 will be increased at the next algorithm iteration), making the contribution of a term bounded by $[0,1]$ likely to be negligible. Since this argument is obviously not rigorous, we proceeded by performing the optimization process twice: once with all these terms ignored, and once with the values of these terms, whenever they arise, set to their maximum value of 1. We found the results numerically indistinguishable, substantiating this approximation. The significance of the approximation cannot be overemphasized: without the inclusion of the term involving $\phi_{m,n}$ in the IPA estimators, these estimators are fully *nonparameteric*, i.e., they require only simple event counters and timers and no traffic rate information whatsoever, since $R_{m,n}$ in (18) is no longer involved.

In the system of Fig. 4, intended to represent the operation of a communication network, the inflow process at the first queue consists of n_1 multiplexed *on–off* data sources generating bursty traffic. When in the *on* state, each source generates a continuous data stream at the rate of α bits per second. These data streams are used to construct 554-byte UDP packets which are forwarded to the buffer at the first queue and thence across the rest of the network. The *on* times and *off* times are iid random variables sampled from the exponential distribution with mean 0.1 seconds. The channel transporting packets from the first queue to the second queue has a capacity of β_1 bps. The inflow process to the second queue consists of the outflow process from the first queue and of traffic from the background generator. The backgrond traffic consists of n_2 independent sources. Each one of these sources has the same statistical characteristics as the sources to the first queue. The outgoing channel from the second queue has a capacity of β_2 bps.

Note that the average bit rate from either one of the independent sources is $\alpha/2$ bps, since the expected durations of the off periods and the on periods are identical. Therefore, the expected bit rate of the aggregate flow to the first queue is $(n_1\alpha/2) \times (554/512)$, where the latter term accounts for the insertion of the headers. Consequently, the traffic intensity at the first queue, denoted by ρ_1, is given by

$$\rho_1 \;=\; n_1 \times \frac{\alpha}{2} \times \frac{554}{512} \times \frac{1}{\beta_1}. \tag{31}$$

Similarly, the traffic intensity of the second queue is denoted by ρ_2. In our simulation experiments we set $n_1 = n_2 = 100$, $\beta_1 = 10\ Mbps$, and $\beta_2 = 20\ Mbps$. Our simulation program was designed to utilize the traffic intensities as simulation input, and we set $\rho_1 = \rho_2 = 0.95$. The program then calculated α according to (31).

Let $\theta = (\boldsymbol{\theta_1}, \boldsymbol{\theta_2})$ denote the two-dimensional parameter vector consisting of the buffer limits at the first and second queue respectively. Recall that the loss volumes and workloads at the two queues are denoted by $L_j(\theta;\mathbf{T})$ and $Q_j(\theta;\mathbf{T})$, $j = 1,2$. Let us define the cost function $F(\theta;\mathbf{T})$ as the following weighted sum of the average loss rate and workload rate.

$$F(\theta;\mathbf{T}) = \frac{1}{T}\left[L_1(\theta;\mathbf{T}) + \mathbf{10Q_1}(\theta;\mathbf{T}) + L_2(\theta;\mathbf{T}) + \mathbf{20Q_2}(\theta;\mathbf{T})\right]. \quad (32)$$

We set the value of T to $T = 1$. We seek to minimize this function using a standard stochastic approximation technique (e.g., see [23]) in conjunction with the IPA derivatives obtained in Section 3. The optimization algorithm iteratively computes a sequence of points, $\theta(\mathbf{i}) = (\theta(\mathbf{i})_\mathbf{1}, \theta(\mathbf{i})_\mathbf{2})$, $i = 1, 2, \ldots,$. Its basic iteration step has the form:

$$\theta(\mathbf{i+1}) \;=\; \theta(\mathbf{i}) - \zeta(\mathbf{i})\mathbf{h}(\mathbf{i}), \quad (33)$$

where $\zeta(i) \geq 0$ is the ith stepsize (we adopted $\zeta(i) = 10/i^{0.6}$), and $h(i)$ is an estimate of the gradient of $F(\theta(\mathbf{i});\mathbf{T})$ obtained via IPA. As already pointed out, because of the simple form of the IPA estimators (28) and (30) for the SFM, all data required for their evaluation can be directly obtained from a sample path of the actual queueing system. In addition, we used a simple heuristic to bound the displacement $\theta(\mathbf{i+1}) - \theta(\mathbf{i})$ along each coordinate by modifying the vector $h(i) = (h(i)_1, h(i)_2)$ as follows. We first computed the partial derivatives $\frac{\partial F(\theta(\mathbf{i});\mathbf{T})}{\partial \theta(\mathbf{i})_\mathbf{j}}$, $j = 1, 2$. If $|\zeta(i)\frac{\partial F(\theta(\mathbf{i});\mathbf{T})}{\partial \theta(\mathbf{i})_\mathbf{j}}| \leq 5$ then we set $h(i)_j = \frac{\partial F(\theta(\mathbf{i});\mathbf{T})}{\partial \theta(\mathbf{i})_\mathbf{j}}$, and if $|\zeta(i)\frac{\partial F(\theta(\mathbf{i});\mathbf{T})}{\partial \theta(\mathbf{i})_\mathbf{j}}| > 5$ then we set $h(i)_j = 5\mathrm{sgn}(\frac{\partial \mathrm{F}(\theta(\mathbf{i});\mathbf{T})}{\partial \theta(\mathbf{i})_\mathbf{j}})/\zeta(\mathrm{i})$.

The parameters $\theta(\mathbf{i})_\mathbf{j}$ ($j = 1, 2$) were considered as real numbers, but the simulation runs were performed at the respective integer values closest to them. Recall that the simulation time horizon at each iteration point $\theta(\mathbf{i})$ was $T = 1.0$ second. The simulation state at the end of each iteration was preserved, and used as the initial state for the simulation at the next iteration point, $\theta(\mathbf{i+1})$. Likewise, we preserved the final state of the process of computing the IPA derivative, and used it as the initial state for the IPA derivative process at the next iteration. Note that only one random seed is called for each optimization experiment.

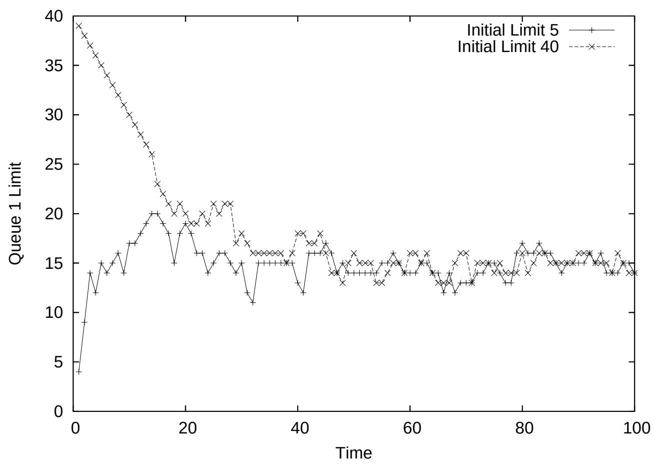

Fig. 5. $\theta(i)_1$ Adjustments vs. Time

We ran the optimization algorithm twice, with two different initial parameters: first with $\theta(\mathbf{1}) = (\mathbf{5}, \mathbf{5})$, and then

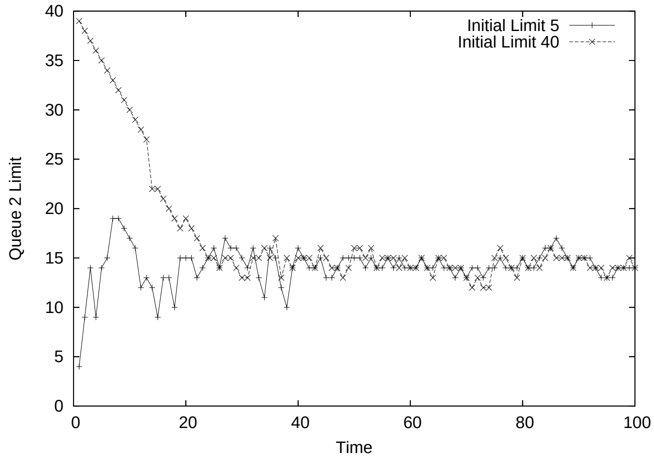

Fig. 6. $\theta(\mathbf{i})_\mathbf{2}$ Adjustments vs. Time

with $\theta(\mathbf{1}) = (\mathbf{40}, \mathbf{40})$. In either case we ran the algorithm for 100 iterations (i.e., 100 seconds). For each experiment, we plotted the evolution of $\theta(\mathbf{i})_\mathbf{1}$ and $\theta(\mathbf{i})_\mathbf{2}$ as a function of i, and show the results in Figs. 5 and 6 respectively. Each of the figures shows one trajectory for the $\theta(\mathbf{1}) = (\mathbf{5}, \mathbf{5})$ initial condition, and a second one for the $\theta(\mathbf{1}) = (\mathbf{40}, \mathbf{40})$ initial condition. The results indicate asymptotic convergence to approximately $\hat{\theta} = (\mathbf{15}, \mathbf{14})$ within approximately 20 seconds. As already mentioned, this optimization process was performed without the term involving $\phi_{m,n}$ in the IPA estimators (28) and (30); it was then repeated with the inclusion of this term set to 1 (its upper bound) in all instances when it arises and the results obtained corresponding to Figs. 5 and 6 were numerically indistinguishable.

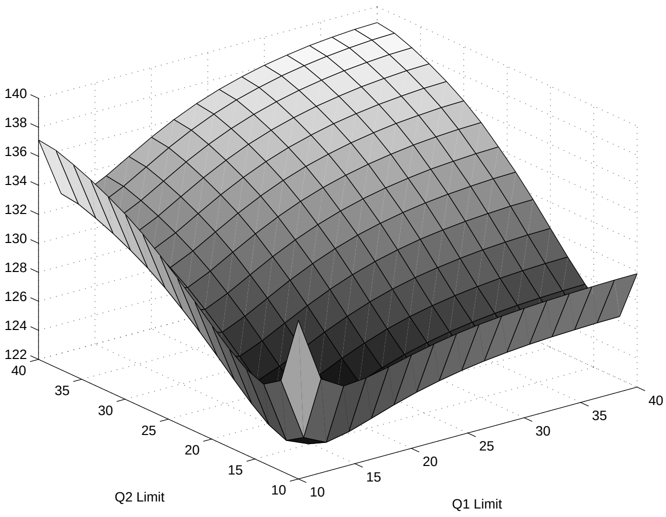

Fig. 7. Cost Function $F(\theta_1, \theta_2; T)$

Finally, to add validity to these results, we plotted the graph of $F(\theta_1, \theta_2; T)$ as shown in Fig. 7. Each point on the plot is the average of 10 separate simulation experiments with $T = 100$ seconds, each with a different seed for the random number generators. However, each set of the 10 simulation experiments uses the same set of 10 random seeds as all other sets of experiments. This graph clearly corroborates the results obtained by the optimization runs, i.e., it shows that $\hat{\theta} = (\mathbf{15}, \mathbf{14})$ is indeed optimal.

V. Conclusions and Future Work

We have considered in this paper a Stochastic Flow Model (SFM) for a communication network of multiple nodes in tandem. Our objective is to control threshold parameters at network nodes so as to optimize performance captured by combining loss and workload metrics. We have developed IPA estimators for these metrics with respect to the threshold and shown them to be unbiased. The simplicity of the estimators derived and the fact they are not dependent on knowledge of the traffic arrival or service processes makes them attractive for on-line control and optimization. This work has extended results applicable to a single-node, single-class SFM in [10], and the next step is to incorporate multiple traffic classes at various nodes, along the lines of [12]. Our ongoing work is also investigating the use of this approach in general topology networks, which we believe to be possible. For example, the presence of cross-traffic at node m in our SFM can be captured by varying the processing rate $\beta_m(t)$ at that node. Finally, and very importantly, ongoing work is also considering how to develop IPA and related control and optimization methods that include network *feedback* effects (i.e., allowing arriving traffic processes to depend on the buffer content in different ways); some related initial results are reported in [21].

Appendix I

Proof of Lemma II.1

Recalling (2), we have $x_{m-1}(\theta;\sigma_{m-1}^-) = 0$ and $x_{m-1}(\theta;\sigma_{m-1}^+) > 0$, which implies that at σ_{m-1} there is a change of sign in $A_{m-1}(\theta;t) = \alpha_{m-1}(\theta;t) - \beta_{m-1}(t)$ from non-positive to positive. For $m = 2$, since $\alpha_1(t)$ and $\beta_1(t)$ are independent of θ, the time of the sign change of $A_1(t)$ is independent of θ too, and it follows that σ_1 is locally independent of θ. For $m > 2$, there are two ways in which a sign change in $A_{m-1}(\sigma_{m-1})$ can take place: continuously or as a result of a jump in either $\alpha_{m-1}(\theta;t)$ or $\beta_{m-1}(t)$ at $t = \sigma_{m-1}$. Let us consider each of these two cases next.

If no discontinuity occurs at $t = \sigma_{m-1}$, then by (2), we have either $\alpha_{m-1}(\theta;t) = \alpha_{m-2}(\theta;t)$ or $\alpha_{m-1}(\theta;t) = \beta_{m-2}(t)$. In the latter case, $A_{m-1}(\theta;\sigma_{m-1}) = \beta_{m-2}(\sigma_{m-1}) - \beta_{m-1}(\sigma_{m-1})$ is clearly independent of θ. In the former case, we have $A_{m-1}(\theta;\sigma_{m-1}) = \alpha_{m-2}(\theta;\sigma_{m-1}) - \beta_{m-1}(\sigma_{m-1})$ where, once again, either $\alpha_{m-2}(\theta;t) = \alpha_{m-3}(\theta;t)$ or $\alpha_{m-2}(\theta;t) = \beta_{m-3}(t)$ at $t = \sigma_{m-1}$. In the latter case, $A_{m-1}(\theta;\sigma_{m-1}) = \beta_{m-3}(\sigma_{m-1}) - \beta_{m-1}(\sigma_{m-1})$ is independent of θ. In the former case, we have $A_{m-1}(\theta;\sigma_{m-1}) = \alpha_{m-3}(\theta;\sigma_{m-1}) - \beta_{m-1}(\sigma_{m-1})$ and the process repeats until we get $A_{m-1}(\theta;\sigma_{m-1}) = \alpha_1(\sigma_{m-1}) - \beta_{m-1}(\sigma_{m-1})$ which is independent of θ. Thus, if the change in sign occurs continuously, we conclude that σ_{m-1} is independent of θ.

This leaves only the possibility that the sign change occurs as a result of a jump in either $\alpha_{m-1}(\theta;t)$ or $\beta_{m-1}(t)$ at $t = \sigma_{m-1}$. Note that $\alpha_{m-1}(\theta;t)$ and $\beta_{m-1}(t)$ may jump simultaneously at $t = \sigma_{m-1}$, but only one of them dominates the sign change, i.e., the jump in the other one alone would not have caused the sign change. The dominating jump in $\beta_{m-1}(t)$ is obviously independent of θ. Therefore, the only possibility is that $\alpha_{m-1}(\theta;t)$ experiences a dominating jump at $t = \sigma_{m-1}$. Moreover, since $\alpha_{m-1}(\theta;t) - \beta_{m-1}(t)$ experiences a sign change from non-positive to positive, $\alpha_{m-1}(\theta;t)$ must switch to a larger value at σ_{m-1}, i.e., $\alpha_{m-1}(\theta;\sigma_{m-1}^-) < \alpha_{m-1}(\theta;\sigma_{m-1}^+)$.

The jump of $\alpha_{m-1}(\theta;t)$ has three possible ways of occurring: (i) switching from $\beta_{m-2}(t)$ to $\alpha_{m-2}(\theta;t)$, (ii) switching from $\alpha_{m-2}(\theta;t)$ to $\beta_{m-2}(t)$, or (iii) having $\alpha_{m-1}(\theta;t) = \alpha_{m-2}(\theta;t)$ because $x_{m-2}(\theta;t) = 0$, and inheriting a jump of $\alpha_{m-2}(\theta;t)$ at that time.

Case (i) is infeasible by the following argument: if σ_{m-1} is a switchover point of $\alpha_{m-1}(\theta;t)$ from $\beta_{m-2}(t)$ to $\alpha_{m-2}(\theta;t)$, then the buffer at node $m-2$ becomes empty at that time, which implies that $\beta_{m-2}(\sigma_{m-1}) - \alpha_{m-2}(\theta;\sigma_{m-1}) > 0$; this contradicts the fact that $\alpha_{m-1}(\theta;t)$ must switch to a larger value at σ_{m-1}.

If case (iii) applies, then $\alpha_{m-2}(\theta;t)$ must switch to a larger value at σ_{m-1}, and we repeat the same argument as the one used above for $\alpha_{m-1}(\theta;t)$ until either case (ii) applies for some $\alpha_{m-i}(\theta;t)$ with $m-i > 2$ or we reach node 2, in which case only case (ii) is possible.

Thus, the proof reduces to considering case (ii), i.e., showing that if σ_{m-1} is a switchover point of $\alpha_{m-1}(\theta;t)$ with $\alpha_{m-1}(\theta;\sigma_{m-1}^-) = \alpha_{m-2}(\theta;\sigma_{m-1}^-)$ and $\alpha_{m-1}(\theta;\sigma_{m-1}^+) = \beta_{m-2}(\sigma_{m-1}^+)$ then, σ_{m-1} is locally independent of θ. Observe that this is precisely the statement of the lemma with m replaced by $m-1$ in $\alpha_m(\theta;t)$ and $\beta_{m-1}(t)$. Therefore, using the same argument as above, this process is repeated until the proof is reduced to showing that if σ_{m-1} is a switchover point of $\alpha_2(\theta;t)$ with $\alpha_2(\theta;\sigma_{m-1}^-) = \alpha_1(\sigma_{m-1}^-)$ and $\alpha_2(\theta;\sigma_{m-1}^+) = \beta_1(\sigma_{m-1}^+)$ then σ_{m-1} is locally independent of θ. This, however, was already established above based on the fact that $\alpha_1(t)$ and $\beta_1(t)$ are both defining processes independent of θ. ■

Proof of Lemma III.1

If $s_{m,i}$ is an active switchover point of $\alpha_m(\theta;t)$, it follows from (2) and **Definition 1** that there are two possible cases: (i) an EP starts at node $m-1$, or (ii) $s_{m,i}$ lies within an EP of node $m-1$ and is an active switchover point of $\alpha_{m-1}(\theta;t)$.

In case (i), an event e_3 occurs at node $m-1$. By **Assumption 1(***c***)**, we can only have $\beta_{m-1}(t) = \alpha_{m-1}(\theta;t)$ at a single time instant and by **Assumption 1(***b***)** that cannot coincide with another event at node $m-1$. Therefore, $\alpha_m(\theta;t)$ must experience a jump from $\beta_{m-1}(t)$ to $\alpha_{m-1}(\theta;t)$ at $t = s_{m,i}$, which is an e_1 event at node m.

In case (ii), $s_{m,i}$ is an active switchover point of $\alpha_{m-1}(\theta;t)$, so either it starts an EP at node $m-2$ or it is an active switchover point of $\alpha_{m-2}(\theta;t)$. Thus, we repeat the previous argument until the only remaining case is that $s_{m,i}$ is an active switchover point of $\alpha_2(\theta;t)$. In this case, $s_{m,i}$ can only be the start of an EP at node 1. By **Assumption 1**(c), we can only have $\beta_1(t)=\alpha_1(t)$ at a single time instant and by **Assumption 1**(b) that cannot coincide with another event at node 1. Therefore, $\alpha_m(\theta;t)$ must again experience a jump from $\beta_1(t)$ to $\alpha_1(\theta;t)$ at $s_{m,i}$, which is an e_1 event at node m. ■

Proof of Lemma III.2

The first part was established in Theorem 6 of [10]. To prove the second part, suppose first that $t\in B_{m,n}$ for some $n=1,...,N_m$. Since t is in the interior of a BP, we have either $x_m(\theta;t)=0$ or $x_m(\theta;t)=b_m$ throughout the BP, therefore,

$$x_m^{'}(\theta;t)=0. \tag{34}$$

Next, suppose $t\in \overline{B}_{m,n+1}$ for some $n=0,...,N_m-1$. In this case,

$$x_m(\theta;t)=x_m(\theta;\sigma_{m,n})+\int_{\sigma_{m,n}}^{t}A_m(\theta;\zeta)d\zeta,$$

therefore,

$$x_m^{'}(\theta;t)=x_m^{\prime}(\theta;\sigma_{m,n})+\frac{d}{d\theta}\int_{\sigma_{m,n}}^{t}A_m(\theta;\zeta)d\zeta. \tag{35}$$

Recall that $\sigma_{m,n}$ is the start of a NBP, so that either $x_m(\theta;\sigma_{m,n})=0$ or $x_m(\theta;\sigma_{m,n})=b_m$. In either case we obtain $x_m^{\prime}(\theta;\sigma_{m,n})=0$. Thus,

$$x_m^{'}(\theta;t)=\frac{d}{d\theta}\int_{\sigma_{m,n}}^{t}A_m(\theta;\zeta)d\zeta$$

Let $\{s_{m,k}\}$, $k=1,...,K_{m,n}(t)$, be the sequence of active switchover points in the interval $(\sigma_{m,n},t)$, where $K_{m,n}(t)$ denotes the total number of such points. First, suppose $n\in\Phi_m$, i.e., from (10) $\sigma_{m,n}$ is an active switchover point and it is, therefore, generally a function of θ. We can then write

$$\begin{aligned}
&\frac{d}{d\theta}\int_{\sigma_{m,n}}^{t}A_m(\theta;\zeta)d\zeta\\
&=\frac{d}{d\theta}\left\{\int_{\sigma_{m,n}}^{s_{m,1}}A_m(\theta;\zeta)d\zeta+\int_{s_{m,1}}^{s_{m,2}}A_m(\theta;\zeta)d\zeta+\right.\\
&\qquad\left.\cdots+\int_{s_{m,K_{m,n}(t)}}^{t}A_m(\theta;\zeta)d\zeta\right\}\\
&=A_m(\theta;s_{m,1}^{-})s_{m,1}^{'}-A_m(\theta;\sigma_{m,n}^{+})\sigma_{m,n}^{'}\\
&+\sum_{k=1}^{K_{m,n}(t)-1}\Big[A_m(\theta;s_{m,k+1}^{-})s_{m,k+1}^{'}-A_m(\theta;s_{m,k}^{+})s_{m,k}^{'}\Big]\\
&-A_m(\theta;s_{m,K_n(t)}^{-})s_{m,K_{m,n}(t)}^{'}
\end{aligned}$$

Since $s_{m,k}$ is an active switchover point of $\alpha_m(\theta;t)$, it follows from Lemma III.1 that it is an e_1 event, so by **Assumption 1**(b) no other event occurs at the same time; in particular, no other e_1 event may take place. Thus $\beta_m(t)$ is continuous at $t=s_{m,k}$, i.e., $\beta_m(s_{m,k}^{+})=\beta_m(s_{m,k}^{-})$. Then, using the definition of $A_m(\theta;t)$ in (4), we get

$$\begin{aligned}
&\frac{d}{d\theta}\int_{\sigma_{m,n}}^{t}A_m(\theta;\zeta)d\zeta\\
&\qquad=-A_m(\theta;\sigma_{m,n}^{+})\sigma_{m,n}^{'}\\
&\qquad\quad+\sum_{k=1}^{K_{m,n}(t)}[\alpha_m(\theta;s_{m,k}^{-})-\alpha_m(\theta;s_{m,k}^{+})]s_{m,k}^{'}\\
&\qquad=-\phi_{m,n}-\sum_{k=1}^{K_{m,n}(t)}\psi_{m,k}
\end{aligned}\tag{36}$$

where we have used (12) and (13). Similarly, if $n\notin\Phi_m$, the only difference is that $\sigma_{m,n}$ is not a function of θ and we get

$$\frac{d}{d\theta}\int_{\sigma_{m,n}}^{t}A_m(\theta;\zeta)d\zeta=-\sum_{k=1}^{K_{m,n}(t)}\psi_{m,k}. \tag{37}$$

Combining (34), (36), and (37) yields (15). ■

Proof of Lemma III.3

Since $s_{m,i}$ is an active switchover point of $\alpha_m(\theta;t)$, it follows from (2) that there are two possible cases: (i) it starts an EP at node $m-1$, or (ii) it lies within an EP of node $m-1$ and is an active switchover point of $\alpha_{m-1}(\theta;t)$. In case (ii), letting $s_{m-1,j}$ (for some j) denote the active switchover point of $\alpha_{m-1}(\theta;t)$, we have $\alpha_m(\theta;t)=\alpha_{m-1}(\theta;t)$ at $t=s_{m,i}=s_{m-1,j}$, therefore (17) immediately follows from (12). Thus, it remains to consider case (i) and prove (16).

If at $s_{m,i}$ an EP $B_{m-1,n+1}$ at node $m-1$ starts, this is an e_3 event at node $m-1$ and we have $s_{m,i}=\tau_{m-1,n+1}$ for some n. Moreover, by **Assumption 1**(b), no other event at node $m-1$ occurs at the same time, so $\alpha_{m-1}(\theta;t)$ and $\beta_{m-1}(t)$ are continuous at $s_{m,i}$. Therefore, $\alpha_m(\theta;s_{m,i}^{+})=\alpha_{m-1}(\theta;s_{m,i})$, while $\alpha_m(\theta;s_{m,i}^{-})=\beta_{m-1}(s_{m,i})$. It follows from (12) that

$$\begin{aligned}
\psi_{m,i}&=[\alpha_{m-1}(\theta;s_{m,i})-\beta_{m-1}(s_{m,i})]\,s_{m,i}^{'}\\
&=A_{m-1}(\theta;s_{m,i})s_{m,i}^{'}
\end{aligned}\tag{38}$$

On the other hand, we have

$$\int_{\sigma_{m-1,n}}^{\tau_{m-1,n+1}}A_{m-1}(\theta;t)dt=0$$

if $\overline{B}_{m-1,n+1}$ is of type (E,E), and

$$\int_{\sigma_{m-1,n}}^{\tau_{m-1,n+1}}A_{m-1}(\theta;t)dt=-b_{m-1}$$

if $\overline{B}_{m-1,n+1}$ is of type (F,E). Regarding the start $\sigma_{m-1,n}$ of the NBP, recall that if it happens to be an active switchover point of $\alpha_{m-1}(\theta;t)$, then it is a function of θ; otherwise, it is independent of θ. Let $\{s_{m-1,k}\}$, $k = 1,...,K$, be the sequence of active switchover points in the interval $\overline{B}_{m-1,n+1}$, where K denotes the total number of such points. Then, differentiating with respect to θ the equations above, we get

$$\begin{aligned} &A_{m-1}(\theta;\tau_{m-1,n+1})\tau_{m-1,n+1}^{'} \\ &\quad -\sum_{k=1}^{K}\left[\alpha_{m-1}(\theta;s_{m-1,k}^{+})-\alpha_{m-1}(\theta;s_{m-1,k}^{-})\right]s_{m-1,k}^{'} \\ &\qquad -\mathbf{1}[n\in\Phi_{m-1}]\cdot A_{m-1}(\theta;\sigma_{m-1,n}^{+})\sigma_{m-1,n}^{'}=0 \end{aligned}$$

where the evaluation of the left-hand-side above is along the same lines as that of (36). In view of the fact that $\tau_{m-1,n+1} = s_{m,i}$ and using (38) and (12)-(13), we get

$$\psi_{m,i}-\sum_{k=1}^{K}\psi_{m-1,k}-\mathbf{1}[n\in\Phi_{m-1}]\cdot\phi_{m-1,n}=0$$

Using (15) in Lemma III.2 with $t = s_{m,i}^{-}$ and $K_{m,n}(t) = K$ above, we obtain (16) and complete the proof. ∎

Proof of Lemma III.4

Since $\sigma_{m,n}$ is an active switchover point of $\alpha_m(\theta;t)$ and $\sigma_{m,n} = s_{m,i}$, (19) immediately follows from (12)-(13).

Next we prove (20). Since $n \in \Phi_m$ we have the end of a FP of node m at $\sigma_{m,n}$, therefore $\alpha_m(\theta;t) - \beta_m(t)$ undergoes a sign change from non-negative to negative, i.e.,

$$\begin{aligned} \alpha_m(\theta;\sigma_{m,n}^{-})-\beta_m(\sigma_{m,n}^{-})\geq 0 \\ \alpha_m(\theta;\sigma_{m,n}^{+})-\beta_m(\sigma_{m,n}^{+})<0. \end{aligned}$$

Since $\sigma_{m,n}$ is also an active switchover point of $\alpha_m(\theta;t)$, then it follows from Lemma III.1 and **Assumption 1(b)** that $\beta_m(t)$ is continuous at $t = \sigma_{m,n}$. Thus we have

$$\beta_m(\sigma_{m,n}^{+})=\beta_m(\sigma_{m,n}^{-})=\beta_m(\sigma_{m,n})$$

Combining this with the previous two inequalities yields

$$\alpha_m(\theta;\sigma_{m,n}^{+})-\alpha_m(\theta;\sigma_{m,n}^{-})\leq\alpha_m(\theta;\sigma_{m,n}^{+})-\beta_m(\sigma_{m,n})<0,$$

from which (20) immediately follows. ∎

Proof of Theorem III.1

The result follows directly from Lemmas III.2-III.4, observing that in using Lemma III.3, $i^* = 1$ corresponds to (16) and $i^* > 1$ corresponds to (17) . ∎

Proof of Lemma III.5

Observe that $\gamma(\theta;t) = \alpha(\theta;t) - \beta(t) = A(\theta;t)$ throughout the interval (τ_n, σ_n). Let $\{s_k\}$, $k = 1,...,K$, be the sequence of active switchover points in (τ_n, σ_n), where K denotes the total number of such points. Also, note that since at τ_n a node m event of type e_3 occurs, by **Assumption 1(b)** no e_1 event can also occur at this node; hence, $A(\theta;t)$ is continuous at $t = \tau_n$. Then,

$$\begin{aligned} &\frac{d}{d\theta}\int_{\tau_n}^{\sigma_n}\gamma(\theta;t)dt \\ &= \frac{d}{d\theta}\int_{\tau_n}^{s_1}A(\theta;t)dt+\sum_{k=1}^{K-1}\frac{d}{d\theta}\int_{s_k}^{s_{k+1}}A(\theta;t)dt \\ &\qquad +\frac{d}{d\theta}\int_{s_K}^{\sigma_n}A(\theta;t)dt \\ &= A(\theta;s_1^{-})s_1^{'}\ -\ A(\theta;\tau_n)\tau_n^{'}+\sum_{k=1}^{K-1}[A(\theta;s_{k+1}^{-})s_{k+1}^{'} \\ &\qquad -A(\theta;s_k^{+})s_k^{'}]+\ A(\theta;\sigma_n^{-})\sigma_n^{'}-\ A(\theta;s_K^{+})s_K^{'} \end{aligned}$$

Since s_k is an active switchover point, it follows from Lemma III.1 and **Assumption 1(b)** that $\beta(t)$ is continuous at $t = s_k$, i.e., $\beta(s_k^{+}) = \beta(s_k^{-})$. Then, using the definition $A(\theta;t) = \alpha(\theta;t) - \beta(t)$ we get

$$\begin{aligned} &\frac{d}{d\theta}\int_{\tau_n}^{\sigma_n}\gamma(\theta;t)dt=A(\theta;\sigma_n^{-})\sigma_n^{'} \\ &\qquad -\sum_{k=1}^{K}[\alpha(s_k^{+})-\alpha(s_k^{-})]s_k^{'}-\ A(\theta;\tau_n)\tau_n^{'} \end{aligned}$$

Finally, noting that $K = |\Psi_n^o|$ using the definition in (8), and recalling (12) yields (24). ∎

Proof of Lemma III.6

Using (1) for the NBP (σ_{n-1}, τ_n) that precedes the FP $[\tau_n, \sigma_n]$,

$$\int_{\sigma_{n-1}}^{\tau_n}[\alpha(\theta;t)-\beta(t)]dt=\int_{\sigma_{n-1}}^{\tau_n}A(\theta;t)dt\ =\ 0\text{ or }b \quad (39)$$

depending on whether this NBP is of type (F, F) or (E, F). In either case, letting $\{s_k\}$, $k = 1,...,K$ be the sequence of active switchover points within (σ_{n-1}, τ_n) and differentiating with respect to θ yields

$$\begin{aligned} &\frac{d}{d\theta}\int_{\sigma_{n-1}}^{s_1}A(\theta;t)dt+\sum_{k=1}^{K-1}\frac{d}{d\theta}\int_{s_k}^{s_{k+1}}A(\theta;t)dt \\ &\qquad +\frac{d}{d\theta}\int_{s_K}^{\tau_n}A(\theta;t)dt=0 \end{aligned}$$

Proceeding as in the proof of the previous lemma to evaluate the middle term above (observing again that s_k is an active switchover point and that, by Lemma III.1 and **Assumption 1(b)**, $\beta(t)$ is continuous at $t = s_k$), we get

$$A(\theta;\tau_n)\tau_n^{'}-\sum_{k=1}^{K}[\alpha(s_k^{+})-\alpha(s_k^{-})]s_k^{'}-\ A(\theta;\sigma_{n-1}^{+})\sigma_{n-1}^{'}=0$$

Noting that $K = \left|\overline{\Psi}_n\right|$ using the definition in (9), and recalling (12) yields (25). ∎

Proof of Lemma III.7

If σ_n is not an active switchover point, i.e., $n \notin \Phi$, then it is independent of θ and $\sigma_n^{'} = 0$. On the other hand, if σ_n is an active switchover point and $\sigma_n = s_i$ for some i, then by Lemma III.1 it is an e_1 event time, hence $\beta(t)$ must be continuous at $t = \sigma_n$ by **Assumption 1(b)**. In this case, the left-hand side of (26) becomes $[A(\theta;\sigma_n^+) - A(\theta;\sigma_n^-)]\sigma_n^{'} = [\alpha(s_i^+) - \alpha(s_i^-)]s_i^{'} = \psi_i$ from (12). ∎

Proof of Theorem III.2

Using (23) and Lemma III.5 we have

$$L_m^{'}(\theta;T) = \sum_{n\in\digamma_m} \frac{d}{d\theta}\int_{\tau_{m,n}}^{\sigma_{m,n}} \gamma_m(\theta;t)dt$$
$$= \sum_{n\in\digamma_m} \Big\{ A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} - A_m(\theta;\tau_{m,n})\tau_{m,n}^{'} - \sum_{i\in\Psi_{m,n}^o} \psi_{m.i} \Big\}$$

Using Lemma III.6 to replace $A_m(\theta;\tau_{m,n})\tau_{m,n}^{'}$ above, we get

$$L_m^{'}(\theta;T) = \sum_{n\in\digamma_m}\Big\{ A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} - A_m(\theta;\sigma_{m,n-1}^+)\sigma_{m,n-1}^{'} - \sum_{i\in\Psi_{m,n}^o} \psi_{m.i} - \sum_{i\in\overline{\Psi}_{m,n}} \psi_{m,i} \Big\}$$

Subtracting and adding the term $A_m(\theta;\sigma_{m,n-1}^-)\sigma_{m,n-1}^{'}$ inside the outer sum we obtain

$$L_m^{'}(\theta;T) = \sum_{n\in\digamma_m} \Big\{ A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} - A_m(\theta;\sigma_{m,n-1}^-)\sigma_{m,n-1}^{'} \Big\}$$
$$+ \sum_{n\in\digamma_m} \Big\{ A_m(\theta;\sigma_{m,n-1}^-)\sigma_{m,n-1}^{'} - A_m(\theta;\sigma_{m,n-1}^+)\sigma_{m,n-1}^{'} \Big\}$$
$$- \sum_{n\in\digamma_m} \Big\{ \sum_{i\in\Psi_{m,n}^o} \psi_{m,i} + \sum_{i\in\overline{\Psi}_{m,n}} \psi_{m,i} \Big\} \qquad (40)$$

Let us consider the first sum above. For $n \in \digamma_m$, there are two possibilities for the NBP that precedes the FP $[\tau_{m,n}, \sigma_{m,n}]$: ($i$) If the NBP is of type (E,F), then $\sigma_{m,n-1}$ is the end of an EP, therefore $\sigma_{m,n-1}^{'} = 0$ by Lemma II.1, and (ii) If the NBP is of type (F,F), then $\sigma_{m,n-1}$ is the end of another FP $[\tau_{m,n-1}, \sigma_{m,n-1}]$. In light of these observations, this sum can be decomposed into groups of terms so that the rth group starts with some $\sigma_{m,r-j}$ initiating a NBP of type (E,F), followed by a FP, followed by a sequence of NBPs of type (F,F) with an ensuing FP, and finally ending at $\sigma_{m,r}$ with a FP $[\tau_{m,r}(\theta), \sigma_{m,r}(\theta)]$ which is followed by an NBP of type (F,E). Adding the terms of any such group we get cancellations of all $A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'}$ leaving only

$$A_m(\theta;\sigma_{m,r}^-)\sigma_{m,r}^{'} - A_m(\theta;\sigma_{m,r-j}^-)\sigma_{m,r-j}^{'}$$

We have already seen that $\sigma_{m,r-j}^{'} = 0$ because this is the start of a NBP of type (E,F). In addition, $\sigma_{m,r}^{'} = 0$ unless it is an active switchover point, i.e., $r \in \Gamma_m$ as defined in (11). It follows that

$$\sum_{n\in\digamma_m} \Big\{ A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} - A_m(\theta;\sigma_{m,n-1}^-)\sigma_{m,n-1}^{'} \Big\} = \sum_{n\in\Gamma_m} A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} \qquad (41)$$

Since $\sigma_{m,n}$ in this sum is an active switchover point of $\alpha_m(\theta;t)$, by Lemma III.1 and **Assumption 1(b)** $\beta_m(t)$ is continuous at $t = \sigma_{m,n}$. Therefore, for all $n \in \Gamma_m$,

$$A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} = [\alpha_m(\theta;\sigma_{m,n}^-) - \beta_m(\sigma_{m,n})]\sigma_{m,n}^{'}$$

Adding and subtracting the term $\alpha_m(\theta;\sigma_{m,n}^+)$ in the bracket above and making use of $R_{m,n}$ as defined in (18), we get

$$\alpha_m(\theta;\sigma_{m,n}^-) - \beta_m(\sigma_{m,n}) = [\alpha_m(\theta;\sigma_{m,n}^+) - \alpha_m(\theta;\sigma_{m,n}^-)](R_{m,n} - 1)$$

and making use of (12) we finally get

$$A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} = (R_{m,n} - 1)\psi_{m,i_n}$$

where $\sigma_{m,n} = s_{m,i_n}$ for some active switchover point index i_n, since $n \in \Gamma_m$. Thus, returning to (41), we can write

$$\sum_{n\in\digamma_m} \Big\{ A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} - A_m(\theta;\sigma_{m,n-1}^-)\sigma_{m,n-1}^{'} \Big\} = -\sum_{n\in\Gamma_m} \psi_{m,i_n} + \sum_{n\in\Gamma_m} R_{m,n}\psi_{m,i_n}$$

Using (19), we have $R_{m,n}\psi_{m,i_n} = \phi_{m,n}$ so that

$$\sum_{n\in\digamma_m} \Big\{ A_m(\theta;\sigma_{m,n}^-)\sigma_{m,n}^{'} - A_m(\theta;\sigma_{m,n-1}^-)\sigma_{m,n-1}^{'} \Big\} = -\sum_{n\in\Gamma_m} \psi_{m,i_n} + \sum_{n\in\Gamma_m} \phi_{m,n} \qquad (42)$$

Next, let us consider the second term in (40). The sum index $(n-1)$ for each $n \in \digamma_m$ implies that we consider the end point of a BP which is followed by a FP, or, in other words, the sum is over the end points of BPs which do not start an (E,E) or (F,E) type of NBP. On the other hand, invoking Lemma III.7, we can see that all terms in the sum are zero unless $(n-1) \in \Phi_m$. This implies that non-zero terms are such that the BP that precedes the nth FP must be another FP; if it is not, then it is an EP and we have already established in Lemma II.1 that in such cases $\sigma_{m,n-1}^{'} = 0$. Thus, by excluding the index of those FPs which start an (F,E) type of NBP from the index set Φ_m, the remaining part of Φ_m contains all the indices of the FPs of interests. Recalling the definition of Γ_m in (11), those excluded points in Φ_m from the set Γ_m, so let n^*

index the non-zero terms in our sum and we can write

$$\sum_{n\in \digamma_m}[A_m(\theta;\sigma^-_{m,n-1})\sigma'_{m,n-1}-A_m(\theta;\sigma^+_{m,n-1})\sigma'_{m,n-1}]$$

$$=-\sum_{n^*\in(\Phi_m-\Gamma_m)}\psi_{m,i_{n^*}} \tag{43}$$

where $\sigma_{m,n^*}=s_{m,i_{n^*}}$ for some active switchover point index i_{n^*}, since $n^*\in\Phi_m$ and $\beta_m(t)$ is continuous by Lemma III.1 and Assumption 1(b). Replacing the index n^* by n and combining (43) and (42) we get from (40):

$$L'_m(\theta;T)=\sum_{n\in\Gamma_m}\phi_{m,n}-\sum_{n\in\Phi_m}\psi_{m,i_n}$$

$$-\sum_{n\in\digamma_m}\left\{\sum_{i\in\Psi^o_{m,n}}\psi_{m,i}+\sum_{i\in\overline{\Psi}_{m,n}}\psi_{m,i}\right\}$$

where we point out the cancellation of the sum $\sum_{n\in\Gamma_m}\psi_{m,i_n}$. Finally, by the definition of Φ_m in (10), the second sum above contains all ψ_{m,i_n} terms such that $n\in\digamma_m$ and $i_n\in\Psi_{m,n}$ corresponds to an active switchover point at the end point of the FP $[\tau_{m,n},\sigma_{m,n}]$. In other words, the second and third sum together include all $\psi_{m,i}$ terms with $i\in\Psi_{m,n}$. Thus, making use of $\Omega_{m,n}$ in (27), we can write

$$L'_m(\theta;T)=\sum_{n\in\Gamma_m}\phi_{m,n}-\sum_{n\in\digamma_m}\sum_{i\in\Omega_{m,n}}\psi_{m,i}$$

which completes the proof. ■

Proof of Theorem III.4

By Lemma III.2, $x'_m(\theta;t)=0$ throughout a BP $B_{m,n}$, so that

$$\int_{\tau_{m,n}}^{\sigma_{m,n}}x'_m(\theta;t)dt=0. \tag{44}$$

For the NBP $\overline{B}_{m,n}$ that precedes the nth BP, let $K=\left|\overline{\Psi}_{m,n}\right|$ and we have

$$\int_{\sigma_{m,n-1}}^{\tau_{m,n}}x'_m(\theta;t)dt=\int_{\sigma_{m,n-1}}^{s_{m,1}}x'_m(\theta;t)dt$$

$$+\sum_{k=1}^{K-1}\int_{s_{m,k}}^{s_{m,k+1}}x'_m(\theta;t)dt+\int_{s_{m,K}}^{\tau_{m,n}}x'_m(\theta;t)dt. \tag{45}$$

By Theorem III.1, for $t\in(\sigma_{m,n-1},s_{m,k})$,

$$x'_m(\theta;t)=-\sum_{j=1}^{k-1}\psi_{m,j}\ -\mathbf{1}\left[(\mathbf{n-1})\in\mathbf{\Phi_m}\right]\cdot\phi_{\mathbf{m,n-1}},$$

and using this in (45) gives

$$\int_{\sigma_{m,n-1}}^{\tau_{m,n}}x'_m(\theta;t)dt=-\int_{\sigma_{m,n-1}}^{s_{m,1}}\mathbf{1}[(n-1)\in\Phi_m]\cdot\phi_{m,n-1}dt$$

$$-\sum_{k=1}^{K-1}\int_{s_{m,k}}^{s_{m,k+1}}\left[\sum_{j=1}^{k-1}\psi_{m,j}\ +\mathbf{1}[(n-1)\in\Phi_m]\cdot\phi_{m,n-1}\right]dt$$

$$-\int_{s_{m,K}}^{\tau_{m,n}}\left[\sum_{j=1}^{K}\psi_{m,j}\ +\mathbf{1}\left[(n-1)\in\Phi_m\right]\cdot\phi_{m,n-1}\right]dt$$

Adding the terms that involve $\psi_{m,j}$, after taking into account several term cancellations we get

$$-\sum_{k=1}^{K-1}\int_{s_{m,k}}^{s_{m,k+1}}\sum_{j=1}^{k-1}\psi_{m,j}dt-\int_{s_{m,K}}^{\tau_{m,n}}\sum_{j=1}^{K}\psi_{m,j}dt$$

$$=-\sum_{k=1}^{K}[\tau_{m,n}-s_{m,k}]\psi_{m,k}$$

and, similarly, adding the terms that involve $\phi_{m,n-1}$ gives

$$-\mathbf{1}\left[(\mathbf{n-1})\in\mathbf{\Phi_m}\right]\cdot\left[\int_{\sigma_{\mathbf{m,n-1}}}^{\mathbf{s_{m,1}}}\phi_{\mathbf{m,n-1}}\mathbf{dt}\right.$$

$$\left.+\sum_{k=1}^{K-1}\int_{s_{m,k}}^{s_{m,k+1}}\sum_{j=1}^{k-1}\phi_{m,n-1}dt+\int_{s_{m,K}}^{\tau_{m,n}}\phi_{m,n-1}dt\right]$$

$$=-\mathbf{1}\left[(\mathbf{n-1})\in\mathbf{\Phi_m}\right]\cdot[\tau_{\mathbf{m,n}}-\sigma_{\mathbf{m,n-1}}]\phi_{\mathbf{m,n-1}}$$

Thus,

$$\int_{\sigma_{m,n-1}}^{\tau_{m,n}}x'_m(\theta;t)dt=-\sum_{k=1}^{K}[\tau_{m,n}-s_{m,k}]\psi_{m,k}$$

$$-\mathbf{1}\left[(\mathbf{n-1})\in\mathbf{\Phi_m}\right]\cdot[\tau_{\mathbf{m,n}}-\sigma_{\mathbf{m,n-1}}]\phi_{\mathbf{m,n-1}} \tag{46}$$

Recall that $\overline{\Psi}_{m,n}$, defined in (9), refers to the NBP $\overline{B}_{m,n}=(\sigma_{m,n-1},\tau_{m,n})$ which precedes the nth BP, so that using (44) and (46) in (29) yields (30). ■

Appendix II

Theorems III.3 and III.5 assert the unbiasedness of the IPA derivatives $L'_m(\theta;T)$ and $Q'_m(\theta;T)$, $m=2,\ldots,M$, and both will be proved in what follows. To set the stage in a general setting, let $\mathcal{L}(\theta)$ be a real-valued random function of a real-valued variable θ, defined on a common probability space $(\Omega,\mathcal{F},\mathcal{P})$. Let θ be confined to a closed and bounded interval Θ. Suppose that, for a given fixed $\theta\in\Theta$, the IPA derivative $\mathcal{L}'(\theta)$ exists w.p.1 (the appropriate one-sided derivative if θ is an end-point of Θ). The IPA derivative is said to be *unbiased* if the operators of expectation in $(\Omega,\mathcal{F},\mathcal{P})$ and differentiation with respect to θ are interchangeable, namely,

$$\frac{d}{d\theta}E[\mathcal{L}(\theta)]\ =\ \mathcal{E}[\mathcal{L}'(\theta)] \tag{47}$$

(see [16],[17]). It is shown in [24] that the following two conditions jointly guarantee the existence of the derivative $\frac{d}{d\theta}E[\mathcal{L}(\theta)]$ and suffice for the unbiasedness of the IPA derivative $\mathcal{L}'(\theta)$.

- **Condition A.1.** For every $\theta\in\Theta$, w.p.1 the derivative $\mathcal{L}'(\theta)$ exists (one-sided derivative, in case θ is an end-point of θ).
- **Condition A.2.** W.p.1 the random function $\mathcal{L}(\theta)$ is Lipschitz continuous throughout Θ, and its Lipschitz constant, K, has a finite first moment, i.e., $E[K]<\infty$.

In what follows we will prove that these conditions are in force for the functions $L_m(\theta;T)$ and $Q_m(\theta;T)$ defined by Eqs. (5) and (6) respectively. To this end, we will rely on an analysis, carried out in [13], of Lipschitz continuity of certain mappings between defining processes and derived processes in the general setting of SFMs. We next present the relevant results.

Consider a single-stage SFM having the inflow rate process $\{\alpha(t)\}$, service rate process $\{\beta(t)\}$, and buffer size θ. As mentioned in Section II, we call these *defining processes* since they define much of the behavior of the SFM, while the processes $\{x(t)\}$, $\{\gamma(t)\}$ and $\{\delta(t)\}$ are called *derived processes* since they can be derived from the defining processes via (1)-(3). Realizations of each one of these processes is a non-negative-valued function defined on the interval $[0,T]$, denoted generically by $u(t)$. These functions will be endowed by two functional norms, namely the L^1 norm defined by

$$||u||_1 \ = \ \int_0^T |u(t)|dt,$$

and the L^∞ norm, defined by

$$||u||_\infty \ = \ \max\{|u(t)| \ : \ t\in[0,T]\}$$

where the functions $u(t)$ are piecewise analytic, hence the essential supremum can be replaced by maximum. The L^1 norm is typically used for the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\delta(t)$ while the L^∞ norm is used for the function $x(t)$. In the forthcoming, we focus on these functions regardless of how they are realized; all that matters is that $x(t)$, $\gamma(t)$ and $\delta(t)$ are derived from $\alpha(t)$, $\beta(t)$ and θ via (1)-(3). Correspondingly, we term these *defining functions* and *derived functions*, as appropriate.

Let us now consider given functions $\alpha(t)$ and $\beta(t)$, and a buffer size θ, and consider the resulting derived functions (via (1)-(3)) denoted by $x(t)$, $\gamma(t)$ and $\delta(t)$, respectively. Next, let us consider the same functions $\alpha(t)$ and $\beta(t)$, but a different buffer size, denoted by $\theta+\Delta\theta$. Correspondingly, the derived functions (computed via (1)-(3)) are denoted by $x(t)+\Delta x(t)$, $\gamma(t)+\Delta\gamma(t)$, and $\delta(t)+\Delta\delta(t)$, respectively. Let K_1 denote the number of EPs in the interval $[0,T]$ that result from the application of the defining functions $\alpha(t)$ and $\beta(t)$, and the buffer size θ, and let K_2 denote the number of EPs in the interval $[0,T]$ that result from the application of the defining functions $\alpha(t)$ and $\beta(t)$, and the buffer size $\theta+\Delta\theta$. Define $K:=\max\{K_1,K_2\}$. Note that $\Delta\delta(t)=[\delta(t)+\Delta\delta(t)]-[\delta(t)]$ can be viewed as a perturbation in the derived outflow-rate function resulting from a perturbation in the buffer size, $\Delta\theta$. Proposition 3.3 in [13] has established the following inequality.

$$||\Delta\delta(t)||_1 \ \leq \ (K+1)|\Delta b|. \tag{48}$$

Consider next functional variations in the inflow rate $\alpha(t)$. Thus, given functions $\alpha(t)$ and $\beta(t)$, and a buffer size θ, let $x(t)$, $\gamma(t)$, and $\delta(t)$ denote the resulting derived functions via (1)-(3); and for a different inflow-rate function, $\alpha(t)+\Delta\alpha(t)$, and the same service-rate function $\beta(t)$ and buffer size θ as before, let the resulting derived functions be $x(t)+\Delta x(t)$, $\gamma(t)+\Delta\gamma(t)$, and $\delta(t)+\Delta\delta(t)$. Proposition 3.1 in [13] has established the following inequalities:

$$||\Delta\gamma(t)||_1 \ \leq \ ||\Delta\alpha(t)||_1, \tag{49}$$

$$||\Delta x(t)||_\infty \ \leq \ ||\Delta\alpha(t)||_1, \tag{50}$$

and

$$||\Delta\delta(t)||_1 \leq ||\Delta\alpha(t)||_1. \tag{51}$$

With these preliminary results we now can prove Theorems III.3 and III.5.

Proof of Theorem III.3 and III.5

We prove the unbiasedness of the IPA derivatives $L_m^{'}(\theta;T)$ and $Q_m^{'}(\theta;T)$ by establishing that Condition A.1 and Condition A.2 above are satisfied for the random functions $L_m(\theta;T)$ and $Q_m(\theta;T)$. Condition A.1 is in force by **Assumption 1**. Regarding Condition A.2, let K be the number of EPs at node 1 in the interval $[0,T]$. Consequently, K is bounded from above by N_1, the total number of BPs at node 1. Regardless of the value of θ, N_1 has a finite first moment by our assumption of $T<\infty$.

Now fix $\theta\in\Theta$ and $\Delta\theta>0$ such that $\theta+\Delta\theta\in\Theta$. By an application of Eq. (48) to node 1, we have

$$||\Delta\delta_1(t)||_1 \ \leq \ (K+1)|\Delta\theta| \leq (N_1+1)|\Delta\theta|. \tag{52}$$

Observe that $\alpha_m(\theta;t)=\delta_{m-1}(\theta;t)$ for all $m=2,...,M$. Consequently, applications of the inequalities (49) and (50) to node m respectively, followed by sequential applications of (51) to nodes from m to 2, yield the following two inequalities,

$$\begin{aligned}||\Delta\gamma_m(t)||_1 \ &\leq ||\Delta\alpha_m(t)||_1 = ||\Delta\delta_{m-1}(t)||_1 \\ &\leq ||\Delta\alpha_{m-1}(t)||_1 = ||\Delta\delta_{m-2}(t)||_1 \\ &\leq \cdots \\ &\leq ||\Delta\alpha_2(t)||_1 = ||\Delta\delta_1(t)||_1 \\ &\leq (N_1+1)|\Delta\theta|,\end{aligned} \tag{53}$$

and

$$\begin{aligned}||\Delta x_m(t)||_\infty \ &\leq ||\Delta\alpha_m(t)||_1 = ||\Delta\delta_{m-1}(t)||_1 \\ &\leq ||\Delta\alpha_{m-1}(t)||_1 = ||\Delta\delta_{m-2}(t)||_1 \\ &\leq \cdots \\ &\leq ||\Delta\alpha_2(t)||_1 = ||\Delta\delta_1(t)||_1 \\ &\leq (N_1+1)|\Delta\theta|\end{aligned} \tag{54}$$

Finally, (5) and (53) imply that $L_m(\theta;T)$ has the Lipschitz constant N_1+1, and (6) and (54) imply that $Q_m(\theta;T)$ has the Lipschitz constant $(N_1+1)T$. As earlier stated, $E[N_1]<\infty$ and $T<\infty$, therefore Condition A.2 is in force for both functions $L_m(\theta;T)$ and $Q_m(\theta;T)$. This completes the proofs. ■

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