# Perturbation Analysis of Feedback-Controlled Stochastic Flow Systems

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***Abstract*— Stochastic flow systems arise naturally or as abstractions of Discrete Event Systems (DES), referred to as Stochastic Flow Models (SFMs). In this paper, we consider such systems operating with a feedback control mechanism, building on earlier work that has studied such SFMs without any feedback. Using Infinitesimal Perturbation Analysis (IPA), we derive gradient estimators for loss and workload related performance metrics with respect to threshold parameters used for buffer control. These estimators are shown to be unbiased. They are also shown to depend only on data observable from a sample path of the actual DES. This renders them computable in on-line environments and easily implementable for control and performance optimization purposes. In the case of linear feedback, we further show that the estimators are nonparametric. Finally, we illustrate the use of these estimators in network control by combining them with standard gradient–based stochastic optimization schemes and providing several simulation-based examples.**

***Index Terms*— Stochastic Flow Model, Discrete Event System, Hybrid System, Perturbation Analysis.**

## I. INTRODUCTION

A NATURAL modeling framework for many stochastic Discrete Event Systems (DES) is provided through queueing theory. However, "real-world" DES become increasingly difficult to handle through queueing theory on an event-by-event basis, especially for telecommunication and computer networks with enormous traffic volumes. Stochastic Flow Models (SFMs) provide an alternative modeling technique to queueing theory with applications including communication networks and manufacturing systems. Fluid models as abstractions of queueing systems were introduced in [1] and later proposed in [2] for the analysis of multiplexed data streams and in [3] for network performance. They have been shown to be especially useful for simulating various kinds of high speed networks [4],[5],[6],[7], as well as manufacturing systems [8]. In a queueing system described by a fluid model, we focus on the behavior of aggregate flows and ignore the identity and dynamics of individual customers. In a *Stochastic Flow Model* (SFM), we further treat flow rates as *stochastic* processes with possible jumps viewed as events, thus capturing a high level of generality for the traffic and service processes involved. While the aggregation property of SFMs brings efficiency to *performance analysis*, the resulting accuracy depends on traffic conditions, the structure of the underlying system, and the nature of the performance metrics of interest. On the other hand, SFMs often capture the critical features of the underlying "real" systems, which is useful in solving *control and optimization* problems. In this case, estimating the gradient of a given performance metric with respect to key parameters becomes an essential task. Perturbation Analysis (PA) methods [9],[10] are therefore suitable, if appropriately adapted to a SFM viewed as a DES [11],[12],[13],[14]. In a single node with threshold-based buffer control, Infinitesimal Perturbation Analysis (IPA) has been shown to yield simple sensitivity estimators for loss and workload metrics with respect to threshold (or buffer size) parameters [13]. In the multiclass case studied in [14], the estimators generally depend on traffic rate information, but not on the stochastic characteristics of the arrival and service processes involved. In addition, the estimators obtained are unbiased under very weak structural assumptions on the defining traffic processes. As a result, they can be evaluated *based on data observed on a sample path of the actual (discrete-event) system* and combined with gradient-based optimization schemes as shown in [13] and [14].

Queueing networks have been studied largely based on the assumption that system state, typically queue length information, has no effect on arrival and service processes, i.e., in the absence of feedback, thus ignoring a potentially important feature of actual system design and operation. For example the Random Early Detection (RED) algorithm in TCP congestion control [15],[16] provides some form of feedback for network management. The same is true for hedging point policies in manufacturing systems [17],[18]. Unfortunately, the presence of feedback significantly complicates analysis. For instance, it is extremely difficult to derive closed-form expressions of performance metrics such as average queue length or mean waiting time, unless stringent assumptions are made [19],[20],[21],[22], let alone developing analytical schemes for performance optimization. It is equally difficult to extend the theory of PA for DES in the presence of feedback. Indeed, such work is absent from the PA literature to the best of our knowledge.

Motivated by the importance of incorporating feedback to stochastic DES as well as their SFM counterparts, and the effectiveness of IPA methods applied to SFMs to date, the purpose of this paper is to tackle the problem of deriving IPA gradient estimators for SFMs with feedback mechanisms. As a starting point, we consider a single-node SFM with threshold-based buffer control as in [13]. An additive feedback mechanism is introduced by setting the inflow rate to $\sigma(t) - p(x(t))$ where $\sigma(t)$ is the maximal external incoming

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flow rate, $x(t)$ is the buffer content (state), and $p(x)$ is a feedback function. The main contribution of the paper is the derivation of IPA gradient estimators for performance metrics related to loss and workload levels with respect to threshold parameters (equivalently, buffer sizes). Even though the presence of feedback in the SFM considerably complicates the task of carrying out IPA, we are able to show that such IPA estimators are indeed possible to obtain for a large class of feedback functions $p(x)$. These IPA estimators depend on data *observable on a sample path of the actual discrete event system* - not just the SFM which can be viewed as an abstraction of the "real system". Moreover, they do not depend on the stochastic characteristics of the arrival and service processes. In the case of linear feedback, we further show that the estimators are *nonparametric*, in the sense that they do not even require any parameter information – only simple event counting and timing data. The presence of feedback also complicates proving the unbiasedness of IPA estimators in that, unlike earlier work, we must now identify and exclude certain parameter values, as will be explained in detail.

The paper is organized as follows. First in Section 2, we present the feedback-based buffer control problem in the SFM setting and define the performance metrics and parameters of interest. In Section 3, we carry out IPA by first deriving sample derivatives of event times in our model and then obtaining the IPA estimators for the gradients of the expected loss rate and average workload with respect to threshold parameters. Section 4 is devoted to proofs of unbiasedness of these estimators under mild technical conditions. The case of linear feedback is discussed in Section 5. In Section 6 we illustrate the use of the estimators by formulating optimization problems and solving them through the use of standard gradient-based stochastic optimization schemes. We finally outline a number of open problems and future research directions in Section 7.

## II. Stochastic Flow Model of a Queueing System with Feedback Control

The stochastic flow system we consider consists of a server with a buffer fed by a source as shown in Fig. 1. The buffer content at time $t$ is denoted by $x(t)$ and it is limited to $\theta$, which may be viewed as a capacity or as a threshold parameter used for buffer control as described in [13]. Thus, $0 \leq x(t) \leq \theta$ and when the buffer level reaches $\theta$ flow loss occurs (i.e., customers are dropped in the underlying queueing system). The maximal processing rate of the server is generally time-varying and denoted by $\mu(t)$. The maximal rate of the source at time $t$ is denoted by $\sigma(t)$, but the actual incoming rate is $\sigma(t)-p(x(t))$, where $p(x)$ is a *feedback function*. We assume that $p(x)$ is a *strictly monotonically increasing function* of $x$ (thus ensuring that the effect of feedback is more pronounced as the buffer level increases) and that it is independent of $\sigma(t)$, $\mu(t)$, or $\theta$. This feedback mechanism implies that $x(t)$ is instantaneously available to the controller (this is true in situations such as manufacturing systems, but unlikely to hold in high-speed environments such as communication networks; we discuss how we propose to deal with this important issue in the last section of the paper). It is also assumed that the stochastic processes $\{\sigma(t)\}$ and $\{\mu(t)\}$ are independent of the buffer level $x(t)$. Finally, we assume that the real-valued parameter $\theta$ is confined to a closed and bounded (compact) interval $\Theta$ and that $\theta > 0$ for all $\theta \in \Theta$.

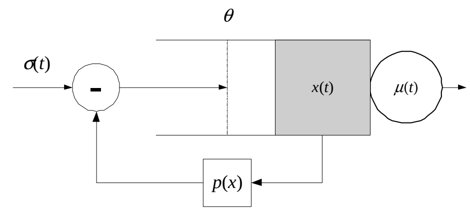


Fig. 1. A SFM with feedback

Setting $\lambda(t)=\sigma(t)-\mu(t)$, the dynamics of the system are described by the following equation:

$$\frac{dx(t)}{dt^+} = \begin{cases} 0 & \text{when } x(t)=0 \\ & \text{and } \lambda(t)-p(0)\leq 0 \\ 0 & \text{when } x(t)=\theta \\ & \text{and } \lambda(t)-p(\theta)\geq 0 \\ \lambda(t)-p(x(t)) & \text{otherwise} \end{cases} \tag{1}$$

Assuming that $\lambda(t)$ and $p(x)$ are both bounded functions, note that $x(t)$ is a continuous function of $t$. Similar to [13], our purpose is to obtain sensitivity information of some performance metrics with respect to key parameters so as to implement stochastic optimization algorithms based on this information. In this paper, we limit ourselves to the threshold $\theta$ as the controllable parameter of interest. For a finite *time horizon* $[0,T]$, we define the *average workload* as:

$$Q_T = \frac{1}{T}\int_0^T x(t)dt \tag{2}$$

and the *loss rate* as:

$$L_T = \frac{1}{T}\int_0^T \mathbf{1}[x(t)=\theta](\lambda(t)-p(\theta))dt \tag{3}$$

where $\mathbf{1}[\cdot]$ is the usual indicator function. A typical optimization problem is to determine $\theta^*$ that minimizes a cost function of the form

$$J_T(\theta)=\gamma E[Q_T(\theta)]+E[L_T(\theta)] \tag{4}$$

where $\gamma$ generally reflects the tradeoff between maintaining proper workload and incurring high loss. We point out here that the presence of feedback also has an effect on the cost function structure, as further discussed in Section 6. Care must also be taken in defining the previous expectations over a finite time horizon, since they generally depend on initial conditions; we shall assume that the queue is empty at time 0.

In order to accomplish this optimization task, we rely on estimates of $dE[Q_T(\theta)]/d\theta$ and $dE[L_T(\theta)]/d\theta$ provided by the sample derivatives $dQ_T(\theta)/d\theta$ and $dL_T(\theta)/d\theta$. Accordingly, the main objective of the following sections is the derivation of $dQ_T(\theta)/d\theta$ and $dL_T(\theta)/d\theta$, which we will pursue through Infinitesimal Perturbation Analysis (IPA) techniques. For any sample performance metric $\mathcal{L}(\theta)$, the IPA gradient estimation technique computes $d\mathcal{L}(\theta)/d\theta$ along an observed sample path. If the IPA-based estimate $d\mathcal{L}(\theta)/d\theta$ satisfies $dE[\mathcal{L}(\theta)]/d\theta =$

$E[d\mathcal{L}/d\theta]$, it is unbiased. Unbiasedness is the principal condition for making the application of IPA practical, since it enables the use of the IPA sample derivative in stochastic gradient-based algorithms. A comprehensive discussion of IPA and its applications can be found in [9],[23] and [10].

## III. IPA ESTIMATION

As already mentioned, our objective is to estimate the derivatives $dE[Q_T(\theta)]/d\theta$ and $dE[L_T(\theta)]/d\theta$ through the sample derivatives $dQ_T(\theta)/d\theta$ and $dL_T(\theta)/d\theta$, which are commonly referred to as IPA estimators. In the process, however, it will be necessary to determine and analyze IPA derivatives for the buffer content $x(t;\theta)$, where we include $\theta$ to stress the dependence on it, as well as for certain event times to be defined, which also depend on $\theta$.

We consider a sample path of the SFM of Fig. 1 over $[0,T]$. For a fixed $\theta$, the interval $[0,T]$ is divided into alternating boundary periods and non-boundary periods. A *Boundary Period* (BP) is defined as the time interval during which $x(t;\theta)=\theta$ or $x(t;\theta)=0$, and a *Non-Boundary Period* (NBP) is defined as the time interval during which $0<x(t;\theta)<\theta$. BPs are further classified as *Empty Periods* (EP) and *Full Periods* (FP). An EP is an interval such that $x(t;\theta)=0$; a FP is an interval such that $x(t;\theta)=\theta$.

For simplicity, we assume that at time $t=0$, $x(0;\theta)=0$. We also assume that there are $N$ NBPs in the interval $[0,T]$, where $N$ is a random number, and index NBPs by $n=1,\ldots,N$. The starting and ending points of a NBP are denoted by $\eta_n$ and $\zeta_n$ respectively. We define the following two random sets of indices:

$$\begin{aligned}&\Psi_F(\theta)\\&=\{n\colon x(t;\theta)=\theta \text{ for all } t\in[\zeta_{n-1},\eta_n),\ n=1,\ldots,N\ \}\end{aligned} \tag{5}$$

$$\begin{aligned}&\Psi_E(\theta)\\&=\{n\colon x(t;\theta)=0 \text{ for all } t\in[\zeta_{n-1},\eta_n),\ n=1,\ldots,N\}\\&=\{1,\ldots,N\}-\Psi_F\end{aligned} \tag{6}$$

Clearly, if $n\in\Psi_F$, the $n$th BP (which immediately precedes the $n$th NBP) is a FP and if $n\in\Psi_E$, the $n$th BP is an EP. A typical sample path is shown in Fig. 2, which includes two NBPs, i.e., $[\eta_n,\zeta_n)$ and $[\eta_{n+1},\zeta_{n+1})$, and three BPs. It is worth noticing that $x(t;\theta)$ is not necessarily differentiable, as shown, for example, at time $\tau$; this may be caused by a discontinuity in $\sigma(t)$ or $\mu(t)$.

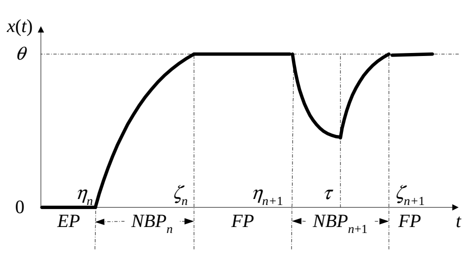


Fig. 2. A Typical Sample Path

### A. Boundedness of Buffer Level Perturbations

In this section we establish an important boundedness property for *buffer level perturbations* defined as

$$\Delta x(t)=x(t;\theta+\Delta\theta)-x(t;\theta),$$

with respect to a perturbation $\Delta\theta$. For simplicity, let us limit ourselves to $\Delta\theta>0$; the case where $\Delta\theta<0$ can be similarly analyzed. We state the boundedness property of $\Delta x(t)$ in the following lemma:

*Lemma 1:* For the system described in (1) and $\Delta\theta>0$,

$$0\le\Delta x(t)\le\Delta\theta,\quad \text{for all } t\in[0,T] \tag{7}$$

*Proof:* See Appendix. ∎

Similarly, for a perturbation $\Delta\theta<0$, we can prove that $\Delta\theta\le\Delta x(t)\le 0$.

*Corollary 1:* If $x(t;\theta)$ is differentiable with respect to $\theta$, then

$$0\le\frac{\partial x(t;\theta)}{\partial\theta}\le 1$$

*Proof:* We have

$$\frac{\partial x(t;\theta)}{\partial\theta^+}=\lim_{\Delta\theta\to0^+}\frac{x(t;\theta+\Delta\theta)-x(t;\theta)}{\Delta\theta}$$

and from Lemma 1 we obtain:

$$0\le\frac{x(t;\theta+\Delta\theta)-x(t;\theta)}{\Delta\theta}\le\frac{\Delta\theta}{\Delta\theta}=1$$

Combining the above two relationships gives $0\le\frac{\partial x(t;\theta)}{\partial\theta^+}\le 1$. Similarly we obtain $0\le\frac{\partial x(t;\theta)}{\partial\theta^-}\le 1$, thus completing the proof. ∎

### B. Queue Content Sample Derivatives

Since we are interested in the sensitivity of performance metrics, which are expressed as functions of $x(t;\theta)$ as in (2)-(3), it is natural to first study $\frac{\partial x(t;\theta)}{\partial\theta}$, the queue content sample derivative. Before proceeding, however, we make some assumptions regarding the class of feedback functions $p(x)$ that we shall consider in our analysis. In particular, we consider continuous piecewise differentiable functions of the form

$$p(x)=\begin{cases}p_1(x) & \text{if } 0\le x\le\theta_1\\ p_2(x) & \text{if } \theta_1\le x\le\theta_2\\ \vdots & \\ p_n(x) & \text{if } \theta_{n-1}\le x\le\theta\end{cases} \tag{8}$$

where $\theta_1<\ldots<\theta_{n-1}$ are real numbers and we make the following assumption:

*Assumption 1:* $p_i(x)$, $i=1,\ldots,n$, are monotonically increasing and continuously differentiable functions. Moreover, there exists a constant $C_p<\infty$ such that for all $x\in[0,\theta]$ and all $i=1,\ldots,n$,

$$\frac{dp_i(x)}{dx}\le C_p$$

**Remark**. If we allow $p(x)$ to be discontinuous at some specific value $x_0$, then it is possible to have $x(t;\theta)=x_0$ for some finite period of time in some sample path. For example, if $x(t_0;\theta)=x_0$ and $p(x_0^-)<\lambda(t)<p(x_0^+)$ for a time interval $[t_0,t_1]$, then $x(t;\theta)=x_0$ for all $t\in[t_0,t_1]$. The discrete

version of this phenomenon is chattering. For simplicity, in this paper we do not deal with discontinuities in $p(x)$. However, since in practice such feedback functions may be of interest, we point out that this situation can be handled in the same framework as the one presented here, along the lines of the work in [24].

The simplest class of feedback functions of the above form is described by a two-segment piecewise differentiable function

$$p(x) = \begin{cases} p_1(x) & \text{if } 0 \le x \le \theta_1 \\ p_2(x) & \text{if } \theta_1 \le x \le \theta \end{cases} \tag{9}$$

in which $\theta_1$ is a known parameter independent of $\theta$ with $0 < \theta_1 < \theta$, and $p_1(x)$, $p_2(x)$ are continuously differentiable monotonically increasing functions with $p_1(\theta_1) = p_2(\theta_1)$. We shall concentrate on feedback functions of this form, and it will become clear that our analysis can be applied to any general piecewise differentiable and monotonically increasing function $p(x)$ in (8). For simplicity, we set $\theta_0 = 0$, $\theta_2 = \theta$, so that we can consistently use the notation $\theta_j$, $j = 0, 1, 2$, to indicate the three *critical values* that $x(t;\theta)$ can take in a typical sample path.

We also assume:

*Assumption 2:* $\lambda(t)$ is a piecewise constant function that can take a finite number of values $\lambda_1, \ldots, \lambda_L$ with $\lambda_i < \lambda_{\max} < \infty$ for all $i = 1, \ldots, L$.

*Assumption 3:* There exists an arbitrarily small positive constant $c_\theta$ such that for all $t$,

$$|\lambda(t) - p(\theta_j)| \ge c_\theta > 0, \quad j = 0, 1, 2. \tag{10}$$

Assumption 3 is mild, but, as we shall see, it is critical in proving the existence of IPA sample derivatives. Combining the two assumptions requires that

$$|\lambda_i - p(\theta_j)| \ge c_\theta \quad \text{for } i = 1, \ldots, L, \quad j = 0, 1, 2 \tag{11}$$

which implies

$$p(\theta_j) \le \lambda_i - c_\theta \;\; \text{or} \;\; p(\theta_j) \ge \lambda_i + c_\theta$$

Since $p(\cdot)$ is a strictly monotonically increasing function, we obtain:

$$\theta_j \le p^{-1}(\lambda_i - c_\theta) \;\; \text{or } \theta_j \ge p^{-1}(\lambda_i + c_\theta) \tag{12}$$

for $i = 1, \ldots, L, \quad j = 0, 1, 2$. Therefore, defining the following $L$ *invalid intervals* $\Delta_i$:

$$\Delta_i = \left[p^{-1}(\lambda_i - c_\theta), p^{-1}(\lambda_i + c_\theta)\right], \quad i = 1, \ldots, L, \tag{13}$$

we conclude that $\theta_j \notin \Delta_i$ for all $i = 1, \ldots, L$, as illustrated in Fig. 3, and note that, for sufficiently small $c_\theta$, there will be no overlap among different $\Delta_i$. This condition imposes a constraint on the controllable parameter $\theta$, which we originally assumed to be defined over a closed and bounded (compact) interval $\Theta$. Letting $\Delta = \cup_{i=1}^{L} \Delta_i$, we now restrict $\theta$ to the set $\tilde{\Theta} = \Theta - \Delta$. We shall also refer to a *valid interval* as the maximal interval between two consecutive invalid intervals. In practice, Assumptions 1-3 are not limiting. For example, Assumption 2 fits common traffic models with on/off sources and fixed service rates which are popular in computer networks. Moreover, for any given $\lambda(t)$, one can always select an appropriate set of values $\lambda_1, \ldots, \lambda_L$ to approximate $\lambda(t)$ to any desirable accuracy level. Lastly, in Assumption 3, by selecting $c_\theta$ to be arbitrarily small the set $\Delta$ of invalid parameter values becomes practically insignificant.

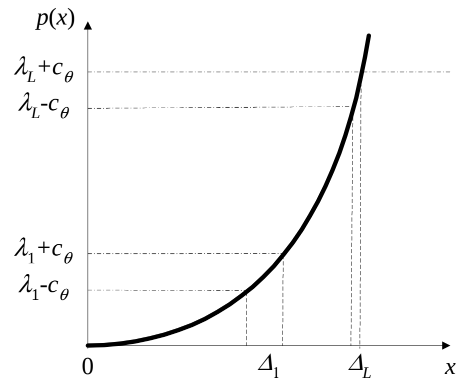


Fig. 3. Illustrating valid and invalid intervals for $\theta_j$, $j = 0, 1, 2$.

Recall that a sample path of the SFM is decomposed into alternating BPs and NBPs. We now refine this decomposition to take into account the structure of the function $p(x)$. To do so, we view the SFM as a DES in which we define the following types of *events*: $(i)$ A jump in $\lambda(t)$, which is termed an *exogenous* event, reflecting the fact that its occurrence time is independent of the controllable parameter $\theta$, and $(ii)$ The buffer content $x(t;\theta)$ reaches any one of the critical values $\theta_j$, $j = 0, 1, 2$; this is termed an *endogenous* event, to reflect the fact that its occurrence time generally depends on $\theta$. Note that the combination of these events and the continuous dynamics in (1) gives rise to a stochastic hybrid system model of the underlying discrete event system of Fig. 1.

Based on these event definitions, we make the following observations. $(i)$ A NBP *end event* is an endogenous event, since its time, $\zeta_n$, generally depends on $\theta$. $(ii)$ A NBP *start event* is an exogenous event: as seen in (1), the end of an EP or FP at time $\eta_n$ is always due to a change in the sign of $\lambda(t) - p(0)$ and $\lambda(t) - p(\theta)$ respectively. This is only feasible when a jump in $\lambda(t)$ occurs at time $t$ according to Assumption 3, which is precisely what we defined as an exogenous event. $(iii)$ The point where the buffer content reaches $\theta_1$ (from either below or above) is an endogenous event (by Assumption 3, this event time is the same at the time when the buffer content also leaves $\theta_1$).

Let us now consider a NBP $[\eta_n, \zeta_n(\theta))$, where we explicitly indicate that its end point depends on $\theta$. Let $\alpha_{n,i}$ denote the $i$th time when $x(t;\theta) = \theta_1$ in this NBP, where $i = 1, \ldots, I_n - 1$, in which $I_n - 1$ is the number of such events. It is possible that $I_n - 1 = 0$ for a NBP, so that to maintain notational consistency we set $\eta_n = \alpha_{n,0}$ and $\zeta_n = \alpha_{n,I_n}$. We can now see that a sample path is decomposed into four sets of intervals that we shall refer to as *modes*: $(i)$ Mode 0 is the set $M_0$ of all EPs contained in the sample path, $(ii)$ Mode 1 is the set $M_1$ of intervals $[\alpha_{n,i}, \alpha_{n,i+1})$ such that $x(\alpha_{n,i}) = 0$ or $\theta_1$ and $0 < x(t) < \theta_1$ for all $t \in (\alpha_{n,i}, \alpha_{n,i+1})$, $n = 1, \ldots, N$, $(iii)$ Mode 2 is the set $M_2$ of intervals $[\alpha_{n,i}, \alpha_{n,i+1})$ such that $x(\alpha_{n,i}) = \theta$ or $\theta_1$ and $\theta_1 < x(t;\theta) < \theta$ for all $t \in [\alpha_{n,i}, \alpha_{n,i+1})$, $n = 1, \ldots, N$, and $(iv)$ Mode 3 is the set $M_3$

of all FPs contained in the sample path. Note that the events occurring at times $\alpha_{n,i}$ are all endogenous for $i = 1, \ldots, I_n$ and we should write $\alpha_{n,i}(\theta)$ to stress this fact; for notational economy, however, we will only write $\alpha_{n,i}$. Finally, recall that for $i = 0$, we have $\alpha_{n,0} = \eta_n$, corresponding to an exogenous event starting the $n$th NBP. The decomposition of a typical NBP is illustrated in Fig. 4. The NBP $[\eta_n, \zeta_n(\theta))$ consists of two $M_1$ intervals and two $M_2$ intervals defined by the presence of $\alpha_{n,1}, \alpha_{n,2}$,and $\alpha_{n,3}$. Moreover, note that at time $t_1$ and $t_2$ exogenous events may occur so that the time derivative of the buffer content $x(t;\theta)$ is discontinuous.

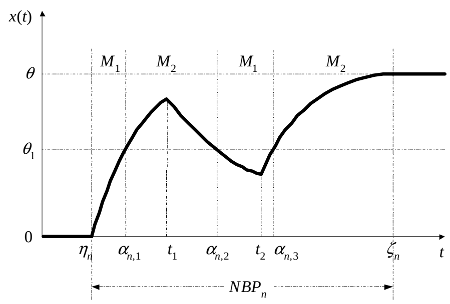


Fig. 4. The Decomposition of a NBP

We shall also make one final assumption:

*Assumption 4:* For every $\theta$, w.p.1, no two events (either exogenous or endogenous) occur at the same time.

This assumption precludes a situation where the queue content reaches the value $\theta$ at the same time $t$ as an exogenous event which might cause it to drop below $\theta$; this would prevent the existence of the derivative of the event time at $t$ (however, one can still carry out perturbation analysis with one-sided derivatives as in [13]). Moreover, by Assumption 4, $N$, the number of NBPs in the sample path, is locally independent of $\theta$ (since no two events may occur simultaneously, and the occurrence of exogenous events does not depend on $\theta$, there exists a neighborhood of $\theta$ within which, w.p.1, the number of NBPs in $[0, T]$ is constant). Hence, the random index sets $\Psi_F$ and $\Psi_E$ defined in (6) are also locally independent of $\theta$.

In what follows, we shall concentrate on a typical NBP $[\eta_n, \zeta_n(\theta))$ and drop the index $n$ from the event times $\alpha_{n,i}$ in order to simplify notation. It is also convenient at this point to define the following:

$$A(t;\theta) = \lambda(t) - p(x(t;\theta)) \qquad (14)$$

Although $A(t;\theta)$ depends on $\theta$, we shall write it as $A(t)$ for the sake of simplicity, unless it is essential to indicate its dependence on $\theta$. In the following lemma we identify the structure of the queue content derivative $\frac{\partial x(t;\theta)}{\partial\theta}$ and show that it depends on the event time derivatives $\frac{\partial\alpha_i}{\partial\theta}$:

*Lemma 2:* Let $[\alpha_i, \alpha_{i+1})$ be an interval in a typical NBP, where $\alpha_i$ is the $i$th time when $x(t;\theta) = \theta_1$. Under Assumptions 1-4, for all $t \in [\alpha_i, \alpha_{i+1})$,

$$\frac{\partial x(t;\theta)}{\partial\theta} = -\frac{\partial\alpha_i}{\partial\theta} A(\alpha_i^+) e^{-\rho(t)} + \mathbf{1}\left[x(\alpha_i;\theta) = \theta\right] e^{-\rho(t)} \qquad (15)$$

where

$$\rho(t) = \int_{\alpha_i}^{t} \frac{dp(x)}{dx} dt \qquad (16)$$

*Proof:* See Appendix. ■

Lemma 2 makes it clear that the queue content sample derivative $\frac{\partial x(t;\theta)}{\partial\theta}$ depends on the event time derivative $\frac{\partial\alpha_i}{\partial\theta}$. Thus, in the next section we address the issue of evaluating these derivatives. The remaining terms in (15) involve: detecting the events such that the buffer content reaches the level $\theta$ or $\theta_1$; the traffic rate information in $A(\alpha_i^+)$ at times when the buffer content reaches the level $\theta_1$; and the evaluation of $\rho(t)$. Specifically, note that $\frac{dp(x)}{dx}$ is generally a function of $x(t;\theta)$, so that an integration of the sample path over an interval $[\alpha_i, t]$ is required to evaluate $\rho(t)$. This evaluation becomes very simple in the case of a linear feedback function $p(x)$, as we will see in Section 5.

Before proceeding with the analysis of the event time derivatives $\frac{\partial\alpha_i}{\partial\theta}$, we provide an alternative way of representing (15) that we will occasionally use:

$$\frac{\partial x(t;\theta)}{\partial\theta} = K_i(\theta) e^{-\rho(t)}$$

where

$$K_i(\theta) = \mathbf{1}\left[x(\alpha_i;\theta) = \theta\right] - A(\alpha_i^+)\frac{\partial\alpha_i}{\partial\theta}.$$

A detailed discussion on the role of $K_i(\theta)$ can be found in the proof of Lemma 2 in the Appendix. Finally, a useful relationship we can derive from the above lemma is the following:

$$\begin{aligned}
\int_{\alpha_i}^{\alpha_{i+1}} \frac{\partial}{\partial\theta}\left[\frac{\partial x(t;\theta)}{\partial t}\right] dt &= \sum_{l=0}^{E_i} \int_{t_l}^{t_{l+1}} \frac{\partial}{\partial\theta}\left[\frac{\partial x(t;\theta)}{\partial t}\right] dt \\
&= \sum_{l=0}^{E_i} \int_{t_l}^{t_{l+1}} \frac{\partial}{\partial t}\left[\frac{\partial x(t;\theta)}{\partial\theta}\right] dt \\
&= \sum_{l=0}^{E_i} \left.\frac{\partial x(t;\theta)}{\partial\theta}\right|_{t_l}^{t_{l+1}} \\
&= \left.\frac{\partial x(t;\theta)}{\partial\theta}\right|_{\alpha_i}^{\alpha_{i+1}} \\
&= K_i(\theta)\left[e^{-\rho(\alpha_{i+1})} - 1\right] \qquad (17)
\end{aligned}$$

where we have used (49) and (16). Recalling (1) and the definition of $A(t;\theta)$ in (14), we can also write the above relationship as

$$\int_{\alpha_i}^{\alpha_{i+1}} \frac{\partial A(t;\theta)}{\partial\theta} dt = K_i(\theta)\left[e^{-\rho(\alpha_{i+1})} - 1\right] \qquad (18)$$

### C. Event Time Sample Derivatives

We derive the sample derivative $\frac{\partial\alpha_i}{\partial\theta}$ through three lemmas which cover the possible values that $x(\alpha_i;\theta)$ can take in an interval $[\alpha_i, \alpha_{i+1})$.

*Lemma 3:* Under Assumptions 1-4, if a FP ends at time $\alpha_i$, i.e., $x(\alpha_i;\theta) = \theta$, then

$$\frac{\partial\alpha_i}{\partial\theta} = 0$$

*Proof:* See Appendix. ■

*Lemma 4:* Under Assumptions 1-4, if an EP ends at time $\alpha_i$, i.e., $x(\alpha_i) = 0$, then

$$\frac{\partial \alpha_i}{\partial \theta} = 0$$

*Proof:* See Appendix. ■

The above two lemmas deal with $\alpha_i$ ending intervals in modes $M_0$ and $M_3$. Next, we obtain the sample derivatives $\frac{\partial \alpha_i}{\partial \theta}$ for the remaining modes $M_1$ and $M_2$.

*Lemma 5:* Under Assumptions 1-4, for an interval $[\alpha_i, \alpha_{i+1})$, $i = 0, \ldots, I_n - 1$, in a NBP:

$$\frac{\partial \alpha_{i+1}}{\partial \theta} = \frac{A(\alpha_i^+) e^{-\rho(\alpha_{i+1})}}{A(\alpha_{i+1})} \cdot \frac{\partial \alpha_i}{\partial \theta} + \frac{\mathbf{1}\left[x(\alpha_{i+1};\theta) = \theta\right]}{A(\alpha_{i+1})} - \frac{\mathbf{1}\left[x(\alpha_i;\theta) = \theta\right] e^{-\rho(\alpha_{i+1})}}{A(\alpha_{i+1})} \tag{19}$$

with $\frac{\partial \alpha_0}{\partial \theta} = 0$.

*Proof:* See Appendix. ■

The combination of Lemmas 3, 4 and 5 provides a simple linear recursive relationship for obtaining the event time sample derivative $\frac{\partial \alpha_i}{\partial \theta}$. In particular, (19) provides this sample derivative over a NBP with initial condition $\frac{\partial \alpha_0}{\partial \theta} = 0$ at the start of any NBP. It also follows from (19) that this derivative remains zero as long as $x(\alpha_i;\theta) \neq \theta$ and $x(\alpha_{i+1};\theta) \neq \theta$. With the help of these lemmas, we can now obtain the sample derivative $\frac{\partial \zeta_n}{\partial \theta}$ that corresponds to the end point of a NBP $[\eta_n, \zeta_n(\theta))$. Let us first extend the definition of $\rho(t)$ in (16) to any interval $[t_1, t_2)$:

$$\rho(t_2, t_1) = \int_{t_1}^{t_2} \frac{dp(x)}{dx} dt \tag{20}$$

and we establish the following result.

*Lemma 6:* Under Assumptions 1-4, for a NBP $[\eta_n, \zeta_n(\theta))$,

$$\frac{\partial \zeta_n}{\partial \theta} = \frac{1}{A(\zeta_n)} \left\{ \mathbf{1}\left[x(\zeta_n;\theta) = \theta\right] - \mathbf{1}\left[x(\eta_n;\theta) = \theta\right] e^{-\rho(\zeta_n, \eta_n)} \right\} \tag{21}$$

*Proof:* See Appendix. ■

We are now in a position to obtain the sample derivatives of the performance metrics defined in (3) and (2).

### D. IPA Sample Derivative of Loss Rate

Recalling the definition of the loss rate $L_T$ in (3), we have

$$L_T(\theta) = \frac{1}{T} \int_0^T \mathbf{1}[x(t;\theta) = \theta](\lambda(t) - p(\theta)) dt = \frac{1}{T} \sum_{n \in \Psi_F} \int_{\zeta_{n-1}}^{\eta_n} A(t;\theta) dt \tag{22}$$

We then establish the following.

*Theorem 1:* Under Assumptions 1-4, the IPA estimator of the loss rate $L_T(\theta)$ with respect to $\theta$ is

$$\frac{dL_T(\theta)}{d\theta} = -\frac{1}{T} \sum_{n \in \Psi_F} \left\{ 1 + (\eta_n - \zeta_{n-1}) p'(\theta) - \mathbf{1}\left[x(\eta_{n-1};\theta) = \theta\right] e^{-\rho(\zeta_{n-1}, \eta_{n-1})} \right\} \tag{23}$$

where

$$p'(\theta) \equiv \left. \frac{dp(x)}{dx} \right|_{x=\theta}$$

*Proof:* Since $[\zeta_{n-1}, \eta_n)$ is a FP, we have $x(t;\theta) = \theta$ for all $t \in [\zeta_{n-1}, \eta_n)$. Thus, as in (40), we have

$$\frac{\partial A(t;\theta)}{\partial \theta} = -\left. \frac{dp(x)}{dx} \right|_{x=\theta} \equiv -p'(\theta)$$

Recalling that $\Psi_F$ is locally independent of $\theta$, it follows from (22) that

$$\begin{aligned} \frac{dL_T(\theta)}{d\theta} &= \frac{1}{T} \sum_{n \in \Psi_F} \left\{ A(\eta_n) \frac{\partial \eta_n}{\partial \theta} - A(\zeta_{n-1}) \frac{\partial \zeta_{n-1}}{\partial \theta} - \int_{\zeta_{n-1}}^{\eta_n} p'(\theta) dt \right\} \\ &= \frac{1}{T} \sum_{n \in \Psi_F} \left\{ A(\eta_n) \frac{\partial \eta_n}{\partial \theta} - A(\zeta_{n-1}) \frac{\partial \zeta_{n-1}}{\partial \theta} - (\eta_n - \zeta_{n-1}) p'(\theta) \right\} \end{aligned}$$

Invoking Lemmas 3 and 4, we have $\frac{\partial \eta_n}{\partial \theta} = 0$. Note that if $n \in \Psi_F$ we must have $x(\zeta_{n-1}) = \theta$, while $x(\eta_{n-1}) = 0$ or $\theta$. Thus, using Lemma 6, (23) immediately follows. ■

It is interesting to observe that this IPA estimator has the important property of being *nonparametric*, in the sense that no information regarding the characteristics of the stochastic processes involved appears in (23), including any flow rate parameters. In fact, for any NBP that starts with an EP and ends with a FP, the only action required is measuring the length of the ensuing FP, $[\eta_n - \zeta_{n-1}]$ (the value of $p'(\theta)$ is known for any $\theta$, given the feedback function $p(x)$). If the NBP that ends at $\zeta_{n-1}$ is one that started with the end of a FP, then the additional term $e^{-\rho(\zeta_{n-1}, \eta_{n-1})}$ needs to be calculated; this simply involves the time instants $\zeta_{n-1}$, $\eta_n$ and the known $p(x)$ for evaluating the integral in (20). More importantly, observe that the information involved in (23) can be *directly obtained from the actual discrete event system*, since all that is needed is detecting a queue level reaching or exceeding a value $\theta$ and then measuring the amount of time that it stays above $\theta$. Therefore, (23) may be used with actual system data, not requiring the implementation of a SFM.

### E. IPA Sample Derivative of Average Workload

Recalling the definition of the average workload $Q_T$ in (2), and making use of the lemmas previously derived, we obtain the following IPA estimator.

*Theorem 2:* Under Assumptions 1-4, the IPA estimator of the workload $Q_T(\theta)$ with respect to $\theta$ is

$$\frac{dQ_T(\theta)}{d\theta} = \frac{1}{T} \left\{ \sum_{n \in \Psi_F} (\eta_n - \zeta_{n-1}) + \sum_{n=1}^{N} \sum_{i=0}^{I_n - 1} \int_{\alpha_{n,i}}^{\alpha_{n,i+1}} K_{n,i}(\theta) e^{-\rho(t, \alpha_{n,i})} dt \right\} \tag{24}$$

where

$$K_{n,i}(\theta) = \mathbf{1}\left[x(\alpha_{n,i};\theta) = \theta\right] - A(\alpha_{n,i}^+) \frac{\partial \alpha_{n,i}}{\partial \theta}$$

*Proof:* Using (2) and the definitions of $\Psi_E$ and $\Psi_F$ in (6) we have:

$$\frac{dQ_T(\theta)}{d\theta} = \frac{1}{T}\frac{d}{d\theta}\left\{\sum_{n\in\Psi_E}\int_{\zeta_{n-1}}^{\eta_n} 0\cdot dt + \sum_{n=1}^{N}\int_{\eta_n}^{\zeta_n} x(t;\theta)dt + \sum_{n\in\Psi_F}\int_{\zeta_{n-1}}^{\eta_n}\theta dt\right\}$$

$$= \frac{1}{T}\frac{d}{d\theta}\left\{\sum_{n=1}^{N}\int_{\eta_n}^{\zeta_n} x(t;\theta)dt + \sum_{n\in\Psi_F}\theta(\eta_n-\zeta_{n-1})\right\}$$

Since $N$ and $\Psi_F$ are locally independent of $\theta$, it follows that

$$\frac{dQ_T(\theta)}{d\theta} = \frac{1}{T}\left\{\sum_{n=1}^{N}\left[x(\zeta_n;\theta)\frac{\partial\zeta_n}{\partial\theta} - x(\eta_n;\theta)\frac{\partial\eta_n}{\partial\theta} + \int_{\eta_n}^{\zeta_n}\frac{\partial x(t;\theta)}{\partial\theta}dt\right] + \sum_{n\in\Psi_F}\left[(\eta_n-\zeta_{n-1}) + \theta(\frac{\partial\eta_n}{\partial\theta} - \frac{\partial\zeta_{n-1}}{\partial\theta})\right]\right\} \tag{25}$$

Invoking Lemmas 3 and 4, we have $\frac{\partial\eta_n}{\partial\theta}=0$. Moreover, since $x(\zeta_n) = \theta\cdot\mathbf{1}\left[(n+1)\in\Psi_F\right]$ we have

$$\sum_{n=1}^{N} x(\zeta_n;\theta)\frac{\partial\zeta_n}{\partial\theta} = \sum_{(n+1)\in\Psi_F}\theta\frac{\partial\zeta_n}{\partial\theta} = \sum_{n\in\Psi_F}\theta\frac{\partial\zeta_{n-1}}{\partial\theta}$$

where we take into account the fact the last NBP may end at time $T$ and have no BP following it, in which case $\zeta_N = T$; hence, $\frac{\partial\zeta_N}{\partial\theta} = 0$ and the above equation still holds. Thus, (25) reduces to

$$\frac{dQ_T(\theta)}{d\theta} = \frac{1}{T}\left\{\sum_{n\in\Psi_F}(\eta_n-\zeta_{n-1}) + \sum_{n=1}^{N}\int_{\eta_n}^{\zeta_n}\frac{\partial x(t;\theta)}{\partial\theta}dt\right\}$$

Let $[\alpha_i,\alpha_{i+1})$, $i=0,\ldots,I_{n-1}$ be the intervals contained in the NBP $[\eta_n,\zeta_n)$. Then, using (49) and $\rho(t,\alpha_{n,i})$ as defined in (20) immediately gives (24). ∎

Similar to the IPA estimator (23), the one in (24) also requires a timer for measuring the length of every FP $[\eta_n-\zeta_{n-1}]$. In addition, however, we need to evaluate $K_{n,i}(\theta)$ at every endogenous event $\alpha_i$ in a NBP. This, in turn, involves the sample derivative $\frac{\partial\alpha_i}{\partial\theta}$, as seen in (45), which requires the recursive evaluation (19). In general, this evaluation involves some rate information in the form of $A(\alpha_i^+)$, i.e., knowledge of the value of $\lambda(t)$ when an event $\alpha_i$ takes place. Thus, unlike (23), the IPA estimator (24) is not nonparametric. Finally, note that the analysis leading to the estimators (23) and (24) can be readily generalized to any piecewise differentiable and monotonically increasing function $p(x)$ in (8) with more than two segments, as long as the events corresponding to the buffer level crossing any one of the thresholds are observed and that the function $\rho(t_2,t_1)$ is evaluated for any interval $[t_1,t_2)$.

**Remark**. By setting $p(x)=0$, the SFM reduces to the one studied in [13] in the absence of feedback. We can readily verify that Theorems 1 and 2 with $p(x)=0$ yield the same results as Theorems 5 and 6 of [13].

## IV. Unbiasedness of IPA Estimators

In this section we establish the unbiasedness of the IPA estimators (23) and (24) for SFM. The presence of feedback in our SFM makes this task somewhat more challenging than in earlier work, such as [13] where no feedback was present.

Normally, the unbiasedness of an IPA derivative $d\mathcal{L}(\theta)/d\theta$ for some performance metric $\mathcal{L}(\theta)$ is ensured by the following two conditions (see [25], Lemma A2, p.70): $(i)$ For every $\theta\in\tilde{\Theta}$, the sample derivative exists w.p.1, and $(ii)$ W.p.1, the random function $\mathcal{L}(\theta)$ is Lipschitz continuous throughout $\tilde{\Theta}$, and the (generally random) Lipschitz constant has a finite first moment. Consequently, establishing unbiasedness reduces to verifying the Lipschitz continuity of $\mathcal{L}(\theta)$ over $\tilde{\Theta}$. In the case of $L_T(\theta)$, however, the existence of invalid intervals in $\Theta$, originating from the presence of feedback, creates a problem that we circumvent in what follows. In order to proceed, we shall need one additional very mild technical condition:

*Assumption 5:* Let $W(\theta)$ be the number of jumps of $\lambda(t)$ in the time interval $[0,T]$. Then, for any $\theta\in\tilde{\Theta}$, $E\left[W(\theta)\right]<\infty$.

*Lemma 7:* Let $\theta$ and $\theta+\Delta\theta$, $\Delta\theta>0$, be in the same valid interval in $\tilde{\Theta}$. Then, under Assumptions 1-5, w.p.1.,

$$|\Delta L_T| \le r\Delta\theta$$

where $r$ is a random variable with a finite expectation.

*Proof:* See Appendix. ∎

A similar result is obtained for the case $\Delta\theta<0$. We can now establish the unbiasedness of the IPA estimator $\frac{dL_T(\theta)}{d\theta}$, as well as that of $\frac{dQ_T(\theta)}{d\theta}$, as follows:

*Theorem 3:* Assume $\theta$ is in a valid interval in $\tilde{\Theta}$. Then, under Assumptions 1-5, the IPA estimators (23) and (24) are unbiased, i.e.,

$$\frac{dE[L_T(\theta)]}{d\theta} = E\left[\frac{dL_T(\theta)}{d\theta}\right], \quad \frac{dE[Q_T(\theta)]}{d\theta} = E\left[\frac{dQ_T(\theta)}{d\theta}\right]$$

*Proof:* In the case of $Q_T(\theta)$, we have, for $\Delta\theta>0$,

$$\Delta Q_T = Q_T(\theta+\Delta\theta) - Q_T(\theta) = \frac{1}{T}\left[\int_0^T x(t;\theta+\Delta\theta)dt - \int_0^T x(t;\theta)dt\right] = \frac{1}{T}\int_0^T \Delta x(t)dt$$

Recalling the boundedness of $\Delta x(t)$ in Lemma 1, we obtain $0\le\Delta Q_T\le\Delta\theta$. Similarly, for $\Delta\theta<0$, we obtain $\Delta\theta\le\Delta Q_T\le 0$. Thus, $Q_T(\theta)$ is Lipschitz continuous and the unbiasedness result follows directly from the known fact (see [25], Lemma A2, p.70) that an IPA derivative $\frac{d\mathcal{L}(\theta)}{d\theta}$ is unbiased if $(i)$ For every $\theta\in\tilde{\Theta}$, the sample derivative exists w.p.1, and $(ii)$ W.p.1, the random function $\mathcal{L}(\theta)$ is Lipschitz continuous throughout $\tilde{\Theta}$, and the (generally random) Lipschitz constant has a finite first moment.

In the case of $L_T(\theta)$, as mentioned earlier, the Lipschitz continuity does not hold generally for all $\theta\in\tilde{\Theta}$ because of the existence of the invalid intervals. But the unbiasedness can still be obtained as follows. Let $\theta$ be in an arbitrary valid interval in $\tilde{\Theta}$ and consider a sequence $\{\theta'_n\} = \{\theta+\Delta\theta_n\}$,

$n = 1, 2, \ldots$, selected so that all its elements belong to the same valid interval, and such that

$$\lim_{n\to\infty} \Delta\theta_n = 0, \quad \lim_{n\to\infty} \theta_n' = \theta$$

Then, define

$$f_n = \frac{L_T(\theta_n') - L_T(\theta)}{\Delta\theta_n}$$

where

$$\lim_{n\to\infty} f_n = \frac{dL_T(\theta)}{d\theta} = f$$

and, by Lemma 7,

$$|f_n| \leq r{=}C_p + \left| \max_{j=1,\ldots,L} [\lambda_j - p(\theta)] \right| \frac{R}{Tc_\theta}$$

Since $E[r] < \infty$, $f_n$ is integrable and the unbiasedness result follows by the Dominated Convergence Theorem [26]. ■

## V. The Linear Feedback Case

In this section we consider the special case of a linear feedback function, i.e.,

$$p(x) = cx \tag{26}$$

for all $x \geq 0$ and $c > 0$. In this case, a NBP $[\eta_n, \zeta_n(\theta))$ is no longer decomposed into subintervals, since there is no endogenous event contained in the NBP. In other words, the sample path now consists of three modes only: $M_0$ corresponding to EPs, $M_2$ corresponds to FPs, and $M_1$ corresponding to NBPs.

Given (26), we obtain for (16):

$$\rho(t) = \int_{\eta_n}^{t} \frac{\partial p(x)}{\partial x} dt = c(t - \eta_n) \tag{27}$$

and (46) reduces to

$$K_n(\theta) = \mathbf{1}[x(\eta_n; \theta) = \theta] \tag{28}$$

since, by Lemmas 3 and 4, we have $\frac{\partial \eta_n}{\partial \theta} = 0$. Note that the indexing is now over NBPs, $n = 1, 2, \ldots$, since there are no endogenous events based on which we previously subdivided a NBP. Accordingly, the queue content derivative in (15) becomes, for any $t \in [\eta_n, \zeta_n(\theta))$,

$$\frac{\partial x(t;\theta)}{\partial \theta} = \mathbf{1}[x(\eta_n; \theta) = \theta]\, e^{-c(t-\eta_n)} \tag{29}$$

Using (27), the result of Lemma 6 becomes

$$\frac{\partial \zeta_n}{\partial \theta} = \frac{1}{A(\zeta_n)} \Big\{ \mathbf{1}[x(\zeta_n; \theta) = \theta] - \mathbf{1}[x(\eta_n; \theta) = \theta]\, e^{-c(\zeta_n - \eta_n)} \Big\}$$

The IPA estimator (23) becomes

$$\frac{dL_T(\theta)}{d\theta} = -\frac{1}{T} \sum_{n \in \Psi_F} \Big\{ 1 + c(\eta_n - \zeta_{n-1}) - \mathbf{1}[x(\eta_{n-1}; \theta) = \theta]\, e^{-c(\zeta_{n-1} - \eta_{n-1})} \Big\} \tag{30}$$

and we can clearly see that this is *nonparametric* and requires only simple timers to evaluate the duration of NBPs $[\eta_{n-1}, \zeta_{n-1})$ and FPs $[\zeta_{n-1}, \eta_n)$. Finally, using (29), the IPA estimator (24) reduces to

$$\begin{aligned} \frac{dQ_T(\theta)}{d\theta} &= \frac{1}{T} \Big\{ \sum_{n \in \Psi_F} (\eta_n - \zeta_{n-1}) + \sum_{n=1}^{N} \int_{\eta_n}^{\zeta_n} \mathbf{1}[x(\eta_n; \theta) = \theta]\, e^{-c(t-\eta_n)} dt \Big\} \\ &= \frac{1}{T} \sum_{n \in \Psi_F} \Big\{ (\eta_n - \zeta_{n-1}) + \frac{1 - e^{-c(\zeta_n - \eta_n)}}{c} \Big\} \end{aligned} \tag{31}$$

Unlike the general case in (24), here we find that the workload IPA estimator is also *nonparametric* and involves only timers for measuring the durations of FPs and their ensuing NBPs. Thus, from an implementation standpoint, using linear feedback enables the use of IPA estimators in control and optimization problems with same ease as in the no feedback case in, for example, [13], as further discussed in the next section.

## VI. Optimal Buffer Control Using SFM-based IPA Estimators

### A. Cost Function Structure

In this section we discuss the influence of feedback on the structure of cost functions defined to formulate optimization problems. Similar to [13], the cost function we consider has the form of (4), i.e., $J(\theta) = \gamma Q_T + L_T$, reflecting the tradeoff between loss and workload (equivalently, throughput and system delay). This tradeoff, however, is complicated by feedback: As the threshold $\theta$ increases, the amount of suppressed flow (the flow not admitted into the system) increases and this has to be either rejected or held at the supply source[1]. This effect is certainly undesirable and has to be taken into account when defining throughput. Therefore, the role played by feedback in this tradeoff needs to be carefully identified.

Let us adopt a pricing-based approach to reveal the effect of feedback on throughput. In an actual (discrete event) queueing system with the proposed feedback mechanism, every customer trying to enter the server belongs to one of the following three classes: $(i)$ it is rejected before entering the system because of the source rate suppression; $(ii)$ it enters the system but is dropped because of overflow; $(iii)$ it is successfully served. Moreover, we assume that $(a)$ two customers of different classes have a price difference, and $(b)$ two customers of the same class have the same price. Our pricing strategy is to *impose a nonnegative price on every packet according to its class* and minimize the total cost. The price is set to 1 for each customer dropped because of loss (overflow), or 0 for each customer successfully served, based on the premise that dropped customers should be penalized the most and those successfully served should be penalized the least. On the other hand, for each customer suppressed before it could enter the system, the price is $s$ with $0 < s < 1$,

[1]Flow supression may be realized by a variety of mechanisms; for example, by letting the supply source hold the flow until the buffer content is below the threshold. As a result, the flow is delayed but not necessarily lost. However, we shall not go into details regarding these mechanisms.

since this justifies the presence of a source rate suppression mechanism through which customers are not necessarily lost but rather delayed. It is worth pointing out that our purpose here is not to discuss a precise pricing scheme, but only to gauge the effect of feedback by a pricing-based approach using the relative price values 0, $s$, 1. The total price for all customers during the time interval $[0, T]$ is then:

$$P(\theta) = [sR_T(\theta) + L_T(\theta)] \cdot T \quad (32)$$

in which $R_T(\theta)$ is the suppression rate of the system and $L_T(\theta)$ is the loss rate. Moreover, let us consider the particular case of linear feedback, so that

$$R_T(\theta) = \frac{1}{T}\int_0^T p(x(t;\theta))dt = \frac{1}{T}\int_0^T cx(t;\theta)dt = cQ_T(\theta) \quad (33)$$

combining (32) and (33) suggests

$$\frac{P(\theta)}{T} = csQ_T(\theta) + L_T(\theta) \quad (34)$$

as the metric for throughput reduction due to overflow and source rate suppression.

On the other hand, taking both throughput and delay (equivalently, workload by Little's Law [10]) into consideration, an overall cost function may be determined as

$$J(\theta) = \frac{P(\theta)}{T} + dQ_T(\theta) = (cs + d)\, Q_T(\theta) + L_T(\theta) \quad (35)$$

in which the parameter $s$ reflects the agreement between the server and the supply source and $d$ reflects the relative importance between throughput efficiency and latency. In summary, the form of this cost function looks similar to that in previous work [13], i.e., it is a linear combination of $Q_T(\theta)$ and $L_T(\theta)$. However, the coefficient of $Q_T(\theta)$ is split into two parts, corresponding to the effect of throughput (affected by feedback through $c$) and delay respectively.

### B. Numerical Results

Based on the preceding discussion of the cost function structure, we present some numerical examples limiting ourselves here to the linear feedback case. As suggested before, the solution to an optimization problem defined for an actual queueing system may be approximated by the solution to the same problem based on a SFM of the system. However, the simple form of the IPA estimators of the loss rate and workload obtained through (24) and (23) actually allows us to use data from the *actual* system in order to estimate sensitivities that, in turn, may be used to solve an optimization problem of interest.

Let us now consider the linear feedback buffer control problem for the actual DES with cost function (35) and illustrate one of several possible means to quantify system performance objectives:

$$J_T^{DES}(\theta) = \gamma E[Q_T^{DES}(\theta)] + E[L_T^{DES}(\theta)] \quad (36)$$

in which $\gamma = cs + d$ as detailed in the previous section. The problem of determining $\theta^*$ to minimize $J_T^{DES}(\theta)$ above may be addressed through a standard stochastic approximation algorithm (details on such algorithms, including conditions required for convergence to an optimum may be found, for instance, in [27]):

$$\theta_{n+1} = \theta_n - \nu_n H_n(\theta_n, \omega_n^{DES}), \qquad n = 0, 1, \ldots \quad (37)$$

where $H_n(\theta_n, \omega_n^{DES})$ is an estimate of $dJ_T^{DES}/d\theta$ evaluated at $\theta = \theta_n$ and based on information obtained from a sample path of the DES denoted by $\omega_n^{DES}$ (to differentiate it from sample paths $\omega_n^{SFM}$ obtained through a SFM of the DES) and $\{\nu_n\}$ is a step size sequence. What we have obtained in (30) and (31) gives us an IPA estimator of $J_T^{SFM}(\theta)$, the cost obtained for the SFM corresponding to the actual DES. Notice, however, that the form of these SFM-based IPA estimators enables their values to be obtained from data of an *actual* (discrete-event) system: The expressions in (30) and (31) simply require (*i*) detecting when the buffer level reaches $\theta$ or 0, and (*ii*) timing the length of BPs and NBPs. These are data available on a sample path of the DES. In other words, the *form* of the IPA estimators is obtained by analyzing the system as a SFM, but the associated *values* can be obtained from real data from the underlying DES. Obviously, the resulting gradient estimator $H_n(\theta_n, \omega_n^{DES})$ is now an approximation leading to a sub-optimal solution of the above optimization problem; however, as extensively discussed in prior work (e.g., [13]) this approximation recovers $\theta^*$ with great accuracy.

Note that $\{\theta_n\}$ in (37) is a sequence of real numbers. Applied to the actual queueing system, we define the control policy as follows: as a customer enters the system (if not suppressed due to the feedback effect), it is accepted if the buffer content $x(t) \le \theta$ or dropped because of overflow when $x(t) > \theta$. Finally, note that, after a control update, the state must be reset to zero, in accordance with our convention that all performance metrics are defined over an interval with an initially empty buffer. In the case of off-line control (as in the numerical examples we present), this simply amounts to simulating the system after resetting its state to 0. In the more interesting case of on-line control, we proceed as follows. Suppose that the $n$th iteration ends at time $\tau_n$ and the state is $x(\theta_n; \tau_n)$ [in general, $x(\theta_n; \tau_n) > 0$]. At this point, the threshold is updated and its new value is $\theta_{n+1}$. Let $\tau_n^0$ be the next time that the buffer is empty, i.e., $x(\theta_{n+1}; \tau_n^0) = 0$. At this point, the $(n+1)$th iteration starts and the next gradient estimate is obtained over the interval $[\tau_n^0, \tau_n^0 + T]$, so that $\tau_{n+1} = \tau_n^0 + T$ and the process repeats. The implication is that over the interval no estimation is carried out while the controller waits for the system to be reset to its proper initial state; therefore, sample path information available over $[\tau_n, \tau_n^0]$ is effectively wasted as far as gradient estimation is concerned.

Figure 5 shows examples of the application of (37) to a single-node system with linear feedback under four different parameter settings (scenarios). The service rate $\mu(t) = 2400$ remains constant throughout the simulation. In all four cases, $\sigma(t)$ is piecewise constant. In scenarios 1 and 2, each interval over which $\sigma(t)$ remains constant is a random variable with a Pareto distribution $Pareto(1/2r, 2)$, i.e., a cdf $F(x; A, B) = 1 - \left(\frac{A}{x}\right)^B$ for $Pareto(A, B)$. In scenarios 3 and 4, each interval is exponentially distributed with parameter $r$. Therefore for all scenarios the expected length of such intervals is $1/r$. At the end of each interval, the next value of $\sigma(t)$ is generated

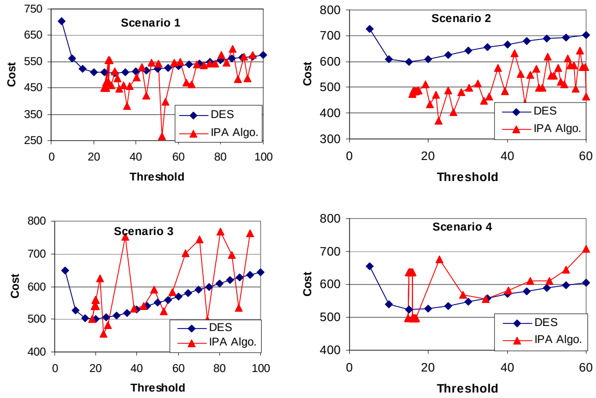


Fig. 5. Numerical results for four different scenarios

according to a transition probability matrix. For simplicity, we assume that all elements of the transition probability matrix are equal and the only feasible value of these elements is $q = 1/m$, in which $m$ is the number of values $\sigma(t)$ can take. For different scenarios, $\sigma(t)$ value sets, the feedback factor $c$ and overflow penalty $\gamma$ also vary. Table 1 summarizes the settings for all four scenarios. Also shown in the table are $\theta_0$, the initial threshold value, and $\theta^*$, the threshold value obtained through (37). In Fig. 5, the curve "DES" denotes the cost function $J_T(\theta)$ obtained through exhaustive simulation for different (discrete) values of $\theta$ with $T = 50000$; the curve "IPA Algo." represents the optimization process (37) with the simulation time horizon for each step of (37) set to $T' = 10$, and with constant step size $\nu = 2.5$. As shown in Fig. 5, the gradient-based algorithm (37) converges to or very near the optimal threshold (the threshold value that corresponds to the minimum of the DES curve) even when the per-step simulation time horizon is so short that the performance with the same time horizon significantly deviates from the "DES" curve. This is an indication of how the optimal control parameter can be recovered through a SFM, even though the corresponding performance estimates obtained by the SFM may be inadequate, as is clearly the case in Scenario 2 of Fig. 5 for example.

In Fig. 6 we also show the effect of $T'$, the simulation time horizon for each step of (37) on its convergence. We observe that the smaller the IPA estimation interval $T'$ becomes, the slower the algorithm converges. When the value is too small, i.e., $T' = 5$ in this case, the threshold finally oscillates between the values 15 and 25, the actual optimum being $\theta^* = 16.4$.

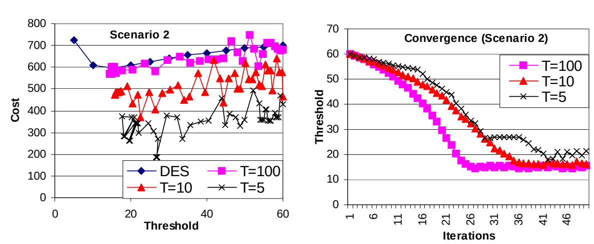


Fig. 6. Convergence as a function of IPA estimation interval

## VII. Conclusions and Future Work

SFMs have recently been used to capture the dynamics of complex DES so as to implement control and optimization methods based on estimating gradients of performance metrics. In doing so, we exploit the simple form of the IPA estimators of these performance gradients and the fact that they depend only on data directly observable along a sample path of the actual system (not just the SFM which is an abstraction of the system). Systems considered to date have not included any feedback mechanism in their operation. In this paper, we have taken the first step towards incorporating feedback by considering a single-node SFM with threshold-based buffer control and controllable inflow rate based on buffer content feedback. We have developed IPA estimators for the loss volume and average workload as functions of the threshold parameter, and shown their unbiasedness, despite the complications brought about by the presence of feedback. The simplicity of the estimators, especially in the linear feedback case, suggests their application to on-line control of networks, as illustrated in Section 6.

The work in this paper opens up a variety of possible extensions. First, the framework presented here can be readily applied to piecewise differentiable feedback functions $p(x)$ with more than two segments or with discontinuities. Moreover, the sensitivity analysis we carried out for the threshold parameter $\theta$ can be extended to all threshold parameters $\theta_1, \ldots, \theta_{n-1}$ of the feedback function (8). On the other hand, there are several issues regarding the feedback mechanism. First of all, the feedback form $\alpha(t) - p(x)$ implies several requirements on the server and the supply source: $(i)$ the form of $p(x)$ requires continuous observation of the buffer content $x(t)$; $(ii)$ this information has to be instantaneously transferred back to the source; $(iii)$ the source has to be able to apply the additive rate control continuously. These requirements may seriously hinder the application of this feedback mechanism in some settings, notably high-speed communication networks. Motivated by this consideration, we are pursuing the study of IPA estimators for alternative feedback mechanisms. For example, of special interest in practice is multiplicative feedback of the form $p(x)\sigma(t)$, where $p(x)$ may be, for instance, a piecewise constant function, making implementation of such a mechanism particularly simple (i.e., by probabilistically dropping incoming customers) [28],[29]. Preliminary work also suggests that a similar analysis may be carried out for an exponential feedback function $p(x) = e^{-cx}$. Along the same lines, by properly selecting $p(x)$, it is possible to emulate other forms of feedback such as the popular TCP congestion control scheme.

Of obvious interest is also the possibility to obtain gradient estimators with respect to parameters of the controller. For example, in the linear feedback case in (26), sensitivity estimates of performance metrics with respect to the "gain" $c$ would be instrumental in tuning such a controller. Our ongoing work suggests that this is indeed possible.

Finally, as mentioned earlier, a critical assumption in this paper is that state information is instantaneously available. In some cases, the delay involved in providing such information

| Scenario | $r$ | $q=\frac{1}{m}$ | $\sigma$ value set ($\times 10^3$) | $c$ | $s$ | $d$ | $\gamma$ | $\theta_0$ | $\theta^*$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 10 | 0.333 | 34,28,22 | 5 | 0.5 | 5 | 7.5 | 95 | 27.3 |
| 2 | 10 | 0.333 | 34,28,22 | 20 | 0.5 | 20 | 30 | 60 | 16.4 |
| 3 | 10 | 0.125 | 34,33,32,29,26,25,19,14 | 5 | 0.5 | 7.5 | 10 | 95 | 20.1 |
| 4 | 10 | 0.125 | 34,33,32,29,26,25,19,14 | 13 | 0.5 | 13.5 | 20 | 95 | 16.1 |

TABLE I
SUMMARY OF PARAMETER SETTINGS FOR FOUR SCENARIOS

to a controller can be significant. To address this problem, alternative feedback mechanisms are needed; for example, a piecewise constant $p(x)$ implies that the controller needs to be notified only when the buffer content reaches one or more critical levels that trigger some action (capturing a "quantization" in the state feedback). In this case, the effect of a communication delay can be incorporated in the proper selection of these critical levels (adjustable controller parameters). This is a crucial issue that we are currently investigating.

# Appendix

*Proof:* [**Lemma** 1] At any time, a state trajectory has to be in one of the following three aggregate states: EP, NBP or FP. Therefore, there are nine possibilities for the joint aggregate states corresponding to the nominal and perturbed sample paths for a given $\theta$ and $\Delta\theta$. For example (EP,NBP) indicates that the nominal sample path is in an EP and the perturbed sample path is in a NBP. Let us consider joint sample paths and let $t_l$ denote the $l$th time instant when either the nominal or the perturbed sample path experiences a transition from one aggregate state to another. Thus, we decompose $[0,T]$ into intervals $[t_l,t_{l+1})$, $l=0,\ldots,B-1$, where $t_0=0$ and $B$ is the total number of such intervals. We now proceed by induction over $l=0,\ldots,B-1$.

For the first time interval corresponding to $l=0$, since, by assumption, $x(0;\theta+\Delta\theta)=x(0;\theta)=0$, $\Delta\theta$ has no effect and we have $\Delta x(t)=0<\Delta\theta$ for all $t\in[t_0,t_1)$. Next, we assume that $0\le\Delta x(t)\le\Delta\theta$ for all $t\in[t_{l-1},t_l)$ with $l>1$, and will establish the same conclusion for all $t\in[t_l,t_{l+1})$. We accomplish this by considering all possible nine cases identified above.

1. (EP,EP): Trivially, $\Delta x(t)=x(t;\theta+\Delta\theta)=x(t;\theta)=0<\Delta\theta$, for all $t\in[t_l,t_{l+1})$.

2. (EP,NBP): Since $x(t;\theta)=0$ for all $t\in[t_l,t_{l+1})$, from (1) we have $\lambda(t)-p(0)\le 0$ for all $t\in[t_l,t_{l+1})$. Moreover, since $p(x)$ is monotonically increasing, $p(x(t;\theta+\Delta\theta))\ge p(x(t;\theta))=p(0)$, which implies

$$\frac{dx(t;\theta+\Delta\theta)}{dt}=\lambda(t)-p(x(t;\theta+\Delta\theta))\le\lambda(t)-p(0)\le 0 \tag{38}$$

Therefore, $\Delta x(t)\le\Delta x(t_l)\le\Delta\theta$, for all $t\in[t_l,t_{l+1})$, where the second inequality is due to the induction hypothesis and the continuity of $x(t;\theta)$. It is also clear that $\Delta x(t)\ge 0$.

3. (EP,FP): Since $x(t_l;\theta+\Delta\theta)=\theta+\Delta\theta$ and $x(t_l;\theta)=0$, we get $\Delta x(t_l)=\theta+\Delta\theta>\Delta\theta$ which (recalling the continuity of $x(t;\theta)$) contradicts the induction hypothesis. Therefore, this case is impossible.

4. (NBP,EP): Since $x(t;\theta+\Delta\theta)=0$ for all $t\in[t_l,t_{l+1})$ and the induction hypothesis and continuity of $x(t;\theta)$ require that $x(t_l;\theta+\Delta\theta)\ge x(t_l;\theta)$, it follows that $x(t_l;\theta)=0$. In addition, since the perturbed path is in an EP, we have $\lambda(t)-p(0)\le 0$. These two facts, imply that $x(t;\theta)=0$ for all $t\in[t_l,t_{l+1})$. This contradicts the assumption that the nominal sample path is in a NBP. Hence, this case is impossible.

5. (NBP,NBP): From (1), we have:

$$\frac{dx(t;\theta)}{dt}=\lambda(t)-p(x)$$

and

$$\frac{dx(t;\theta+\Delta\theta)}{dt}=\lambda(t)-p(x(t;\theta+\Delta\theta)),$$

which implies

$$\frac{d\Delta x(t)}{dt}=-\left[p(x(t;\theta+\Delta\theta))-p(x(t;\theta))\right] \tag{39}$$

for $t\in[t_l,t_{l+1})$. Combining the above equation with the induction hypothesis that $\Delta x(t_l)\ge 0$ (recalling the continuity of $x(t;\theta)$) and the monotonicity of $p(\cdot)$, we obtain $p(x(t_l;\theta+\Delta\theta))-p(x(t_l;\theta))\ge 0$. It follows that

$$\frac{d\Delta x(t)}{dt}\le 0$$

for $t\ge t_l$ as long as $\Delta x(t)\ge 0$, $t\in[t_l,t_{l+1})$, hence $\Delta x(t)\le\Delta x(t_l)\le\Delta\theta$. Suppose that $\Delta x(\tau)=0$ for some $\tau\in[t_l,t_{l+1})$. Then, by (39), $\frac{d\Delta x(t)}{dt}=0$ for $t\ge\tau$, i.e., $x(t;\theta+\Delta\theta))=x(t;\theta)$, which implies that both sample paths coincide until the next transition to an EP or FP at $t_{l+1}$. Therefore, $\Delta x(t)\ge 0$ for all $t\in[t_l,t_{l+1})$.

6. (NBP,FP): Since $x(t_l;\theta+\Delta\theta)=\theta+\Delta\theta$, the induction hypothesis $\Delta x(t_l)\le\Delta\theta$ (recalling the continuity of $x(t;\theta)$) and the fact that $x(t;\theta)\le\theta$ imply that $x(t_l;\theta)=\theta$. On the other hand, since the perturbed sample path is in a FP, $\lambda(t)-p(\theta+\Delta\theta)\ge 0$. The monotonicity of $p(x)$ then implies that $\lambda(t)-p(\theta)>0$, $t\in[t_l,t_{l+1})$. Since $x(t_l;\theta)=\theta$, it follows that the nominal sample path is also in a FP, which contradicts the assumption that it is in a NBP. So this case is infeasible.

7. (FP,EP): Since $x(t_l;\theta)=\theta$ and $x(t_l;\theta+\Delta\theta)=0$, we get $\Delta x(t_l)<0$, which contradicts the induction assumption $\Delta x(t_l)\ge 0$, and this case is also infeasible.

8. (FP,NBP): In this case, $x(t;\theta)=\theta$ for all $t\in[t_l,t_{l+1})$ and the induction hypothesis $\Delta x(t_l)\ge 0$ (recalling the continuity of $x(t;\theta)$) implies that $x(t_l;\theta+\Delta\theta)\ge\theta$. Moreover, we show next that $x(t;\theta+\Delta\theta)\ge\theta$ for all $t\in[t_l,t_{l+1})$. If $x(t_l;\theta+\Delta\theta)=\theta$ for some $\tau\in[t_l,t_{l+1})$, we get form (1):

$$\frac{dx(\tau;\theta+\Delta\theta)}{d\tau^+}=\lambda(\tau)-p(\theta)\ge 0$$

where the inequality is due to the fact that the nominal sample path is in a FP, i.e., $\lambda(t) - p(\theta) \geq 0$ for all $t \in [t_l, t_{l+1})$. Therefore, the perturbed sample path is constrained in this case by $\theta \leq x(t;\theta+\Delta\theta) \leq \theta + \Delta\theta$, and it follows that $0 \leq \Delta x(t) \leq \Delta\theta$ for all $t \in [t_l, t_{l+1})$.

9. (FP,FP): Since $x(t;\theta) = \theta$ and $x(t;\theta+\Delta\theta) = \theta + \Delta\theta$, it is clear that for all $t \in [t_l, t_{l+1})$ we have $0 < \Delta x(t) = \Delta\theta$.

This completes the inductive proof. ■

*Proof:* [**Lemma** 2] We begin the proof by further decomposing the interval $[\alpha_i, \alpha_{i+1})$ into subintervals defined by all exogenous events in $(\alpha_i, \alpha_{i+1})$. Let $t_l$ be the time instant when the $l$th exogenous event occurs in $(\alpha_i, \alpha_{i+1})$, $l = 1, \ldots, E_i$, where $E_i$ is the (random) number of these exogenous events. For notational simplicity we set $\alpha_i = t_0$ and $\alpha_{i+1} = t_{E_i+1}$. In the interval $(t_l, t_{l+1})$, $l = 1, 2, \ldots$, $\lambda(t)$ is constant and $p(x)$ is differentiable and we can see that $\frac{\partial x(t;\theta)}{\partial\theta}$ exists and is continuous with respect to $t$. For all $t \in (t_l, t_{l+1})$, (1) gives

$$\frac{\partial x(t;\theta)}{\partial t} = A(t;\theta)$$

Differentiating with respect to $\theta$ and recalling that $\lambda(t)$ is independent of $\theta$, we have from (14)

$$\frac{\partial A(t;\theta)}{\partial\theta} = -\frac{\partial p(x)}{\partial\theta} = -\frac{dp(x)}{dx}\frac{\partial x(t;\theta)}{\partial\theta} \tag{40}$$

so that

$$\frac{\partial}{\partial\theta}\left[\frac{\partial x(t;\theta)}{\partial t}\right] = -\frac{dp(x)}{dx}\frac{\partial x(t;\theta)}{\partial\theta} \tag{41}$$

Define

$$\rho_l(t) = \int_{t_l}^{t} \frac{dp(x)}{dx} dt. \tag{42}$$

Then, (41) can be rewritten as:

$$\frac{\partial}{\partial t}\left[\frac{\partial x(t;\theta)}{\partial\theta}\right] = -\frac{\partial\rho_l(t)}{\partial t}\frac{\partial x(t;\theta)}{\partial\theta} \tag{43}$$

Solving this equation we obtain for $t \in (t_l, t_{l+1})$:

$$\frac{\partial x(t;\theta)}{\partial\theta} = k_l e^{-\rho_l(t)} \tag{44}$$

in which

$$k_l = \left.\frac{\partial x(t;\theta)}{\partial\theta}\right|_{t=t_l^+}$$

Since $t_l$ for $l \geq 1$ is the occurrence time of an exogenous event (when $\lambda(t)$ switches from one value to another), it is locally independent of $\theta$. By Assumption 4 only this exogenous event occurs at $t_l$, and $\frac{\partial x(t;\theta)}{\partial\theta}$ is continuous at $t_l$, thus,

$$\left.\frac{\partial x(t;\theta)}{\partial\theta}\right|_{t=t_l^+} = \frac{\partial x(t_l;\theta)}{\partial\theta} = \left.\frac{\partial x(t;\theta)}{\partial\theta}\right|_{t=t_l^-}$$

On the other hand, when $l = 0$, $t_0 = \alpha_i$ depends on $\theta$. Let us define

$$K_i(\theta) \equiv \left.\frac{\partial x(t;\theta)}{\partial\theta}\right|_{t=\alpha_i^+} \tag{45}$$

which can be determined as follows. Since

$$x(t;\theta) = x(\alpha_i;\theta) + \int_{\alpha_i}^{t} A(\tau;\theta)d\tau,$$

differentiating with respect to $\theta$, we obtain:

$$\frac{\partial x(t;\theta)}{\partial\theta} = \frac{\partial x(\alpha_i;\theta)}{\partial\theta} + \int_{\alpha_i}^{t}\frac{\partial A(\tau;\theta)}{\partial\theta}d\tau - A(\alpha_i^+)\frac{\partial\alpha_i}{\partial\theta}$$

where, recalling the definition of the event times $\alpha_i$, $i = 0, \ldots, I$, there are three possible values that $x(\alpha_i)$ can take: $\theta$, $\theta_1$ or 0. Thus,

$$\frac{\partial x(\alpha_i;\theta)}{\partial\theta} = \mathbf{1}\left[x(\alpha_i) = \theta\right]$$

Since the boundedness of $\frac{\partial p(x)}{\partial x}$ and $\frac{\partial x(t;\theta)}{\partial\theta}$ is guaranteed by Assumption 1 and Corollary 1 respectively, from (40) we obtain:

$$\left|\frac{\partial A(t)}{\partial\theta}\right| = \left|\frac{dp(x)}{dx}\right|\left|\frac{\partial x}{\partial\theta}\right| \leq C_p$$

Therefore,

$$K_i(\theta) = \mathbf{1}\left[x(\alpha_i;\theta) = \theta\right] - A(\alpha_i^+)\frac{\partial\alpha_i}{\partial\theta} \tag{46}$$

Returning to (44) for all $l = 0, 1, 2, \ldots$ we now have:

$$\frac{\partial x(t_1;\theta)}{\partial\theta} = K_i(\theta)e^{-\rho_0(t_1)} \tag{47}$$

$$\frac{\partial x(t_l;\theta)}{\partial\theta} = \frac{\partial x(t_{l-1};\theta)}{\partial\theta}e^{-\rho_{l-1}(t_l)}, \quad \text{for } l > 1 \tag{48}$$

Recalling the definition of $\rho_l(t)$ in (42), let

$$\rho(t) = \int_{\alpha_i}^{t}\frac{dp(x)}{dx}dt$$

and note that for $t \in (t_l, t_{l+1})$, $\rho(t) = \sum_{i=1}^{l}\rho_{i-1}(t_i) + \rho_l(t)$. Thus, combining (47)-(48) with (44) gives

$$\frac{\partial x(t;\theta)}{\partial\theta} = K_i(\theta)e^{-\rho(t)} \quad \text{for all } t \in [\alpha_i, \alpha_{i+1}) \tag{49}$$

which is precisely (15). ■

*Proof:* [**Lemma** 3] If $x(t;\theta)$ decreases from $\theta$ at time $\alpha_i$, this defines the start of a NBP which, as already seen, is an exogenous event, independent of $\theta$. Specifically, from (1) we must have $\lambda(\alpha_i^-) - p(\theta) \geq 0$ and $\lambda(\alpha_i^+) - p(\theta) < 0$ where $\lambda(t)$ is independent of $\theta$. ¿From Assumption 3, $\lambda(\alpha_i^-) - p(\theta) > c_\theta$, $\lambda(\alpha_i^+) - p(\theta) < -c_\theta$. Therefore, there exists a neighborhood of $\theta$ within which a change of $\theta$ does not affect $\alpha_i$. This implies that $\alpha_i$ is locally independent of $\theta$ and the result follows. ■

*Proof:* [**Lemma** 4] The proof is similar to that of the previous lemma, with $\lambda(\alpha_i^-) - p(0) \leq 0$ and $\lambda(\alpha_i^+) - p(0) > 0$. ■

*Proof:* [**Lemma** 5] We have

$$\int_{\alpha_i}^{\alpha_{i+1}} A(t;\theta)dt = x(\alpha_{i+1};\theta) - x(\alpha_i;\theta) \tag{50}$$

Depending on the values of $x(\alpha_i;\theta)$ and $x(\alpha_{i+1};\theta)$, we have the following possible cases:

*Case 1*: $x(\alpha_i;\theta) = x(\alpha_{i+1};\theta)$, so that $\frac{\partial}{\partial\theta}\left[x(\alpha_{i+1};\theta) - x(\alpha_i;\theta)\right] = 0$.

*Case 2*: $x(\alpha_i;\theta) = 0$, $x(\alpha_{i+1};\theta) = \theta_1$, so that $\frac{\partial}{\partial\theta}\left[x(\alpha_{i+1};\theta) - x(\alpha_i;\theta)\right] = 0$.

*Case 3*: $x(\alpha_i;\theta) = \theta_1$, $x(\alpha_{i+1};\theta) = \theta$, so that $\frac{\partial}{\partial\theta}\left[x(\alpha_{i+1};\theta) - x(\alpha_i;\theta)\right] = 1$.

*Case 4*: $x(\alpha_i;\theta) = \theta$, $x(\alpha_{i+1};\theta) = \theta_1$, so that $\frac{\partial}{\partial\theta}[x(\alpha_{i+1};\theta) - x(\alpha_i;\theta)] = -1$.

*Case 5*: $x(\alpha_i;\theta) = \theta_1$, $x(\alpha_{i+1};\theta) = 0$, so that $\frac{\partial}{\partial\theta}[x(\alpha_{i+1};\theta) - x(\alpha_i;\theta)] = 0$.

Combining these cases yields $\frac{\partial}{\partial\theta}[x(\alpha_{i+1};\theta) - x(\alpha_i;\theta)] = \mathbf{1}[x(\alpha_{i+1};\theta) = \theta] - \mathbf{1}[x(\alpha_i;\theta) = \theta]$, which can take three possible values: $1, 0$, and $-1$. Thus, from (50) we get

$$\frac{\partial}{\partial\theta}\left[\int_{\alpha_i}^{\alpha_{i+1}} A(t;\theta)dt\right] = \mathbf{1}[x(\alpha_{i+1};\theta) = \theta] - \mathbf{1}[x(\alpha_i;\theta) = \theta] \tag{51}$$

Let $t_l$ be the time instant when the $l$th exogenous event occurs in $(\alpha_i, \alpha_{i+1})$, $l = 1, \ldots, E_i$, where $E_i$ is the (random) number of these exogenous events. Then,

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left[\int_{\alpha_i}^{\alpha_{i+1}} A(t;\theta)dt\right] = \frac{\partial}{\partial\theta}\left[\sum_{l=0}^{E_i}\int_{t_l}^{t_{l+1}} A(t;\theta)dt\right] \\
&= \sum_{l=0}^{E_i}\frac{\partial}{\partial\theta}\int_{t_l}^{t_{l+1}} A(t;\theta)dt \\
&= \sum_{l=0}^{E_i}\left\{A(t_{l+1}^-)\frac{\partial t_{l+1}}{\partial\theta} - A(t_l^+)\frac{\partial t_l}{\partial\theta}\right. \\
&\left. + \int_{t_l}^{t_{l+1}} \frac{\partial A(t;\theta)}{\partial\theta}dt\right\} \\
&= A(\alpha_{i+1}^-)\frac{\partial\alpha_{i+1}}{\partial\theta} - A(\alpha_i^+)\frac{\partial\alpha_i}{\partial\theta} \\
&+ K_i(\theta)\left[e^{-\rho(\alpha_{i+1})} - 1\right]
\end{aligned}$$

where we have made use of (18). Therefore, using (51), the above equation yields

$$\begin{aligned}
\frac{\partial\alpha_{i+1}}{\partial\theta} &= \frac{1}{A(\alpha_{i+1}^-)}\left\{A(\alpha_i^+)\frac{\partial\alpha_i}{\partial\theta} - K_i(\theta)\left[e^{-\rho(\alpha_{i+1})} - 1\right]\right. \\
&\left. + \mathbf{1}[x(\alpha_{i+1};\theta) = \theta] - \mathbf{1}[x(\alpha_i;\theta) = \theta]\right\}
\end{aligned}$$

Note that for all $i > 0$, $\alpha_i$ corresponds to an endogenous event, therefore, by Assumption 4, there can be no jump in $A(t)$ at $t = \alpha_i$, $i > 0$, i.e., $A(\alpha_{i+1}^-) = A(\alpha_{i+1})$ for all $i = 0, \ldots, I_n - 1$ (we must still write $A(\alpha_i^+)$, however, to account for the jump at $t = \alpha_0$ which initiates the NBP). Then, recalling the definition of $K_i(\theta)$ in (45), the expression above reduces to

$$\begin{aligned}
\frac{\partial\alpha_{i+1}}{\partial\theta} &= \frac{1}{A(\alpha_{i+1})}\left\{A(\alpha_i^+)\frac{\partial\alpha_i}{\partial\theta}e^{-\rho(\alpha_{i+1})}\right. \\
&\left. + \mathbf{1}[x(\alpha_{i+1};\theta) = \theta] - \mathbf{1}[x(\alpha_i;\theta) = \theta]\,e^{-\rho(\alpha_{i+1})}\right\}
\end{aligned}$$

which is precisely (19). The fact that the initial condition for this recursion is $\frac{\partial\alpha_0}{\partial\theta} = 0$ follows from Lemmas 3, 4. ■

*Proof:* [**Lemma** 6] Assume there are $I_n - 1$ endogenous events in $(\eta_n, \zeta_n)$ and recall that $\alpha_0 = \eta_n$ and $\alpha_{I_n} = \zeta_n$. Applying Lemma 5 to all intervals $[\alpha_i, \alpha_{i+1})$ contained in $[\eta_n, \zeta_n)$ we obtain:

$$\begin{aligned}
&A(\alpha_1^-)\frac{\partial\alpha_1}{\partial\theta} - A(\alpha_0^+)\frac{\partial\alpha_0}{\partial\theta}e^{-\rho(\alpha_1,\alpha_0)} \\
&= \mathbf{1}[x(\alpha_1;\theta) = \theta] - \mathbf{1}[x(\alpha_0;\theta) = \theta]\,e^{-\rho(\alpha_1,\alpha_0)} \\
&A(\alpha_2^-)\frac{\partial\alpha_2}{\partial\theta} - A(\alpha_1^+)\frac{\partial\alpha_0}{\partial\theta}e^{-\rho(\alpha_2,\alpha_1)} \\
&= \mathbf{1}[x(\alpha_2;\theta) = \theta] - \mathbf{1}[x(\alpha_1;\theta) = \theta]\,e^{-\rho(\alpha_2,\alpha_1)} \\
&\vdots \\
&A(\alpha_{I_n}^-)\frac{\partial\alpha_{I_n}}{\partial\theta} - A(\alpha_{I_n-1}^+)\frac{\partial\alpha_{I_n-1}}{\partial\theta}e^{-\rho(\alpha_{I_n},\alpha_{I_n-1})} \\
&= \mathbf{1}[x(\alpha_{I_n};\theta) = \theta] - \mathbf{1}[x(\alpha_{I_n-1};\theta) = \theta]\,e^{-\rho(\alpha_{I_n},\alpha_{I_n-1})}
\end{aligned}$$

We notice that for $0 < i < I_n$, $\mathbf{1}[x(\alpha_i;\theta) = \theta] = 0$, otherwise the NBP would end before time $\alpha_{I_n}$. Moreover, by Assumption 4, no exogenous event occurs at time $\alpha_i$ for $i > 0$, therefore $A(\alpha_i^-) = A(\alpha_i^+) = A(\alpha_i)$ for $i > 0$. In view of these facts, we get

$$\begin{aligned}
&A(\alpha_1)\frac{\partial\alpha_1}{\partial\theta} - A(\alpha_0^+)\frac{\partial\alpha_0}{\partial\theta}e^{-\rho(\alpha_1,\alpha_0)} \\
&= -\mathbf{1}[x(\alpha_0;\theta) = \theta]\,e^{-\rho(\alpha_1,\alpha_0)} \\
&A(\alpha_2)\frac{\partial\alpha_2}{\partial\theta} - A(\alpha_1)\frac{\partial\alpha_0}{\partial\theta}e^{-\rho(\alpha_2,\alpha_1)} = 0 \\
&\vdots \\
&A(\alpha_{I_n-1})\frac{\partial\alpha_{I_n-1}}{\partial\theta} - A(\alpha_{I_n-2})\frac{\partial\alpha_{I_n-2}}{\partial\theta}e^{-\rho(\alpha_{I_n-1},\alpha_{I_n-2})} = 0 \\
&A(\alpha_{I_n})\frac{\partial\alpha_{I_n}}{\partial\theta} - A(\alpha_{I_n-1})\frac{\partial\alpha_{I_n-1}}{\partial\theta}e^{-\rho(\alpha_{I_n},\alpha_{I_n-1})} \\
&= \mathbf{1}[x(\alpha_{I_n};\theta) = \theta]
\end{aligned}$$

Multiplying the $i$th equation above by $e^{\rho(\alpha_i,\alpha_1)}$ and summing up the results we obtain:

$$\begin{aligned}
&A(\alpha_{I_n})\frac{\partial\alpha_{I_n}}{\partial\theta}e^{\rho(\alpha_{I_n},\alpha_1)} - A(\alpha_0^+)\frac{\partial\alpha_0}{\partial\theta}e^{-\rho(\alpha_1,\alpha_0)} \\
&= \mathbf{1}[x(\alpha_{I_n};\theta) = \theta]\,e^{\rho(\alpha_{I_n},\alpha_1)} - \mathbf{1}[x(\alpha_0;\theta) = \theta]\,e^{-\rho(\alpha_1,\alpha_0)}
\end{aligned}$$

Recall that $\alpha_0 = \eta_n$, $\alpha_{I_n} = \zeta_n$ and $\frac{\partial\eta_n}{\partial\theta} = 0$ (Lemmas 3, 4). Observing, in addition, that $\rho(\alpha_{I_n}, \alpha_1) + \rho(\alpha_1, \alpha_0) = \rho(\zeta_n, \eta_n)$ we obtain:

$$A(\zeta_n)\frac{\partial\zeta_n}{\partial\theta} = \mathbf{1}[x(\zeta_n;\theta) = \theta] - \mathbf{1}[x(\eta_n;\theta) = \theta]\,e^{-\rho(\zeta_n,\eta_n)}$$

which gives (21). ■

*Proof:* [**Lemma** 7] Similar to (22), for the perturbed sample path with $\Delta\theta > 0$, we have

$$\begin{aligned}
&L_T(\theta + \Delta\theta) \\
&= \frac{1}{T}\int_0^T \mathbf{1}[x(t;\theta+\Delta\theta) = \theta + \Delta\theta][\lambda(t) - p(\theta+\Delta\theta)]dt
\end{aligned}$$

Setting $\Delta L_T = L_T(\theta + \Delta\theta) - L_T(\theta)$, we get

$$\begin{aligned}
\Delta L_T = \frac{1}{T}\int_0^T &\{\mathbf{1}[x(t;\theta+\Delta\theta) = \theta + \Delta\theta][\lambda(t) - p(\theta+\Delta\theta)] \\
&- \mathbf{1}[x(t;\theta) = \theta][\lambda(t) - p(\theta)]\}\,dt
\end{aligned}$$

Considering the possible values of $x(t;\theta)$ and $x(t;\theta + \Delta\theta)$ at any time $t$, there are four cases:

1. $x(t;\theta) = \theta$, $x(t;\theta+\Delta\theta) = \theta+\Delta\theta$: In this case, $\mathbf{1}\left[x(t;\theta+\Delta\theta) = \theta+\Delta\theta\right] = \mathbf{1}\left[x(t;\theta) = \theta\right] = 1$.

2. $x(t;\theta) = \theta$, $x(t;\theta+\Delta\theta) < \theta+\Delta\theta$: In this case, $\mathbf{1}\left[x(t;\theta+\Delta\theta) = \theta+\Delta\theta\right] = 0$, $\mathbf{1}\left[x(t;\theta) = \theta\right] = 1$.

3. $x(t;\theta) < \theta$, $x(t;\theta+\Delta\theta) = \theta+\Delta\theta$: This case is infeasible because it implies that $\Delta x(t) > \Delta\theta$ which violates Lemma 1.

4. $x(t;\theta) < \theta$, $x(t;\theta+\Delta\theta) < \theta+\Delta\theta$: In this case, $\mathbf{1}\left[x(t;\theta+\Delta\theta) = \theta+\Delta\theta\right] = \mathbf{1}\left[x(t;\theta) = \theta\right] = 0$.

Thus, the expression for $\Delta L_T$ above is non-zero only under cases 1 and 2, corresponding to $x(t;\theta) = \theta$. Let us decompose $[0,T]$ into intervals corresponding to the feasible cases above and denote these intervals by $V_i$, $i = 1, 2, \ldots$, and their lengths by $|V_i|$. Moreover, let

$$\Psi_1 = \{i : \ x(t;\theta) = \theta, \ x(t;\theta+\Delta\theta) = \theta+\Delta\theta \text{ for all } t \in V_i\}$$
$$\Psi_2 = \{i : \ x(t;\theta) = \theta, \ x(t;\theta+\Delta\theta) < \theta+\Delta\theta \text{ for all } t \in V_i\}$$

Thus,

$$\Delta L_T = \frac{1}{T}\left\{\sum_{i\in\Psi_1}\int_{V_i}\left[p(\theta) - p(\theta+\Delta\theta)\right]dt + \sum_{i\in\Psi_2}\int_{V_i}\left[p(\theta) - \lambda(t)\right]dt\right\} \tag{52}$$

We will prove that both terms in the right-hand-side bracket of the above equation are bounded. For the first term, we have

$$\begin{aligned}&\left|\sum_{i\in\Psi_1}\int_{V_i}\left[p(\theta) - p(\theta+\Delta\theta)\right]dt\right| \\ &= |p(\theta) - p(\theta+\Delta\theta)|\sum_{i\in\Psi_1}|V_i| \\ &\le C_p\Delta\theta\sum_{i\in\Psi_1}|V_i| \le C_pT\Delta\theta\end{aligned} \tag{53}$$

where the first inequality is due to Assumption 1 and the Generalized Mean Value Theorem applied to $p(x)$ with $x \in [\theta, \theta+\Delta\theta]$.

For the second term, since $i \in \Psi_2$, we have

$$x(t;\theta) = \theta \text{ and } \lambda(t) - p(\theta) \ge 0 \text{ for all } t \in V_i$$

By Assumption 2, $|\lambda(t) - p(\theta)| \ge c_\theta$, so we get

$$\lambda(t) - p(\theta) \ge c_\theta \tag{54}$$

Since $\theta$ and $\theta+\Delta\theta$ are in the same valid interval in $\tilde{\Theta}$, we can also show that

$$\lambda(t) - p(\theta+\Delta\theta) \ge 0$$

To establish this inequality, suppose that $\lambda(t) - p(\theta+\Delta\theta) < 0$. Then, from the continuity of $p(\cdot)$, there exist $x^*$ and $t^*$ such that

$$\lambda(t^*) - p(x^*) = 0, \qquad \theta \le x^* \le \theta+\Delta\theta, \qquad t^* \in V_i \tag{55}$$

This implies an invalid interval between $\theta$ and $\theta+\Delta\theta$ which contradicts our assumption that $\theta$ and $\theta+\Delta\theta$ are in the same valid interval. Thus, in view of this inequality and Assumption 2 which requires that $|\lambda(t) - p(\theta+\Delta\theta)| \ge c_\theta$, we get

$$\lambda(t) - p(\theta+\Delta\theta) \ge c_\theta \tag{56}$$

Since $p(x)$ is a monotonically increasing function, combining (54) and (56) we obtain:

$$\lambda(t) - p(x) \ge c_\theta \quad \text{for all } x \in [\theta, \theta+\Delta\theta] \tag{57}$$

Recall that for $i \in \Psi_2$, we have $x(t;\theta) = \theta$ and $x(t;\theta+\Delta\theta) < \theta+\Delta\theta$. In addition, by Lemma 1, we have $\Delta x \ge 0$, so that $x(t;\theta+\Delta\theta) \ge \theta$. Thus, for all $t \in V_i$, $\theta \le x(t;\theta+\Delta\theta) < \theta+\Delta\theta$, and it follows from (1) and (57) that

$$\frac{\partial x(t;\theta+\Delta\theta)}{\partial t} = \lambda(t) - p(x(t;\theta+\Delta\theta)) \ge c_\theta, \tag{58}$$

Since $\theta \le x(t;\theta+\Delta\theta) < \theta+\Delta\theta$, the starting point of such an interval $V_i$ is either $(i)$ the start of a FP in the nominal path, or $(ii)$ the end of a FP in the perturbed path. However, due to (58), the latter case is not possible, since starting with the end of a FP requires $\frac{\partial x(t;\theta+\Delta\theta)}{\partial t^+} < 0$. Thus, for $i \in \Psi_2$, the starting point of $V_i$ is the start of a FP in the nominal path at some time $\tau_{i,0}$.

Regarding the end of $V_i$, it can occur if either $(i)$ the end of a FP occurs in the nominal path, i.e., $x(t;\theta)$ ceases to be at $\theta$ at some time $\tau_{i,1}$, or $(ii)$ the start of a FP occurs in the perturbed path, i.e., $x(t;\theta+\Delta\theta)$ reaches $\theta+\Delta\theta$ at some time $\tau_{i,2}$.

Combining the above observations, we have

$$|V_i| = \min(\tau_{i,1}, \tau_{i,2}) - \tau_{i,0} \le \tau_{i,2} - \tau_{i,0}$$

where $\tau_{i,2} - \tau_{i,0}$ is upper-bounded by the time needed for the perturbed sample path to evolve from $\theta$ to $\theta+\Delta\theta$ under (58), i.e., $\tau_{i,2} - \tau_{i,0} \le \Delta\theta/(\frac{\partial x(t;\theta+\Delta\theta)}{\partial t}) \le \Delta\theta/c_\theta$. Thus,

$$|V_i| \le \frac{\Delta\theta}{c_\theta}$$

Then, returning to the second term of (52), we have

$$\begin{aligned}&\left|\sum_{i\in\Psi_2}\int_{V_i}\left[p(\theta) - \lambda(t)\right]dt\right| \\ &\le \left|\sum_{i\in\Psi_2}\int_{V_i}\left|\max_{j=1,\ldots,L}\left[\lambda_j - p(\theta)\right]\right|dt\right| \\ &\le \left|\max_{j=1,\ldots,L}\left[\lambda_j - p(\theta)\right]\right|\left|\sum_{i\in\Psi_2}\int_{V_i}dt\right| \\ &\le \left|\max_{j=1,\ldots,L}\left[\lambda_j - p(\theta)\right]\right|\sum_{i\in\Psi_2}|V_i| \\ &\le \left|\max_{j=1,\ldots,L}\left[\lambda_j - p(\theta)\right]\right|\frac{R\Delta\theta}{c_\theta}\end{aligned} \tag{59}$$

where $R \equiv |\Psi_2|$ is the number of intervals in $[0,T]$ that belong to the set $\Psi_2$. As mentioned earlier, the end of such a $V_i$ interval corresponds to either the end of a FP in the nominal sample path or the start of a FP in the perturbed sample path, i.e.,

$$R \le F(\theta) + F(\theta+\Delta\theta)$$

where $F(\theta)$, $F(\theta+\Delta\theta)$ are the numbers of FPs in the nominal and the perturbed sample paths respectively. Moreover, $F(\theta)$ is bounded by $W(\theta)$, the number of switches of $\lambda(t)$ in $[0,T]$;

similarly, $F(\theta + \Delta\theta)$ is bounded by $W(\theta + \Delta\theta)$. Recalling Assumption 5, we have

$$E[R] \leq E\left[W(\theta) + W(\theta + \Delta\theta)\right] < \infty$$

Combining (53) and (59) we finally obtain:

$$|\Delta L_T| \leq \frac{1}{T}\left\{C_p T + \left|\max_{j=1,\ldots,L}\left[\lambda_j - p(\theta)\right]\right| \frac{R}{c_\theta}\right\}\Delta\theta$$

and by setting

$$r{=}C_p + \left|\max_{j=1,\ldots,L}\left[\lambda_j - p(\theta)\right]\right| \frac{R}{Tc_\theta}$$

the proof is complete. ∎

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