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On the Convergence Rate of Ordinal Optimization for a Class of Stochastic Discrete Resource Allocation Problems

Liyi Dai, Christos G. Cassandras, and Christos G. Panayiotou

***Abstract*—In [1], stochastic discrete resource allocation problems were considered which are hard due to the combinatorial explosion of the feasible allocation search space, as well as the absence of closed-form expressions for the cost function of interest. An ordinal optimization algorithm for solving a class of such problems was then shown to converge in probability to the global optimum. In this paper, we show that this result can be strengthened to almost sure convergence, under some additional mild conditions, and we determine the associated rate of convergence. In the case of regenerative systems, we further show that the algorithm converges exponentially fast.**

***Index Terms*—Convergence, ordinal optimization, resource allocation, stochastic optimization.**

I. Introduction

In a recent paper [1], we presented an algorithm for a class of stochastic discrete resource allocation problems which are hard due to the combinatorial explosion of the feasible allocation search space, as well as the absence of closed-form expressions for the cost function of interest. The algorithm was shown to converge in probability to the global optimum. An important feature of this algorithm is that it is driven by ordinal estimates of a cost function, i.e., simple comparisons of estimates, rather than their cardinal values. One can therefore exploit the fast convergence properties of ordinal comparisons [2]. In this paper, we address explicitly this feature by analyzing the convergence rate properties of the algorithm.

In the resource allocation problems studied in [1], there are r identical *resources* to be allocated over N *user classes* so as to optimize some system performance measure (objective function). Let $\boldsymbol{s} = [n_1, \cdots, n_N]$ denote an allocation and $\mathcal{S} = \{[n_1, \cdots, n_N]\colon \sum_{i=1}^N n_i = r,\, n_i \in \{1, \cdots, r\}\}$ be the finite set of feasible allocations, where n_i is simply the number of resources allocated to user class i. Let $L_i(\boldsymbol{s})$ be the class i cost associated with the allocation vector $\boldsymbol{s}$. Assuming that $L_i(\boldsymbol{s})$ depends only on the number of resources assigned to class i, we can write $L_i(\boldsymbol{s}) = L_i(n_i)$. Then, the basic resource allocation problem is

$$(\mathrm{RA}) \min_{\boldsymbol{s}\in\mathcal{S}} \sum_{i=1}^{N} \beta_i L_i(n_i)$$

where, without loss of generality for the purpose of our analysis, we may set $\beta_i = 1$ for all i.

In a *stochastic* setting, the cost function $L_i(\boldsymbol{s})$ is usually an expectation whose exact value is generally difficult to obtain and one resorts to estimates of $L_i(\boldsymbol{s})$ which may be obtained through simulation or through direct on-line observation of a system. We denote by $\tilde{L}_i^t(\boldsymbol{s})$ an estimate of $L_i(\boldsymbol{s})$ based on observing a sample path for a time period of length t. In [1], an on-line optimization algorithm for solving (RA) in such a setting was developed and shown to converge in probability to the global optimum under certain conditions. The contribution of this paper is twofold: first, we show that under certain additional mild technical conditions the algorithm converges almost surely as well; in addition, for a class of problems with regenerative structure, convergence is shown to be exponentially fast, in the sense that the probability of finding the optimal allocation converges exponentially to 1 if the simulation length increases linearly. This is a highly desirable property from a practical point of view.

In Section II we review the stochastic optimization algorithm of [1] and identify a number of properties based on which we establish almost sure convergence and determine the associated rate of convergence. In Section III, we concentrate on resource allocation applied to regenerative systems and further show that the algorithm converges exponentially fast.

II. Convergence of On-Line Stochastic Optimization Algorithm

The following is the stochastic resource allocation algorithm given in [1], represented as the Markov process $\{(\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k)\}$ where the vector $\tilde{\boldsymbol{s}}_k = [\tilde{n}_{1,k}, \tilde{n}_{2,k}, \cdots, \tilde{n}_{N,k}]$ is the allocation after the kth step, $\tilde{\mathcal{C}}_k$ is a subset of user indices after the kth step updated as shown below, and the "$\tilde{\cdot}$" notation is used to indicate that all quantities involved are based on estimates $\tilde{L}_i^t(\boldsymbol{s})$ of $L_i(\boldsymbol{s})$, obtained from a sample path of length t; in particular, the length of the sample path at the kth iteration of the process is denoted by $f(k)$ and the corresponding cost estimates by $\tilde{L}_i^{f(k)}(\cdot)$. With $\mathcal{C}_0 = \{1, \cdots, N\}$, after proper initialization, we have:

$$\tilde{n}_{i,k+1} = \begin{cases} \tilde{n}_{i,k} - 1 & \text{if } i = \tilde{i}_k^* \text{ and } \tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*) > 0 \\ \tilde{n}_{i,k} + 1 & \text{if } i = \tilde{j}_k^* \text{ and } \tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*) > 0 \\ \tilde{n}_{i,k} & \text{otherwise} \end{cases} \tag{1}$$

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where

$$\tilde{i}_k^* = \arg \max_{i \in \tilde{\mathcal{C}}_k} \{\Delta \tilde{L}_i^{f(k)}(\tilde{n}_{i,k})\} \tag{2}$$

$$\tilde{j}_k^* = \arg \min_{i \in \tilde{\mathcal{C}}_k} \{\Delta \tilde{L}_i^{f(k)}(\tilde{n}_{i,k})\} \tag{3}$$

$$\tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*) = \Delta \tilde{L}_{\tilde{i}_k^*}^{f(k)}(\tilde{n}_{\tilde{i}_k^*,k}) - \Delta \tilde{L}_{\tilde{j}_k^*}^{f(k)}(\tilde{n}_{\tilde{j}_k^*,k}+1) \tag{4}$$

$$\tilde{\mathcal{C}}_{k+1} = \begin{cases} \tilde{\mathcal{C}}_k - \{\tilde{j}_k^*\} & \text{if } \tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*) \le 0 \\ \mathcal{C}_0 & \text{if } |\tilde{\mathcal{C}}_k| = 1 \\ \tilde{\mathcal{C}}_k & \text{otherwise} \end{cases} \tag{5}$$

where $\Delta \tilde{L}_i^{f(k)}(n_i) = \tilde{L}_i^{f(k)}(n_i) - \tilde{L}_i^{t(k)}(n_i - 1)$. It is clear that (1)–(5) define a Markov process $\{(\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k)\}$, whose state transition probability matrix is determined by $\tilde{i}_k^*$, $\tilde{j}_k^*$, and $\tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*)$. For a detailed discussion and interpretation of this process, the reader is referred to [1]. Briefly, $\tilde{i}_k^*$ and $\tilde{j}_k^*$ estimate the users "most sensitive" and "least sensitive" to the removal of a resource among those users in the set $\tilde{\mathcal{C}}_k$. Then, (1) forces the exchange of a resource from the most to the least sensitive user at the kth step of this process, provided the quantity $\tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*)$ is strictly positive; otherwise, the allocation is unaffected, but the user with index $\tilde{j}_k^*$ is removed from the set $\tilde{\mathcal{C}}_k$ through (5). When this set is reduced to a single user, it is reset to $\mathcal{C}_0$ and the procedure repeats. The quantity $\tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*)$ is the estimated "potential cost reduction" when an actual change in allocation takes place.

We will make the following three assumptions, as in [1]:

- *A1:* For all $i = 1, \cdots, N$, $L_i(n_i)$ is such that $\Delta L_i(n_i + 1) > \Delta L_i(n_i)$, where

  $$\Delta L_i(n_i) = L_i(n_i) - L_i(n_i - 1), \qquad n_i = 1, \cdots, r \tag{6}$$

  with $\Delta L_i(0) \equiv -\infty$ and $\Delta L_i(N+1) \equiv \infty$.
- *A2:* For every i, and every n_i, the estimate $\tilde{L}_i^t(n_i)$ is ergodic as the sample path length increases in the sense that

  $$\lim_{t \to \infty} \tilde{L}_i^t(n_i) = L_i(n_i), \qquad \text{a.s.}$$
- *A3:* Let $\delta_k(i, j) = \Delta L_i(\tilde{n}_{i,k}) - \Delta L_j(\tilde{n}_{j,k} + 1)$. For every $\delta_k(\tilde{i}_k^*, \tilde{j}_k^*) = 0$, there is a constant p_0 such that

  $$P[\tilde{\delta}_k(\tilde{i}_k^*, \tilde{j}_k^*) \le 0 | \delta_k(\tilde{i}_k^*, \tilde{j}_k^*) = 0, (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k)] \ge p_0 > 0$$

  for any k and any pair $(\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k)$.

For a detailed discussion of these assumptions, see [1]. Note that *A3* is a technical condition guaranteeing that an estimate does not always give one-sided-biased, incorrect information. Also note that the results in this paper do not require the technical condition that the optimal allocation be unique (*A3* in [1]). If several allocations exhibit optimal performance, the proposed scheme will converge to a *set* of allocations and will oscillate among the members of the set.

Under *A1–A3*, it was proven in [1] that $\tilde{\boldsymbol{s}}_k$ converges in probability to the optimal solution $\boldsymbol{s}^*$ as long as $f(k) \to \infty$ as $k \to \infty$. Our main result in this section is that $\tilde{\boldsymbol{s}}_k$ also converges *almost surely* under some additional mild assumptions. In order to derive this result (Theorem 1), we need to first revisit some properties of the process (1)–(5) which were derived in [1].

A. Properties of Stochastic Resource Allocation Process

We present five inequalities as properties P1)–P5) that were either derived in [1] or are direct consequences of results in [1] which are explicitly referenced in what follows. Let $L(\boldsymbol{s}) = \sum_{i=1}^N L_i(n_i)$. First, define as in [1]:

$$d_k(\boldsymbol{s}, \mathcal{C}) = 1 - P[L(\tilde{\boldsymbol{s}}_{k+1}) \le L(\tilde{\boldsymbol{s}}_k) | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \tag{7}$$

so that $[1 - d_k(\boldsymbol{s}, \mathcal{C})]$ is the probability that either some cost reduction or no change in cost results from the kth transition in our process (1)–(5) (i.e., the new allocation has at most the same cost). It was proven in [1, Lemma 4.2] that the probability of this event is asymptotically 1, i.e., the process (1)–(5) corresponds to an asymptotic descent resource allocation algorithm. The following inequality is (65) established in the proof of Lemma 4.2 in [1]:

P1)

$$d_k(\boldsymbol{s}, \mathcal{C}) \le \sum_{\{(i,j) | \delta_k(i,j) < 0\}} P[\Delta \tilde{L}_i^{f(k)}(\tilde{n}_i) > \Delta \tilde{L}_j^{f(k)}(\tilde{n}_j + 1)].$$

Note that the set $\{(i, j) | \delta_k(i, j) < 0\}$ is finite.

Next, given any state $(\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k)$ reached by the process (1)–(5), define

$$A_k^{\max} = \{j | \Delta L_j(\tilde{n}_{j,k}) = \max_i \{\Delta L_i(\tilde{n}_{i,k})\}\}, \tag{8}$$

$$A_k^{\min} = \{j | \Delta L_j(\tilde{n}_{j,k}) = \min_i \{\Delta L_i(\tilde{n}_{i,k})\}\}. \tag{9}$$

Observe that $A_k^{\max}$ and $A_k^{\min}$ are, respectively, the sets of indices i_k^* and j_k^* defined in (2) and (3) if exact measurements were available (deterministic case). Note that i_k^*, j_k^* *need not be unique* at each step k, hence the need for these sets. We then define

$$a_k(\boldsymbol{s}, \mathcal{C}) = 1 - P[\tilde{i}_k^* \in A_k^{\max} | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})], \tag{10}$$

$$b_k(\boldsymbol{s}, \mathcal{C}) = 1 - P[\tilde{j}_k^* \in A_k^{\min} | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})]. \tag{11}$$

Here, $[1 - a_k(\boldsymbol{s}, \mathcal{C})]$ is the probability that our stochastic resource allocation process at step k correctly identifies an index $\tilde{i}_k^*$ as belonging to the set $A_k^{\max}$ (similarly for $[1 - b_k(\boldsymbol{s}, \mathcal{C})]$). We can then obtain the following inequality:

P2)

$$\begin{aligned} a_k(\boldsymbol{s}, \mathcal{C}) &= P[\tilde{i}_k^* \notin A_k^{\max} | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\le P[\max_{j \notin A_k^{\max}} \{\Delta \tilde{L}_j^{f(k)}(\tilde{n}_{j,k})\} \ge \\ &\qquad \max_{i \in A_k^{\max}} \{\Delta \tilde{L}_i^{f(k)}(\tilde{n}_{i,k})\} | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\le \min_{i \in A_k^{\max}} P[\max_{j \notin A_k^{\max}} \{\Delta \tilde{L}_j^{f(k)}(\tilde{n}_{j,k})\} \ge \\ &\qquad \Delta \tilde{L}_i^{f(k)}(\tilde{n}_{i,k}) | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\le \min_{i \in A_k^{\max}} \left\{ \sum_{j \notin A_k^{\max}} P[\Delta \tilde{L}_j^{f(k)}(\tilde{n}_{j,k}) \ge \right. \\ &\qquad \left. \Delta \tilde{L}_i^{f(k)}(\tilde{n}_{i,k}) | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \right\}. \end{aligned}$$

Note that $\Delta L_j(\tilde{n}_{j,k}) < \Delta L_i(\tilde{n}_{i,k})$ for all $j \notin A_k^{\max}$ and $i \in A_k^{\max}$. Similarly, we get

P3)

$$\begin{aligned} b_k(\boldsymbol{s}, \mathcal{C}) \le \min_{i \in A_k^{\min}} &\left\{ \sum_{j \notin A_k^{\min}} P[\Delta \tilde{L}_j^{f(k)}(\tilde{n}_{j,k}) \le \right. \\ &\left. \Delta \tilde{L}_i^{f(k)} \cdot (\tilde{n}_{i,k}) | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \right\} \end{aligned}$$

and $\Delta L_j(\tilde{n}_{j,k}) > \Delta L_i(\tilde{n}_{i,k})$ for all $j \notin A_k^{\min}$ and $i \in A_k^{\min}$.

Next, we define

$$a_k = \sup_{i \ge k} \max_{(\boldsymbol{s}, \mathcal{C})} a_i(\boldsymbol{s}, \mathcal{C}), \qquad b_k = \sup_{i \ge k} \max_{(\boldsymbol{s}, \mathcal{C})} b_i(\boldsymbol{s}, \mathcal{C}),$$
$$d_k = \sup_{i \ge k} \max_{(\boldsymbol{s}, \mathcal{C})} d_i(\boldsymbol{s}, \mathcal{C}) \tag{12}$$

and we choose a sequence of integers $\{\alpha_k\}$ satisfying

$$\lim_{k\to\infty} \alpha_k = \infty, \qquad \lim_{k\to\infty} \alpha_k(a_k + b_k) = 0, \\ \lim_{k\to\infty} (1 - d_{\lfloor k/2 \rfloor})^{\alpha_k} = 1 \tag{13}$$

where, for any x, $\lfloor x \rfloor$ is the greatest integer smaller than x. With this sequence of $\{\alpha_k\}$, we define as in [1]

$$e_k(\boldsymbol{s}, \mathcal{C}) = 1 - P[L(\tilde{\boldsymbol{s}}_{k+\alpha_k}) < L(\tilde{\boldsymbol{s}}_k) | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \tag{14}$$

and observe that $[1 - e_k(\boldsymbol{s}, \mathcal{C})]$ is the probability that strict improvement (i.e., strictly lower cost) results when transitioning from a state such that the allocation is not optimal to a future state α_k steps later.

For any α_k, consider the event $[L(\tilde{\boldsymbol{s}}_{i+1}) \le L(\tilde{\boldsymbol{s}}_i),\, i = k, \cdots, k + \alpha_k - 1]$ and observe that

$$\begin{aligned} &P[L(\tilde{\boldsymbol{s}}_{i+1}) \le L(\tilde{\boldsymbol{s}}_i),\, i = k, \cdots, k + \alpha_k - 1 | \\ &\quad (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\quad = P[\exists\, h,\, k \le h < k + \alpha_k \text{ s.t. } L(\tilde{\boldsymbol{s}}_{h+1}) < L(\tilde{\boldsymbol{s}}_h), \\ &\text{and } L(\tilde{\boldsymbol{s}}_{i+1}) \le L(\tilde{\boldsymbol{s}}_i),\, i = k, \cdots, k + \alpha_k - 1,\, i \ne h | \\ &\quad (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\quad + P[L(\tilde{\boldsymbol{s}}_{i+1}) = L(\tilde{\boldsymbol{s}}_i),\, i = k, \cdots, k + \alpha_k - 1 | \\ &\quad (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\quad \le P[L(\tilde{\boldsymbol{s}}_{k+\alpha_k}) < L(\tilde{\boldsymbol{s}}_k) | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\quad + P[L(\tilde{\boldsymbol{s}}_{i+1}) = L(\tilde{\boldsymbol{s}}_i),\, i = k, \cdots, k + \alpha_k - 1 | \\ &\quad (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})]. \end{aligned} \tag{15}$$

Moreover, it was shown in [1, eq. (81) of Lemma 4.5] that

$$\begin{aligned} &P[L(\tilde{\boldsymbol{s}}_{i+1}) \le L(\tilde{\boldsymbol{s}}_i),\, i = k, \cdots, k + \alpha_k - 1 | \\ &\quad (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \ge (1 - d_k)^{\alpha_k}. \end{aligned} \tag{16}$$

In addition, it follows from [1, eqs. (67), (72), (74), and (77) of Lemma 4.4] that

$$\begin{aligned} &P[L(\tilde{\boldsymbol{s}}_{h+1}) = L(\tilde{\boldsymbol{s}}_h),\, h = k, \cdots, k + \alpha_k - 1 | \\ &\quad (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\quad \le p_0^{-1}(1 - p_0)^{\alpha_k - (|\mathcal{C}| + N)} + (\alpha_k - 1)(a_k + b_k) \\ &\quad + \sum_{M=k}^{k+\alpha_k-1} P[\tilde{\delta}_M(\tilde{i}_M^*, \tilde{j}_M^*) \le 0, \\ &\quad \delta_M(\tilde{i}_M^*, \tilde{j}_M^*) > 0 | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})] \\ &\quad \le p_0^{-1}(1 - p_0)^{\alpha_k - (|\mathcal{C}| + N)} + (\alpha_k - 1)(a_k + b_k) \\ &\quad + \sum_{M=k}^{k+\alpha_k-1} \sum_{\{(i,j) \in \tilde{\mathcal{C}}_M,\, \delta_M(i,j) > 0\}} \\ &\quad P[\tilde{\delta}_M(i, j) \le 0 | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})]. \end{aligned} \tag{17}$$

We can now combine (15)–(17) to establish the following inequality for any $(\boldsymbol{s}, \mathcal{C})$:

P4)

$$\begin{aligned} e_k(\boldsymbol{s}, \mathcal{C}) \le{}& [1 - (1 - d_k)^{\alpha_k}] + p_0^{-1}(1 - p_0)^{\alpha_k - (|\mathcal{C}| + N)} \\ &+ (\alpha_k - 1)(a_k + b_k) + \sum_{M=k}^{k+\alpha_k-1} \sum_{\{(i,j) \in \tilde{\mathcal{C}}_M,\, \delta_M(i,j) > 0\}} \\ &P[\tilde{\delta}_M(i, j) \le 0 | (\tilde{\boldsymbol{s}}_k, \tilde{\mathcal{C}}_k) = (\boldsymbol{s}, \mathcal{C})]. \end{aligned}$$

The last property we will use was established in [1, eq. (63) of Theorem 1]:

P5)

$$P[\tilde{\boldsymbol{s}}_k = \boldsymbol{s}^*] \ge (1 - e_{\lfloor k/2 \rfloor})^q [(1 - d_{\lfloor k/2 \rfloor})^{\alpha_k}]^q$$

where q is a finite constant determined by the parameters of the resource allocation problem (RA) and e_k is defined as

$$e_k = \sup_{i \ge k} \max_{\boldsymbol{s} \in \mathcal{S},\, \mathcal{C}} e_i(\boldsymbol{s}, \mathcal{C}). \tag{18}$$

B. Main Convergence Result

As already stated, under assumptions *A1–A3*, it was proven in [1] that $\tilde{\boldsymbol{s}}_k$ converges in probability to the optimal solution $\boldsymbol{s}^*$ as long as $f(k) \to \infty$ as $k \to \infty$. By proper selection of the sample path length $f(k)$, i.e., the kth estimation period, and under some additional mild assumptions we can now show that the process (1)–(5) converges almost surely; we can also determine the associated rate of convergence. For this purpose, we first recall a result of [2].

Lemma 2.1: Suppose that $\{\tilde{x}_t,\, t \ge 0\}$ is a stochastic process satisfying a) $\lim_{t\to\infty} \tilde{x}_t = x$, a.s.; b) $\lim_{t\to\infty} E[\tilde{x}_t] = x$; c) $\text{Var}[\tilde{x}_t] = O(1/t)$. If $x > 0$, then $P[\tilde{x}_t \le 0] = O(1/t)$.

The assumption in Lemma 2.1 is very mild and almost always satisfied in the simulation or direct sample path observation of discrete-event dynamic systems. Lemma 2.1 establishes the rate of convergence for comparing $\tilde{x}_t$ against 0. Using this result, we can prove the following lemma which will be needed for our main result.

Lemma 2.2: Assume that, for every i, the estimate $\tilde{L}_i^t(n_i)$ satisfies the assumptions of Lemma 2.1. Then, for any $i,\, j,\, i \ne j$,

$$P[\Delta\tilde{L}_i^t(n_i) \ge \Delta\tilde{L}_j^t(n_j)] = O\left(\frac{1}{t}\right)$$

and

$$P[\Delta\tilde{L}_i^t(n_i) < \Delta\tilde{L}_j^t(n_j)] = 1 - O\left(\frac{1}{t}\right)$$

provided that $\Delta L_i(n_i) < \Delta L_j(n_j)$.

Proof: Let $\tilde{x}_k = \Delta\tilde{L}_j^t(n_j) - \Delta\tilde{L}_i^t(n_i)$ and $x = \Delta L_j(n_j) - \Delta L_i(n_i)$. Then, $\{\tilde{x}_k\}$ satisfies the assumptions of Lemma 2.1 and $x > 0$. Thus, the conclusion holds.

Theorem 1: Suppose A1–A3 and the assumptions of Lemma 2.2 hold. If $f(k) \ge k^{1+c}$ for some constant $c > 0$, then the process described by (1)–(5) converges almost surely to the global optimal allocation.

Proof: If $f(k) \ge k^{1+c}$ for some $c > 0$ and if the assumptions of Lemma 2.2 are satisfied, we know from Lemma 2.2, the definition in (12) and property (P1) in Section II

$$d_k = O\left(\frac{1}{f(k)}\right) = O\left(\frac{1}{k^{1+c}}\right).$$

Furthermore, since the space of $(\boldsymbol{s}, \mathcal{C})$ is finite, Lemma 2.2, the definition in (12), and properties P2)–P3) imply that

$$a_k = O\left(\frac{1}{f(k)}\right) = O\left(\frac{1}{k^{1+c}}\right), \\ b_k = O\left(\frac{1}{f(k)}\right) = O\left(\frac{1}{k^{1+c}}\right).$$

Next, choose

$$\alpha_k = \frac{1 + c}{-\ln(1 - p_0)} \ln(k), \qquad k = 1, 2, \cdots$$

and observe that $\{\alpha_k\}$ above satisfies (13) and that $(1 - p_0)^{\alpha_k} = (1/k^{1+c})$. Then, property P4) gives

$$\begin{aligned} e_k &= O(1 - (1 - d_k)^{\alpha_k}) + O((1 - p_0)^{\alpha_k}) \\ &\quad + O((\alpha_k - 1)(a_k + b_k)) + O\left(\frac{1}{f(k)}\right) \\ &= O\left(\frac{\ln(k)}{k^{1+c}}\right) + O((1 - p_0)^{\alpha_k}) = O\left(\frac{\ln(k)}{k^{1+c}}\right). \end{aligned}$$

Finally, from property P5) we get

$$P[\tilde{\boldsymbol{s}}_k = \boldsymbol{s}^*] = 1 - O(1 - (1 - e_{\lfloor k/2 \rfloor})^q (1 - d_{\lfloor k/2 \rfloor})^{q\alpha_k}) = 1 - O(e_{\lfloor k/2 \rfloor} + \alpha_k d_{\lfloor k/2 \rfloor}) = 1 - O\left(\frac{\ln(k)}{k^{1+c}}\right).$$

Since $\tilde{\boldsymbol{s}}_k$ can take only a finite set of values, the previous equation can be rewritten as

$$P[|\tilde{\boldsymbol{s}}_k - \boldsymbol{s}^*| \geq \epsilon] = O\left(\frac{\ln(k)}{k^{1+c}}\right) \tag{19}$$

for any sufficiently small $\epsilon > 0$. Since $\sum_k (\ln(k)/k^{1+c}) < \infty$, we know from the Borel–Cantelli Lemma [5, pp. 255–256] that $\{\tilde{\boldsymbol{s}}_k\}$ converges almost surely to the optimum allocation $\boldsymbol{s}^*$.

The inequality P5) establishes the convergence in probability of the process (1)–(5), while Theorem 1 proves the almost sure convergence. For any sample path length $f(k)$, the proof of Theorem 1 shows that the rate of convergence in probability is

$$P[\tilde{\boldsymbol{s}}_k = \boldsymbol{s}^*] = 1 - O\left(\frac{\ln(k)}{f(k)}\right)$$

provided that $\alpha_k = O(\ln(f(k)))$.

III. Convergence of Algorithm for Regenerative Systems

In many cases, knowledge of the structure of a specific discrete-event dynamic system allows us to gain further insight on the convergence of the process (1)–(5). To be precise, let $\{X_t(\theta) \in R, t \geq 0\}$ be a parameterized stochastic process that describes the discrete-event dynamic system that we are interested in. We use θ to indicate different "designs" (e.g., parameter settings). In many discrete-event dynamic systems, estimates of performance measures based on the simulation or observation over a time period t can often be constructed in the form

$$\tilde{L}^t(\theta) = \frac{1}{t}\int_0^t \ell(X_u(\theta))\,du \tag{20}$$

where $\ell(.)$: $R \to R$. Although other forms of estimators are possible, the form of (20) is typical and we shall limit ourselves to it (other forms are considered in [2]). We consider next the convergence of (1)–(5) for the important class of *regenerative* systems.

Definition 3.1 [4, p. 19]: A stochastic process $\{X_t(\theta)\}$ is said to be regenerative (in a classic sense) if there is an increasing sequence of nonnegative finite random times $\{\tau_i(\theta), i \geq 0\}$ such that for each $i \geq 0$

i) $\{X_{\tau_i(\theta)+t}(\theta), \tau_k(\theta) - \tau_i(\theta), t \geq 0, k \geq i\}$ are identically distributed.

ii) $\{X_{\tau_i(\theta)+t}(\theta), \tau_k(\theta) - \tau_i(\theta), t \geq 0, k \geq i\}$ does not depend on $\{X_t(\theta), \tau_j(\theta), t \leq \tau_i, 0 \leq j \leq i\}$.

$\{\tau_i(\theta), i \geq 1\}$ is a sequence of regeneration points and $T_i(\theta) = \tau_i(\theta) - \tau_{i-1}(\theta), i \geq 1,$ is the cycle time of the ith regenerative cycle. Then $\{T_i(\theta), i \geq 1\}$ is a sequence of i.i.d. random variables. We also define $T_0(\theta) = \tau_0(\theta)$.

Let $\{X_t(\theta)\}$ be a regenerative process with cycle times $\{T_i(\theta)\}$. Then for estimators of performance measures of the form (20), Dai in [2] shows the following result on the convergence rate of comparing two performance measures $L(\theta_1)$ and $L(\theta_2)$.

Lemma 3.1: For $\theta = \theta_1$ or θ_2 and for $i = 0, 1$, assume that

a) $\lim_{t\to\infty} \tilde{L}^t(\theta) = L(\theta)$, a.s.

b) The cycle time $T_i(\theta)$ has finite, continuous moment generating function $E[e^{sT_i(\theta)}]$ in a neighborhood of $s = 0$.

c) The cycle time $T_1(\theta)$ is not degenerate in the sense that $\lim_{t\to\infty} P[T_1(\theta) \leq t] < 1$.

d) The function $\ell(.)$ is bounded and $|\ell(.)| \leq B, 0 < B < \infty$.

If $L(\theta_1) > L(\theta_2)$, then there exists a constant $\alpha > 0$ such that

$$P[\tilde{L}^t(\theta_1) > \tilde{L}^t(\theta_2)] = 1 - O(e^{-\alpha t})$$

and

$$P[\tilde{L}^t(\theta_1) \leq \tilde{L}^t(\theta_2)] = O(e^{-\alpha t}), \tag{21}$$

in other words, the rate of convergence for comparing the two sample estimators $\tilde{L}^t(\theta_1)$ and $\tilde{L}^t(\theta_2)$ is exponential.

We can now prove the following theorem regarding the almost sure convergence of the process (1)–(5) and the associated rate of convergence.

Theorem 2: Suppose that assumptions A1–A3 hold and for every pair $i, j, i \neq j, \Delta\tilde{L}_i^t(n_i) - \Delta\tilde{L}_j^t(n_j)$ is a regenerative process with regenerative cycle times satisfying b) and c) of Lemma 3.1. Then, there exists a $c > 0$ large enough such that if $f(k) \geq c\ln(k)$ for sufficiently large k, the process described by (1)–(5) converges almost surely to the global optimal allocation. Furthermore,

$$P[\tilde{\boldsymbol{s}}_k = \boldsymbol{s}^*] = 1 - O(\ln(k)e^{-\alpha f(k)}) \tag{22}$$

for some $\alpha > 0$.

Proof: Assumption A2 guarantees that $\Delta\tilde{L}_i^t(n_i) - \Delta\tilde{L}_j^t(n_j)$ satisfies a) of Lemma 3.1. Since there are only a finite number of feasible allocations, d) of Lemma 3.1 automatically holds. Under the assumptions of the theorem, Lemma 3.1 applies. Thus, there exists an $\alpha > 0$ such that

$$a_k = O(e^{-\alpha f(k)}), \quad b_k = O(e^{-\alpha f(k)}), \quad d_k = O(e^{-\alpha f(k)}).$$

We know from the proof of Theorem 1 that

$$P[\tilde{\boldsymbol{s}}_k = \boldsymbol{s}^*] = 1 - O(\alpha_k e^{-\alpha f(k)}) + O((1-p_0)^{\alpha_k}).$$

Choose c such that $c\alpha > 1$ and set

$$\alpha_k = \frac{c\alpha}{-\ln(1-p_0)}\ln(k), \qquad k = 1, 2, \cdots.$$

Then $e^{\alpha f(k)} \geq k^{1+c'}$, $c' = c\alpha - 1 > 0$. Consequently, we know from the proof of Theorem 1 that $\{\tilde{\boldsymbol{s}}_k\}$ converges almost surely to the optimum, and (22) holds.

Finally, note that if we set $f(k) = k$, then (22) becomes $P[\tilde{\boldsymbol{s}}_k = \boldsymbol{s}^*] = 1 - O(\ln(k)e^{-\alpha k})$, which converges exponentially.

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