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Perturbation Analysis of Multiclass Stochastic Fluid Models

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Abstract. We use a stochastic fluid model (SFM) to capture the operation of finite-capacity queueing systems with multiple customer classes. We derive gradient estimators for class-dependent loss and workload related performance metrics with respect to any one of several threshold parameters used for buffer control. These estimators are shown to be unbiased and directly observable from a sample path without any knowledge of underlying stochastic characteristics of the traffic processes. This renders them computable in on-line environments and easily implementable in settings such as communication networks.

Keywords: perturbation analysis, stochastic fluid model, unbiased estimator, quality of service

1. Introduction

In this paper, we use the framework of stochastic fluid models (SFM), together with perturbation analysis (PA) techniques, in order to develop gradient estimators of various performance metrics for queueing system with multiple customer classes. Such estimators can be used for on-line control and performance optimization of communication networks and manufacturing systems, as described in some of our previous work (Cassandras et al., 2002; Yu and Cassandras, 2002). In particular, we are interested in using threshold-based controllers and have applied infinitesimal perturbation analysis (IPA) to obtain sensitivity estimators for loss and workload metrics with respect to controllable threshold parameters. In Cassandras et al. (2002), a single-node single-class SFM was analyzed. In Cassandras et al. (2003), we considered a node with infinite buffer capacity which processes two traffic streams: one traffic stream is uncontrolled and the other is subject to threshold-based buffer control. In Sun et al. (2002), we provided some results for a similar two-class model with finite buffer capacity. In this paper, we analyze a finite-capacity model with $M \geq 2$ traffic classes and associated thresholds.

SFMs provide an alternative to queueing models when the latter become impractical due to huge traffic volumes and complex stochastic processes that cannot be handled by tractable analytical derivations. The SFM paradigm allows the aggregation of multiple events, associated with the movement of individual customers (e.g., packets or parts), into a single event associated with a rate change. It foregoes the identity and dynamics of individual customers and focuses instead on the aggregate flow rate. SFMs have recently been shown to be especially useful for analyzing various kinds of high-speed networks (Cassandras et al., 2002, 2003; Wardi and Melamed, 2002; Kesidis et al., 1996; Kumaran and Mitra, 1998; Miyoshi, 1998; Liu et al., 1999; Yan and Gong, 1999). As argued in Cassandras et al. (2002, 2003), such models may not always be accurate for the purpose of performance analysis, but they capture the salient features of the underlying ''real'' system in a way which is often sufficient to solve control and optimization problems. In this case, estimating the gradient of a given cost function with respect to key parameters becomes an essential task. PA methods (Ho and Cao, 1991; Cassandras and Lafortune, 1999) are therefore suitable, if appropriately adapted to a SFM viewed as a discrete-event system (Cassandras et al., 2002, 2003; Wardi et al., 2002; Liu and Gong, 2002). From a technical standpoint, performance metrics associated with a SFM can often be shown to be Lipschitz continuous with respect to controllable parameters of interest. This generally results in unbiased IPA estimators associated with a SFM, whereas the same is usually not true for the corresponding discrete event system. For example, IPA estimators have been known to be biased when applied to queueing models with customer blocking or with multiple customer classes (Cao, 1987). In addition, the estimators are generally simple to implement and can be evaluated based on data observed on a single sample path of the actual (discrete-event) system. Thus, we use the SFM to derive the form of a gradient estimator, but can then implement it using actual system data. Even though using SFM-based estimators on an actual system may introduce a bias, simulation results indicate that gradient estimators obtained in this fashion work well with gradient-based optimization algorithms.

In a single-class SFM of a communication network node with threshold-based buffer control, IPA was shown to yield simple nonparametric sensitivity estimators for packet loss and workload metrics with respect to the threshold parameter (Cassandras et al., 2002). In the infinite-capacity two-class case studied in Cassandras et al. (2003), the estimators generally depend on some traffic rate information, but not on the stochastic characteristics of the arrival and service processes involved. In this paper, we consider a finite-capacity SFM with multiple ($M \geq 2$) classes viewed as different traffic streams (see Figure 1). Each stream is associated with a threshold and is subject to threshold-based control. The finite buffer capacity and the presence of $M > 2$ classes cannot be handled as straightforward generalizations of Cassandras et al. (2002) and our analysis leads to more complex estimator forms, as we shall see, although the basic theoretical setting for IPA is similar to previous work such as Cassandras et al. (2002) and Yu and Cassandras (2002). Thus, the main contributions of this paper are (i) the derivation of IPA gradient estimators for multiple class-dependent loss metrics and a workload metric with respect to the threshold parameters in the model and (ii) proving that these estimators are unbiased. In the general case where $M > 2$, the IPA estimators we derive are obtained through recursive equations. In the case where $M = 2$, we are further able to exploit the structure of the sample paths of the SFM and obtain explicit closed-form expressions for the IPA estimators that involve readily observable sample path data.

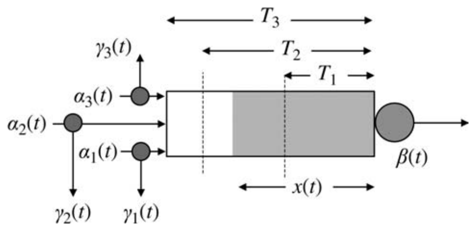

Figure 1. Stochastic fluid model (SFM) with $M = 3$ classes.

The paper is organized as follows. First, in Section 2, we present our model and define performance metrics and parameters of interest. We also present a partition of a sample path into time intervals defined by specific events that facilitate the PA task. In Section 3, we derive IPA estimators for the sensitivities of the traffic loss volumes for all classes and for the system workload with respect to any one controllable threshold parameter. In Section 4, we analyze the simpler two-class model exploiting the structure of its sample paths and derive closed-form (as opposed to recursively obtained) IPA estimators. In Section 5, we establish the unbiasedness of the derived estimators and conclude with Section 6.

2. SFM of a Multiclass Queueing System

We consider a single node with a common FIFO buffer fed by M sources. Each source defines a ''class'' $m, m = 1, \ldots, M$, and is associated with a threshold T_m, where we assume $0 < T_1 < \cdots < T_m < \cdots < T_M$. By convention, we set $T_0 \equiv 0$. If the buffer content is above the value T_m, all traffic from class m is rejected (which implies that all traffic from classes labeled $i \leq m$ are also rejected). In the context of communication networks, it is worth noting that this model captures the operation of the differentiated services (DS) protocol that has been proposed for supporting quality of service (QoS) requirements (Blake et al., 1998; Heinanen et al., 1999; Panayiotou and Cassandras, 2001). Class M corresponds to the top-priority traffic and if all such traffic is to be accepted, then T_M is simply the physical buffer capacity of the node.

The inflow rate of class m at time t is denoted by $\alpha_m(t)$ and the corresponding loss rate by $\gamma_m(t)$. The service rate is denoted by $\beta(t)$ and let $x(t)$ be the buffer level at time t. For the purpose of our analysis, we choose any one of the thresholds, say $T_{\overline{m}}$, as the one with respect to which we wish to carry out sensitivity analysis and denote this parameter by θ. We then write the m-th class loss rate as $\gamma_m(\theta; t)$ and the buffer content as $x(\theta; t)$ to express their dependence on the choice of θ. We also assume that the processes $\{\alpha_m(t)\}, m = 1, \ldots, M$, and $\{\beta(t)\}$ are independent of θ and cannot take negative values; other than that, these processes are only assumed to be bounded and right-continuous piecewise continuously differentiable w.p.1. The time variable t is confined to an interval $[0, T]$ for a given fixed $0 < T < \infty$. It will also be convenient in our analysis to make use of the following definition, for any $m = 1, \ldots, M$:

$$A_m(t) \equiv \sum_{n=m}^{M} \alpha_n(t) - \beta(t) \tag{1}$$

and observe that

$$A_m(t) \geq A_{m+1}(t), \qquad m = 1, \ldots, M-1 \tag{2}$$

Figure 1 depicts the SFM described above with $M = 3$ and T_3 set to the physical capacity of the buffer.

We assume that the parameter θ is confined to a bounded (compact) interval $\Theta = (T_{\overline{m}-1}, T_{\overline{m}+1})$ and let $\mathscr{L}(\theta) : \Theta \to \mathbb{R}$ be any random function defined over the underlying probability space $(\Omega, \mathscr{F}, P)$. Strictly speaking, we write $\mathscr{L}(\theta, \omega)$ to indicate that this sample function depends on the sample point $\omega \in \Omega$, but will suppress ω unless it is necessary to stress this fact. In what follows, we consider two performance metrics, the m-th class loss volume $L_{m,T}(\theta), m = 1, \ldots, M$, and the cumulative workload (or just work) $Q_T(\theta)$, defined on the interval $[0, T]$ as follows:

$$L_{m,T}(\theta) = \int_0^T \gamma_m(\theta; t)dt, \qquad m = 1, \ldots, M \tag{3}$$

$$Q_T(\theta) = \int_0^T x(\theta; t)dt \tag{4}$$

Note that from the workload metric it is possible to obtain a delay metric using appropriate forms of Little's law (see, for example, Wardi and Melamed, 2001).

Viewed as a discrete-event system, an event in a sample path of the above SFM may be either exogenous or endogenous. An exogenous event is one that causes a change in the dynamics of $x(\theta; t)$ due to changes in one or more of the processes $\{\alpha_m(t)\},\ m = 1, \ldots, M$, and $\{\beta(t)\}$, which are independent of θ. In particular, we will be interested in changes in the sign of $A_m(t)$ in (1) for some $m = 1, \ldots, M$ and define an associated exogenous event e_1 as one where the buffer content leaves the value $x(\theta; t) = T_m$, for some $m = 0, \ldots, M$, after it has maintained it for some finite length of time. An endogenous event on the other hand, denoted by e_2, is defined to occur whenever the buffer content reaches the value $x(\theta; t) = T_m$, for any $m = 0, \ldots, M$. Note that $x(t; \theta) = T_m$ for all t in some interval $[s_1, s_2]$ implies that $A_m(t) > 0$ and $A_{m+1} < 0$. In other words, the arriving flow rate from the $M - m - 1$ highest priority classes is less than the capacity $\beta(t)$; thus, a portion of the $\alpha_m(t)$ flow is admitted, while there is some excess which is rejected. As a result, we observe the period with constant $x(t; \theta) = T_m$. At $t = s_2$, if the sign of either $A_m(t)$ or $A_{m+1}(t)$ changes, then this interval will end with an e_1 event.

For the purpose of our analysis, we partition the sample path into cycles defined by successive exogenous events e_1 (the term ''cycle'', however, should not be interpreted as implying any sort of regenerative property). Suppose that a sample path includes K such cycles, where K is a random number which is locally independent of θ. Denote the

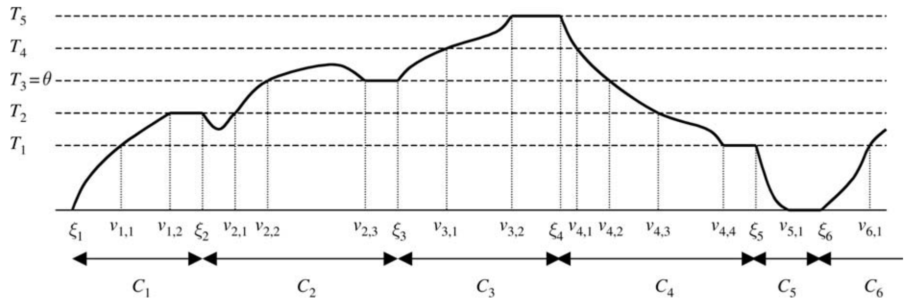

Figure 2. Typical sample path segment for a SFM with $M = 5$ classes.

corresponding exogenous event times by $\xi_k, k = 1, \ldots, K$ and let $\xi_{K+1} = T$, so we can express cycles as intervals $\mathscr{C}_k = [\xi_k, \xi_{k+1}), k = 1, \ldots, K$. We further denote all endogenous event (e_2) times within $\mathscr{C}_k$ by $v_{k,i}, i = 1, \ldots, S_k$. By convention, let $v_{k,0} = \xi_k$ and $v_{k,S_k+1} = \xi_{k+1}$, so $\mathscr{C}_k$ is divided into periods

$$p_{k,i} \equiv [v_{k,i}, v_{k,i+1}), \qquad i = 0, \ldots, S_k$$

and the corresponding open interval $(v_{k,i}, v_{k,i+1})$ is denoted by $p^o_{k,i}$. From our definition of e_1 and e_2, we can immediately see that throughout the last period, p_{k,S_k}, of $\mathscr{C}_k$, $x(\theta; t)$ is ''flat'', that is, $x(\theta; t) = T_m$ for some $m = 0, \ldots, M$; we shall refer to this as a Type I interval. During all other periods $p_{k,i}, \ i = 0, \ldots, S_k - 1, \ x(\theta; t)$ takes values strictly between any two adjacent thresholds; we shall refer to these as Type II intervals. A typical sample path for the case of $M = 5$ is shown in Figure 2.

During Type II intervals $p_{k,i}, i = 0, \ldots, S_k - 1$, if $T_{m-1} < x(\theta; t) < T_m$ for $t \in p^o_{k,i}$, the buffer content dynamics are

$$\frac{dx(t)}{dt^+} = A_m(t) \tag{5}$$

On the other hand, during Type I intervals p_{k,S_k}, the buffer content dynamics are

$$\frac{dx(\theta; t)}{dt^+} = 0 \tag{6}$$

With this discussion in mind, we may also view the SFM as a hybrid dynamic system in which the buffer content satisfies the dynamics (5)–(6) with switches between $M + 1$ ''modes'' caused by the discrete events e_1 and e_2 we have defined above.

3. IPA with Respect to Thresholds

Our objective here is to estimate the derivatives $dE[L_{m,T}(\theta)]/d\theta, \ m = 1, \ldots, M$ and $dE[Q_T(\theta)]/d\theta$, through the sample derivatives $dL_{m,T}(\theta)/d\theta, \ m = 1, \ldots, M$ and

$dQ_T(\theta)/d\theta$ which are commonly referred to as the IPA estimators (Ho and Cao, 1991; Cassandras and Lafortune, 1999). Henceforth, we shall use the ''prime'' notation to denote derivatives with respect to θ, so the sample derivatives above are denoted by $L'_{m,T}(\theta)$ and $Q'_T(\theta)$, respectively. For any sample function $\mathscr{L}(\theta)$, an IPA-based estimate $\mathscr{L}'(\theta)$ of the performance metric derivative $dE[\mathscr{L}(\theta)]/d\theta$ is unbiased if $dE[\mathscr{L}(\theta)]/d\theta = E[\mathscr{L}'(\theta)]$ (see Ho and Cao, 1991; Cassandras and Lafortune, 1999). Unbiasedness is the principal condition for making the application of IPA useful in practice, for reliable sensitivity analysis purposes or for use in conjunction with control and optimization methods that employ stochastic gradient-based techniques. We proceed by first evaluating the sample derivatives $L'_{m,T}(\theta)$ and $Q'_T(\theta)$ in terms of event time derivatives $dv_{k,i}/d\theta$, which we also denote by $v'_{k,i}$, and then concentrate on the evaluation of these event time derivatives based on observable quantities along a given sample path.

Similar to our definition of $A_m(t)$ in (1), let us also define

$$A_{m,k,i} \equiv \sum_{n=m}^{M} \alpha_n(v_{k,i}) - \beta(v_{k,i}) \tag{7}$$

Before proceeding, let us identify conditions under which the sample derivatives involved exist. Recall that any exogenous event time is locally independent of θ, whereas any endogenous event time is generally a function of θ. Excluding the possibility of the simultaneous occurrence of exogenous and endogenous events, the only situation preventing the existence of sample derivatives involves some t such that $x(\theta; t) = T_{\overline{m}} = \theta$ and $A_{\overline{m}}(t) = 0$ or $A_{\overline{m}+1}(t) = 0$; in such cases, the one-sided derivative still exists and can be obtained through a finite difference analysis (as in Cassandras et al., 2002). However, to keep the analysis simple, we focus only on the differentiable case by proceeding under the following technical conditions:

Assumption 1:

a. $\alpha_m(t) < \infty,\ m = 1, \ldots, M$ and $\beta(t) < \infty$ for all $t \in [0, T]$.

b. For every $\theta \in \Theta$, w.p. 1, exogenous and endogenous events may not occur at the same time.

c. W.p. 1, there exists no interval $(v_{k,i}, v_{k,i} + \tau),\ \tau > 0$, such that $x(\theta; t) = T_m$ for all $t \in (v_{k,i}, v_{k,i} + \tau)$, and either $A_m(t) = 0$ or $A_{m+1}(t) = 0$.

All three parts of Assumption 1 are mild technical conditions to ensure the existence of sample derivatives. Regarding part c, as already pointed out, one-sided derivatives may still be used if a sample path happens to contain a period with constant $x(t; \theta) = T_m$ for some $m = 1, \ldots, M$ in which $A_m(t) = 0$ or $A_{m+1}(t) = 0$. Specifically, suppose we change θ by some $\Delta\theta > 0$. Then, if $A_m(t) = 0$, the change in $x(t), \Delta x(t) = x(t; \theta + \Delta\theta) - x(t; \theta) = 0$. On the other hand, if $\Delta\theta < 0$, then $\Delta x(t) < 0$. As a result, the left and right derivatives with respect to θ are different.

3.1. Class m Loss IPA Derivatives

Using the sample path partition into cycles, we may write (3) as follows:

$$L_{m,T}(\theta) = \sum_{k=1}^{K} \int_{\xi_k}^{\xi_{k+1}} \gamma_m(\theta; t)dt \tag{8}$$

Let us now define

$$\lambda_{m,k}(\theta) = \int_{\xi_k}^{\xi_{k+1}} \gamma_m(\theta; t)dt \tag{9}$$

so that we can write

$$L'_{m,T}(\theta) = \sum_{k=1}^{K} \lambda'_{m,k}(\theta) \tag{10}$$

and our objective is to evaluate $\lambda'_{m,k}(\theta)$ for any $k = 1, \ldots, K$. For simplicity, we will drop the index k as we focus on a specific cycle $\mathscr{C}_k$ in the following.

For the purpose of our analysis, a useful way of grouping periods p_i within a typical cycle is by defining sets associated with each class $m = 1, \ldots, M$ as follows:

1. *Partial Loss Period Set* U_m. For any $p_i \in U_m$, the buffer content is $x(\theta; t) = T_m$ for all $t \in p_i$, and class m traffic experiences partial loss. In particular, the traffic flows satisfy

   $$A_m(t) > 0 \quad \text{and} \quad A_{m+1}(t) < 0 \tag{11}$$

   so that the processing capacity $\beta(t)$ can accommodate the cumulative incoming flow $\sum_{n=m+1}^{M} \alpha_n(t)$ due to classes $m+1, \ldots, M$, but not the flow $\sum_{n=m}^{M} \alpha_n(t)$ that includes the next lower priority class m. In this case, the system accepts only the portion of the class m traffic that can be accommodated and incurs a ''partial'' loss rate

   $$\gamma_m(\theta; t) = A_m(t) \tag{12}$$

   The dynamics of $x(\theta; t)$ during this period are given by

   $$\frac{dx(\theta; t)}{dt^+} = 0$$

   Formally, we define U_m as follows:

   $$U_m := \{ p_i : x(\theta; t) = T_m, \ t \in p_i \}. \tag{13}$$

Note that the starting point v_i of such a period corresponds to an endogenous event e_2, whereas the ending point v_{i+1} corresponds to an exogenous event e_1 and is, therefore, locally independent of θ. Also note any one element of U_m is always the last interval of a cycle; conversely, the last interval of a cycle, p_S, provided it is such that $x(\theta;t) \neq 0$ for all $t \in p_S$, must be an element of U_m for some $m = 1, \ldots, M$.

2. *Full Loss Period Set* V_m. During such periods, the buffer content is $x(\theta;t) > T_m$ (possibly excluding the starting point v_i; for instance, in Figure 2 $(v_{3,1}, v_{3,2}) \in V_4$, but $[v_{3,1}, v_{3,2}) \in V_3$). In this case, all class m traffic is lost:

$$\gamma_m(\theta;t) = \alpha_m(t) \tag{14}$$

Formally, we define V_m as follows:

$$V_m := \{p_i : x(\theta;t) > T_m,\ t \in p_i^o\} \tag{15}$$

3. *No Loss Period Set* W_m. During such periods the buffer content is $x(\theta;t) < T_m$ (possibly excluding the starting point v_i) and no class m loss occurs:

$$\gamma_m(\theta;t) = 0 \tag{16}$$

Formally, we define W_m as follows:

$$W_m := \{p_i : x(t) < T_m,\ t \in p_i^o\} \tag{17}$$

Note that each of the sets above is locally independent of θ (by a random function $f(\theta)$ being ‘‘locally independent’’ of θ we mean that for a given θ there exists $\Delta\theta > 0$ such that for every $\bar{\theta} \in (\theta - \Delta\theta, \theta + \Delta\theta)$, w.p.1 $f(\bar{\theta}) = f(\theta)$, where $\Delta\theta$ may depend on both θ and on the sample path, and that for any particular m, $U_m \cup V_m \cup W_m = [0, T)$ with all sets being mutually exclusive). Moreover, recall that p_S is the only possible period which belongs to U_m and $v_{S+1} = \xi_{k+1}$. Thus, (9) can be written as

$$\lambda_m(\theta) = \sum_{j=0}^{S} \mathbf{1}\{p_j \in V_m\} \int_{v_j}^{v_{j+1}} \alpha_m(t)dt + \mathbf{1}\{p_S \in U_m\} \int_{v_S}^{v_{S+1}} A_m(t)dt \tag{18}$$

where $\mathbf{1}\{\cdot\}$ is the usual indicator function. By differentiating with respect to θ, we obtain

$$\lambda'_m(\theta) = \sum_{j=0}^{S} \mathbf{1}\{p_j \in V_m\} \left[\alpha_m(t)v'_{j+1} - \alpha_m(t)v'_j\right] - \mathbf{1}\{p_S \in U_m\} \cdot A_{m,S} v'_S \tag{19}$$

where the sample derivatives $v'_i(\theta)$ exist under Assumption 1. From (19), we observe that the task of evaluating $L'_{m,T}(\theta)$ reduces to evaluating $v'_i(\theta),\ i = 1, \ldots, S$, provided the flow rates involved in this expression are known.

3.2. Work IPA Derivative

Similar to the analysis in Section 3.1, we may write (4) as follows:

$$Q_T(\theta) = \sum_{k=1}^{K} \int_{\xi_k}^{\xi_{k+1}} x(\theta; t)dt \tag{20}$$

Define

$$q_k(\theta) = \int_{\xi_k}^{\xi_{k+1}} x(\theta; t)dt \tag{21}$$

so that we can write

$$Q'_T(\theta) = \sum_{k=1}^{K} q'_k(\theta) \tag{22}$$

in which

$$q'_k(\theta) = \frac{d}{d\theta} \int_{\xi_k}^{\xi_{k+1}} x(\theta; t)dt = \int_{\xi_k}^{\xi_{k+1}} x'(\theta; t)dt \tag{23}$$

where we use the fact that ξ_k, ξ_{k+1} are independent of θ. Again, we drop the index k in the following analysis of a typical cycle $\mathscr{C}_k$.

Let us first consider the last period p_S in a cycle, which is a Type I period. If $x(\theta; t) = T_m \neq \theta$ for $t \in p_S$, we have

$$x'(\theta; t) = 0 \tag{24}$$

If, on the other hand, $x(\theta; t) = \theta$ for $t \in p_S$, we have

$$x'(\theta; t) = 1 \tag{25}$$

Next, let us consider the possible values $x(\theta; t)$ may take in any other interval $p_i, \ i = 0, \ldots, S-1$ in the cycle. There are four cases.

Case 1: $x(\theta; v_i) = T_m \neq \theta$ and $T_m < x(\theta; t) < T_{m+1}$ for $t \in p_i^o$. In this case,

$$x(\theta; t) = T_m + \int_{v_i}^{t} A_{m+1}(t)dt$$

and, upon differentiating, we have

$$x'(\theta;t) = -A_{m+1,i}v_i' \tag{26}$$

Case 2: $x(\theta;v_i) = T_m \neq \theta$ and $T_{m-1} < x(\theta;t) < T_m$ for $t \in p_i^o$. We have

$$x(\theta;t) = T_m + \int_{v_i}^{t} A_m(t)dt$$

and, upon differentiating, we have

$$x'(\theta;t) = -A_{m,i}v_i' \tag{27}$$

where $A_{m,i}$ was defined in (7).

Case 3: $x(\theta;v_i) = \theta \equiv T_{\overline{m}}$ and $T_{\overline{m}} < x(\theta;t) < T_{\overline{m}+1}$ for $t \in p_i^o$. We have

$$x(\theta;t) = \theta + \int_{v_i}^{t} A_{\overline{m}+1}(t)dt$$

and, upon differentiating, we have

$$x'(\theta;t) = 1 - A_{\overline{m}+1,i}v_i' \tag{28}$$

Case 4: $x(\theta;v_i) = \theta \equiv T_{\overline{m}}$ and $T_{\overline{m}-1} < x(\theta;t) < T_{\overline{m}}$ for $t \in p_i^o$. We have

$$x(\theta;t) = \theta + \int_{v_i}^{t} A_{\overline{m}}(t)dt$$

and, upon differentiating, we have

$$x'(\theta;t) = 1 - A_{\overline{m},i}v_i' \tag{29}$$

Summarizing this analysis, it is clear that the complete evaluation of $x'(\theta;t)$, and hence of $Q_T'(\theta)$ through (23), relies on the event time derivatives.

3.3. Event Time Derivatives

As already discussed, the starting point of $\mathscr{C}_k$ is independent of θ because it is an exogenous event, so $\xi_k' = v_{k,0}' = 0$. Therefore, we next concentrate on a typical $\mathscr{C}_k$ and, dropping the index k, we shall show how an endogenous event time derivative v_i', $i = 1, \ldots, S$, is related to v_{i-1}' so as to derive appropriate recursive relationships. Recall that $v_i, i = 1, \ldots, S$, are the event times when the buffer content reaches (and possibly crosses) the thresholds $T_0, \ldots, T_M$. In addition, it is clear that the value of $L_{m,T}'(\theta)$

or $Q'_T(\theta)$ depends only on the event time derivatives $v'_i,\ i = 1, \ldots, S$ and the arrival and processing rates at the instants v_i. Therefore, we only need to focus on these particular event instants and ignore all system activity in between them. This also explains why the IPA estimators derived in this paper are not dependent on the stochastic characteristics of the arrival and service processes; only arrival and service rates at selected event times v_i are involved. Furthermore; if these rates are known or they can be measured on line, then evaluating $L'_{m,T}(\theta)$ and $Q'_T(\theta)$ reduces to the evaluation of the event time derivatives v'_i which is what we do next.

To do so, we consider all possible intervals $p_i \equiv [v_i, v_{i+1}),\ i = 0, \ldots, S-1$. There are eight possible cases.

Case 1: $x(\theta; v_i) = T_m \neq \theta$ and $x(\theta; v_{i+1}) = T_{m+1} \neq \theta$. In this case,

$$\int_{v_i}^{v_{i+1}} A_{m+1}(t)dt = T_{m+1} - T_m$$

and, upon differentiating,

$$A_{m+1,i+1} v'_{i+1} - A_{m+1,i} v'_i = 0$$

so that

$$v'_{i+1} = \frac{A_{m+1,i}}{A_{m+1,i+1}} \cdot v'_i \tag{30}$$

Case 2: $x(\theta; v_i) = T_m \neq \theta$ and $x(\theta; v_{i+1}) = T_{m-1} \neq \theta$. Recall that since $x(t;\theta) > T_{m-1}$, $t \in [v_i, v_{i+1})$, flows from all low priority classes $1, \ldots, m-1$ are dropped and only the higher priority classes $m, \ldots, M$ are accommodated (see also (7)). Thus,

$$\int_{v_i}^{v_{i+1}} A_m(t)dt = T_{m-1} - T_m$$

and, upon differentiating,

$$A_{m,i+1} v'_{i+1} - A_{m,i} v'_i = 0$$

so that

$$v'_{i+1} = \frac{A_{m,i}}{A_{m,i+1}} \cdot v'_i \tag{31}$$

Case 3: $x(\theta; v_i) = \theta = T_{\overline{m}}$ and $x(\theta; v_{i+1}) = T_{\overline{m}+1}$. We have

$$\int_{v_i}^{v_{i+1}} A_{\overline{m}+1}(t)dt = T_{\overline{m}+1} - \theta$$

and, upon differentiating,

$$A_{\overline{m}+1,i+1}v'_{i+1} - A_{\overline{m}+1,i}v'_i = -1$$

so that

$$v'_{i+1} = \frac{A_{\overline{m}+1,i}}{A_{\overline{m}+1,i+1}} \cdot v'_i - \frac{1}{A_{\overline{m}+1,i+1}} \tag{32}$$

Case 4: $x(\theta; v_i) = \theta = T_{\overline{m}}$ and $x(\theta; v_{i+1}) = T_{\overline{m}-1}$. We have

$$\int_{v_i}^{v_{i+1}} A_{\overline{m}}(t)dt = T_{\overline{m}-1} - \theta$$

and, upon differentiating,

$$A_{\overline{m},i+1}v'_{i+1} - A_{\overline{m},i}v'_i = -1$$

so that

$$v'_{i+1} = \frac{A_{\overline{m},i}}{A_{\overline{m},i+1}} \cdot v'_i - \frac{1}{A_{\overline{m},i+1}} \tag{33}$$

Case 5: $x(\theta; v_i) = T_{\overline{m}-1}$ and $x(\theta; v_{i+1}) = \theta = T_{\overline{m}}$. We have

$$\int_{v_i}^{v_{i+1}} A_{\overline{m}}(t)dt = \theta - T_{\overline{m}-1}$$

and, upon differentiating,

$$A_{\overline{m},i+1}v'_{i+1} - A_{\overline{m},i}v'_i = 1$$

so that

$$v'_{i+1} = -\frac{A_{\overline{m},i}}{A_{\overline{m},i+1}} \cdot v'_i + \frac{1}{A_{\overline{m},i+1}} \tag{34}$$

Case 6: $x(\theta; v_i) = T_{\overline{m}+1}$ and $x(\theta; v_{i+1}) = \theta = T_{\overline{m}}$. We have

$$\int_{v_i}^{v_{i+1}} A_{\overline{m}+1}(t)dt = \theta - T_{\overline{m}+1}$$

and, upon differentiating,

$$A_{\overline{m}+1,i+1}v'_{i+1} - A_{\overline{m}+1,i}v'_i = 1$$

so that

$$v'_{i+1} = -\frac{A_{\overline{m}+1,i}}{A_{\overline{m}+1,i+1}} \cdot v'_i + \frac{1}{A_{\overline{m}+1,i+1}} \tag{35}$$

Case 7: $x(\theta; v_i) = x(\theta; v_{i+1}) = T_m$ and $x(\theta; t) > T_m$ for $t \in p_i^o$. Note that it is possible to have $T_m = \theta$. We have

$$\int_{v_i}^{v_{i+1}} A_{m+1}(t)dt = 0$$

and, upon differentiating, we obtain (30).

Case 8: $x(\theta; v_i) = x(\theta; v_{i+1}) = T_m$ and $x(\theta; t) < T_m$ for $t \in p_i^o$. Once again, it is possible to have $T_m = \theta$. We have

$$\int_{v_i}^{v_{i+1}} A_m(t)dt = 0$$

Upon differentiating, we obtain (31).

By combining (30)–(35), we can see that $v'_{i+1}, i = 0, \ldots, S-1$ is obtained through a simple recursive relationship. We can summarize this analysis as follows:

THEOREM 3.1 *For any endogenous event occurring at time $v_{i+1},\ i = 0, \ldots, S-1$, we have*

$$v'_{i+1} = F_i \cdot v'_i + G_i \tag{36}$$

where F_i, G_i are given in (30)–(35) depending on the values of $x(\theta; v_i)$ and $x(\theta; v_{i+1})$. Moreover, $G_i \neq 0$ if and only if (i) $x(\theta; v_i) \neq x(\theta; v_{i+1}) = \theta$ or (ii) $x(\theta; v_{i+1}) \neq x(\theta; v_i) = \theta$.

Proof: The recursive relationship (36) follows directly from the analysis leading to (30)–(35). The necessary and sufficient condition for $G_i \neq 0$ also follows from (30)–(35). ■

Note that if an exogenous event occurs at ξ, we get $v_j' = 0$ for all $j \geq 1$ as long as the condition $G_i \neq 0$ above is not met. Thus, event time derivatives accumulate according to (36) following an endogenous event; they are periodically reset to 0 and may maintain this value for considerable portions of a sample path as long as $G_i = 0$. Finally, based on (36), we can return to (19) and (23) in order to fully evaluate the loss volume and work IPA derivatives.

4. IPA for the Two-Class Case

In the previous section, we saw that IPA estimators for the class-dependent loss volumes and the workload with respect to a given threshold parameter among $T_1, \ldots, T_M$ are obtained through recursive relationships based on (36). For the special case where $M = 2$, we are able to obtain these estimators through closed-form expressions by exploiting the specific structure of a sample path when only two classes are involved. Thus, we focus on the SFM shown in Figure 3 and simplify notation by letting $T_2 = b$ be the buffer capacity and setting $T_1 = T_{\overline{m}} = \theta$, where $0 < T_1 < b$. In this case, there are three performance metrics involved, the loss volume of class 1, $L_{1,T}(\theta)$, the loss volume of class 2, $L_{2,T}(\theta)$, and the work, $Q_T(\theta)$.

A sample path example of this SFM is shown in Figure 4. As in the general M-class case, the sample path of the two-class system is divided into cycles $\mathscr{C}_k, k = 1, \ldots, K$. In the remainder of this section, we derive the IPA estimators $L_{1,T}'(\theta)$, $L_{2,T}'(\theta)$, and $Q_T'(\theta)$. Our main results may be summarized as follows: (i) We show that in this case these estimators are given by closed-form expressions and derive these expressions; (ii) we show that $L_{2,T}'(\theta)$ is easy to obtain from $L_{1,T}'(\theta)$ without requiring any additional computation; and (iii) we obtain a simple upper and lower bound for $L_{1,T}'(\theta)$.

As seen in Section 3, $L_{1,T}'(\theta)$, $L_{2,T}'(\theta)$, and $Q_T'(\theta)$ are based on event time derivatives. Theorem 3.1 shows how to evaluate $v_{i+1}, \; i = 0, \ldots, S-1$, for a particular cycle $\mathscr{C}_k$. For the two-class case, there are several simplifications taking place in the eight cases considered in Section 3.3. Specifically, Cases 1, 2 are clearly infeasible; Cases 3, 4 only apply to period p_{S-1}; Cases 5, 6 only apply to period p_0; and only Cases 7, 8 are always applicable. Taking advantage of these observations, (36) in Theorem 3.1 can be considerably simplified. In particular, the following lemma shows that all event time derivatives that turn out to be of interest in evaluating $L_{1,T}'(\theta), L_{2,T}'(\theta)$, and $Q_T'(\theta)$ are either 0 or can be expressed in terms of $A_{1,i}$ and $A_{2,i}$ (defined in (7) with the index k dropped). By

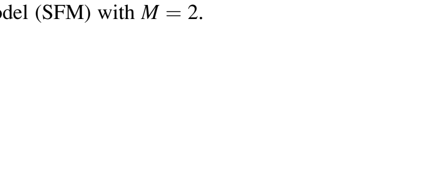

Figure 3. Stochastic fluid model (SFM) with $M = 2$.

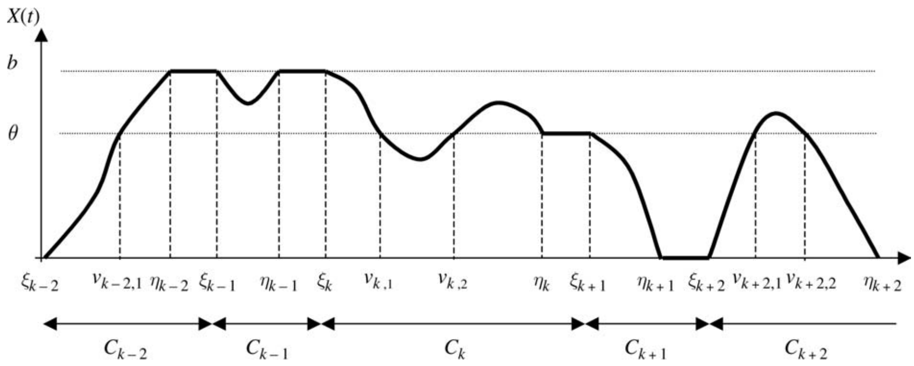

Figure 4. Typical sample path segment with $M = 2$.

convention, we shall set $A_{1,0} \equiv 1$ and $A_{2,0} \equiv 1$. We will also use the notation $\lceil x \rceil$ to denote the smallest integer greater than or equal to x.

LEMMA 4.1 *Within a cycle that starts with ξ, for all v_i, $i = 1, \ldots, S$, which satisfy $x(\theta; v_i) = \theta$:*

1. *If* $x(\theta; \xi) = 0$:

$$v'_1 = \frac{A_{1,0}}{A_{1,1}} \tag{37}$$

$$v'_{2n} = \prod_{i=1}^{n} \frac{A_{2,2i-1}}{A_{2,2i}} \cdot \frac{A_{1,2i-2}}{A_{1,2i-1}} \tag{38}$$

where $1 \leq n \leq \lceil S - 1/2 \rceil$ *and* $S > 1$; *and*

$$v'_{2n+1} = \frac{A_{1,2n}}{A_{1,2n+1}} \prod_{i=1}^{n} \frac{A_{2,2i-1}}{A_{2,2i}} \cdot \frac{A_{1,2i-2}}{A_{1,2i-1}} \tag{39}$$

where $1 \leq n \leq \lfloor S - 1/2 \rfloor$, *and* $S > 2$.

2. *If* $x(\theta; \xi) = b$:

$$v'_1 = \frac{A_{2,0}}{A_{2,1}} \tag{40}$$

$$v'_{2n} = \prod_{i=1}^{n} \frac{A_{1,2i-1}}{A_{1,2i}} \cdot \frac{A_{2,2i-2}}{A_{2,2i-1}} \tag{41}$$

where $1 \leq n \leq \lceil S - 1/2 \rceil$ *and* $S > 1$; *and*

$$v'_{2n+1} = \frac{A_{2,2n}}{A_{2,2n+1}} \prod_{i=1}^{n} \frac{A_{1,2i-1}}{A_{1,2i}} \cdot \frac{A_{2,2i-2}}{A_{2,2i-1}} \tag{42}$$

where $1 \leq n \leq \lfloor S - 1/2 \rfloor$ *and* $S > 2$.

3. *If* $x(\theta; \xi) = \theta$:

$$v'_i = 0 \tag{43}$$

Proof: See Appendix. ■

4.1. Class 1 Loss Derivatives

Based on the analysis in Section 3.1, to evaluate $L'_{1,T}(\theta)$ we need to evaluate $\lambda'_{1,k}(\theta)$ for any $k = 1, \ldots, K$. Focusing on a typical cycle $\mathscr{C}_k$ and dropping the index k in the following analysis, we get from (19):

$$\lambda'_1(\theta) = \sum_{i=0}^{S} \mathbf{1}\{p_i \in V_1\}\left[\alpha_1(v_{i+1})v'_{i+1} - \alpha_1(v_i)v'_i\right] - \mathbf{1}\{p_S \in U_1\} \cdot A_{1,S}v'_S \tag{44}$$

Regarding the last period p_S, we next make an observation which simplifies our analysis because it allows us to proceed without needing to evaluate v'_S beyond the case already covered by Lemma 4.1. In particular, since the last partial loss period of each cycle can only take the values θ, b, and 0, this allows us to identify three possible cases:

i. If $p_S \in U_1$, then $x(\theta; v_S) = \theta$ and Lemma 4.1 applies allowing us to obtain v'_S.

ii. If $p_S \in V_1$ then $x(\theta; v_S) = b$ and we cannot obtain v'_S via Lemma 4.1. However, by the continuity of the sample path, we must have $p_{S-1} \in V_1$. Thus, the last two terms contributed by the sum on the right-hand-side of (44) are due to p_{S-1} in the amount

$$\alpha_1(v_S)v'_S - \alpha_1(v_{S-1})v'_{S-1}$$

and by p_S in the amount

$$-\alpha_1(v_S)v'_S$$

where we use the fact that $v'_{S+1} = 0$. Therefore, the combined contribution of p_{S-1} and p_S is

$$-\alpha_1(v_{S-1})v'_{S-1} \tag{45}$$

and we observe that v'_S is not needed for evaluating $\lambda'_1(\theta)$.

iii. If $p_S \in W_1$, then $x(\theta; v_S) = 0$ and Lemma 4.1 does not apply. However, again by the continuity of the sample path, we must have $p_{S-1} \in W_1$. It is clear from (44) that v'_S is again not involved in the evaluation of $\lambda'_1(\theta)$.

The next four lemmas provide the expressions for the derivative $\lambda'_1(\theta)$ in (44) over a cycle, depending on the state of the SFM when the cycle starts and ends. In the first lemma, we show that $\lambda'_1(\theta) = 0$ whenever the cycle starts from θ (e.g., $\mathscr{C}_{k+1}$ in Figure 4) regardless of how it ends. In the second lemma, we show that $\lambda'_1(\theta) = -1$ whenever the cycle ends with θ (e.g., $\mathscr{C}_k$ in Figure 4) if it starts with $x(\theta; \xi) \neq \theta$.

LEMMA 4.2 *For a cycle that starts with ξ, if $x(\theta; \xi) = \theta$, then:*

$$\lambda'_1(\theta) = 0 \tag{46}$$

Proof: See Appendix. ■

LEMMA 4.3 *For a cycle that starts with ξ and contains events occurring at $v_i, i = 1, \ldots, S$, if $x(\theta; \xi) \neq \theta$ and $x(\theta; v_S) = \theta$, then:*

$$\lambda'_1(\theta) = -1 \tag{47}$$

Proof: See Appendix. ■

The next two lemmas consider the cases where a cycle starts with $x(\theta; \xi) = 0$ and b, respectively, and does not end with θ (e.g., $\mathscr{C}_{k-2}$, $\mathscr{C}_{k-1}$, and $\mathscr{C}_{k+2}$ in Figure 4). In these cases, $\lambda'_1(\theta)$ is given by flow rate information contained in $A_{1,i}$ and $A_{2,i}$ (defined in (7) with the index k dropped).

LEMMA 4.4 *For a cycle that starts with ξ and contains events occurring at $v_i, i = 1, \ldots, S$:*

1. If $x(\theta; \xi) = 0$, $x(\theta; v_S) = 0$:

For $S = 1$: $$\lambda'_1(\theta) = 0 \tag{48}$$

For $S > 1$: $$\lambda'_1(\theta) = -1 + \prod_{i=1}^{(S-1)/2} \frac{A_{1,2i}}{A_{2,2i}} \cdot \frac{A_{2,2i-1}}{A_{1,2i-1}}. \tag{49}$$

2. If $x(\theta; \xi) = 0$, $x(\theta; v_S) = b$:

For $S = 2$: $$\lambda'_1(\theta) = -1 + \frac{A_{2,1}}{A_{1,1}} \tag{50}$$

For $S > 2$: $$\lambda'_1(\theta) = -1 + \frac{A_{2,S-1}}{A_{1,S-1}} \prod_{i=1}^{(S-2)/2} \frac{A_{1,2i}}{A_{2,2i}} \cdot \frac{A_{2,2i-1}}{A_{1,2i-1}}. \tag{51}$$

Proof: See Appendix. ■

LEMMA 4.5 *For a cycle that starts with* ξ *and contains events occurring at* $v_i, i = 1, \ldots, S$:

1. *If* $x(\theta;\xi) = b$, $x(\theta; v_S) = 0$:

$$\text{For } S = 2:\ \lambda_1'(\theta) = -1 + \frac{A_{1,1}}{A_{2,1}} \tag{52}$$

$$\text{For } S > 2:\ \lambda_1'(\theta) = -1 + \frac{A_{1,S-1}}{A_{2,S-1}} \prod_{i=1}^{(S-2)/2} \frac{A_{2,2i}}{A_{1,2i}} \cdot \frac{A_{1,2i-1}}{A_{2,2i-1}}. \tag{53}$$

2. *If* $x(\theta;\xi) = b$, $x(\theta; v_S) = b$:

$$\text{For } S = 1:\ \lambda_1'(\theta) = 0 \tag{54}$$

$$\text{For } S > 1:\ \lambda_1'(\theta) = -1 + \prod_{i=1}^{(S-1)/2} \frac{A_{2,2i}}{A_{1,2i}} \cdot \frac{A_{1,2i-1}}{A_{2,2i-1}}. \tag{55}$$

Proof: See Appendix. ■

Finally, we can also establish some simple bounds on $\lambda_1'(\theta)$ as shown in the following lemma.

LEMMA 4.6 *The class 1 loss over any cycle satisfies:* $-1 \leq \lambda_1'(\theta) \leq 0$.

Proof: See Appendix. ■

Motivated by our analysis thus far, let us partition the set of all cycles as follows. First, define

$$\Phi = \{k \in \{1, \ldots, K\} : x(\theta; \xi_k) = \theta\}$$

to be the set of cycles that start from θ, corresponding to Lemma 4.2. Let $\bar{\Phi}(\theta)$ denote the complement set of Φ and define

$$\bar{\Phi}_{*\theta} = \{k : k \in \bar{\Phi}, x(\theta; \xi_{k+1}) = \theta\}$$
$$\bar{\Phi}_{00} = \{k : k \in \bar{\Phi}, x(\theta; \xi_k) = 0,\ x(\theta; \xi_{k+1}) = 0\}$$
$$\bar{\Phi}_{0b} = \{k : k \in \bar{\Phi}, x(\theta; \xi_k) = 0,\ x(\theta; \xi_{k+1}) = b\}$$
$$\bar{\Phi}_{b0} = \{k : k \in \bar{\Phi}, x(\theta; \xi_k) = b,\ x(\theta; \xi_{k+1}) = 0\}$$
$$\bar{\Phi}_{bb} = \{k : k \in \bar{\Phi}, x(\theta; \xi_k) = b,\ x(\theta; \xi_{k+1}) = b\}$$

which cover the remaining cases, corresponding to Lemmas 4.3–4.5. Moreover, note that all the above sets are locally independent of θ. The expressions in Lemmas 4.4–4.5 can also be simplified by defining for any $k = 1, \ldots, K$,

$$R_k, n = \prod_{i=1}^{n/2} \frac{A_{1,k,2i}}{A_2, k, 2i} \cdot \frac{A_{2,k,2i-1}}{A_{1,k,2i-1}}, \qquad n = 2, 4, \ldots \quad \text{and} \quad R_{k,0} = 1 \tag{56}$$

$$\bar{R}_{k,n} = \prod_{i=1}^{n/2} \frac{A_{2,k,2i}}{A_{1,k,2i}} \cdot \frac{A_{1,k,2i-1}}{A_{2,k,2i-1}}, \qquad n = 2, 4, \ldots \quad \text{and} \quad \bar{R}_{k,0} = 1 \tag{57}$$

Then, using Lemmas 4.2–4.5 and the definitions above, we get

$$\lambda'_{1,k}(\theta) = -1 + \begin{cases} 0 & \text{if } k \in \bar{\Phi}_{*\theta} \\ R_{k,S_k-1} & \text{if } k \in \bar{\Phi}_{00} \\ \frac{A_{2,k,S_k-1}}{A_{1,k,S_k-1}} R_{k,S_k-2} & \text{if } k \in \bar{\Phi}_{0b} \\ \frac{A_{1,k,S_k-1}}{A_{2,k,S_k-1}} \bar{R}_{k,S_k-2} & \text{if } k \in \bar{\Phi}_{b0} \\ \bar{R}_{k,S_k-1} & \text{if } k \in \bar{\Phi}_{bb} \\ 1 & \text{if } k \in \Phi \end{cases} \tag{58}$$

We may now summarize our results as follows.

THEOREM 4.1 *The sample derivative $L'_{1,T}(\theta)$ is given by*

$$L'_{1,T}(\theta) = -\ K^* + \sum_{k \in \bar{\Phi}_{00}} R_{k,S_k-1} + \sum_{k \in \bar{\Phi}_{0b}} \frac{A_{2,k,S_k-1}}{A_{1,k,S_k-1}} R_{k,S_k-2}$$
$$+ \sum_{k \in \bar{\Phi}_{b0}} \frac{A_{1,k,S_k-1}}{A_{2,k,S_k-1}} \bar{R}_{k,S_k-2} + \sum_{k \in \bar{\Phi}_{bb}} \bar{R}_{k,S_k-1} \tag{59}$$

where K^* is the (random) cardinality of the set $\bar{\Phi}_{*\theta}$ contained in $[0, T]$, including a possibly incomplete last cycle.

Proof: The result follows immediately from (10) and (58). ■

The expression in (59) provides the IPA estimator for the loss metric $L_{1,T}(\theta)$. Unlike the $M > 2$ case studied in Section 3, here we are able to obtain a closed-form expression in terms of $A_{1,k,i}$ and $A_{2,k,i}$. Again, we note that $L'_{1,T}(\theta)$ does not depend on any distributional information regarding the arrival and service processes and involves only flow rates at event times $v_{k,i}$ which may be estimated on line.

As far as implementation is concerned, the evaluation of $\lambda'_{1,k}(\theta)$ in Lemmas 4.2–4.5 requires observing events e_1 and e_2 and the corresponding rates of α_1, α_2, and β at their occurrence times, so that we can evaluate $A_{1,k,i}$ and $A_{2,k,i}$. If a cycle starts from an empty buffer or full buffer and ends with a partial loss period at θ, then the only implementation requirement is that such a cycle be detected and the contribution of this cycle is simply -1. It is also worth mentioning the following simple bounds we can obtain:

COROLLARY $L'_{1,T}(\theta)$ is bounded as follows:

$$-K^* \leq L'_{1,T}(\theta) \leq 0$$

Proof: Follows directly from (59) and Lemma 4.6, in view of (10). ■

4.2. Class 2 Loss Derivatives

Since $T_2 = b$, class 2 experiences partial loss only during the period through which $x(\theta; t) = b$, and the loss rate is $\gamma_2(\theta; t) = \alpha_2(t) - \beta(t)$. From (19) we have

$$\lambda'_2(\theta) = -\mathbf{1}\{p_S \in U_2\} \cdot A_{2,S} v'_S \tag{60}$$

The following lemma greatly simplifies the task of evaluating the sample derivative $L'_{2,T}(\theta)$, since it can be obtained as a byproduct of the evaluation of $L'_{1,T}(\theta)$ in Theorem 4.1.

LEMMA 4.7 *For a cycle that starts with ξ and contains events occurring at $v_i, i = 1, \ldots, S$:*

$$\lambda'_2(\theta) = \begin{cases} 0 & \text{if } p_S \notin U_2 \\ 1 & \text{if } p_S \in U_2 \text{ and } x(\theta; \xi) = \theta \\ -\lambda'_1(\theta) & \text{if } p_S \in U_2 \text{ and } x(\theta; \xi) \neq \theta \end{cases} \tag{61}$$

where $\lambda_1(\theta)$ is the class 1 loss incurred over the cycle.

Proof: See Appendix. ■

4.3. Work Derivative

As we saw in Section 3.2, $Q'_T(\theta)$ requires the evaluation of

$$q'_k(\theta) = \int_{\xi_k}^{\xi_{k+1}} x'(\theta; t) dt \tag{62}$$

for every $k = 1, \ldots, K$. Again, we drop the index k for convenience, as we consider a typical cycle $\mathscr{C}_k$. Regarding the first period, p_0, in the cycle, there are two possibilities. If

$x(\theta;\xi) \neq \theta$, Cases 1 and 2 in Section 3.2 apply. In view of the fact that $\xi' = 0$, by (26) and (27) we have

$$x'(\theta;t) = 0 \tag{63}$$

If, on the other hand, $x(\theta;\xi) = \theta$, then either Case 3 or Case 4 of Section 3.2 applies. Either way, from (28) and (29) we get

$$x'(\theta;t) = 1 \tag{64}$$

If $S > 1$, observe that all endogenous events contained in a cycle are constrained to be such that $x(\theta;v_i) = \theta$ (see also Figure 4). Thus, for all $p_i, i = 1, \dots, S-1$, either Case 3 or Case 4 of Section 3.2 applies. If $x(\theta;t) > \theta$ for $t \in p_i^o, i > 0$, by (28) we have

$$x'(\theta;t) = 1 - A_{2,i}v_i' \tag{65}$$

Otherwise, if $x(\theta;t) < \theta$ for $t \in p_i^o$, $i > 0$, by (29) we have

$$x'(\theta;t) = 1 - A_{1,i}v_i' \tag{66}$$

Finally, consider the last period, p_S, in the cycle. From (24) and (25), if $p_S \notin U_1$ then we have

$$x'(\theta;t) = 0 \tag{67}$$

and, otherwise,

$$x'(\theta;t) = 1 \tag{68}$$

We can summarize this analysis in the following:

LEMMA 4.8 *For a cycle that starts with ξ and contains events occurring at $v_i, i = 1, \dots, S$:*

$$q'(\theta) = \mathbf{1}\{x(\theta;\xi) = \theta\} \cdot (v_1 - \xi) + \sum_{i=1}^{S-1}(v_{i+1} - v_i)\phi_i + \mathbf{1}\{p_S \in U_1\} \cdot (v_{S+1} - v_S) \tag{69}$$

where

$$\phi_i = \begin{cases} 1 - A_{1,i}v_i', & \text{if } x(\theta;t) > \theta, \quad t \in p_i^o \\ 1 - A_{2,i}v_i', & \text{if } x(\theta;t) < \theta, \quad t \in p_i^o \end{cases} \tag{70}$$

Proof: The proof follows from (62) by combining (63)–(68). ■

Note that if $x(\theta;\xi) = \theta$, by (43) we have $v_i' = 0$ for $i = 1,\ldots,S-1$, and (70) implies $\phi_i = 1$ for $i = 1,\ldots,S-1$. Therefore, in this case, by combining the first two terms of (69) the total contribution of the first S periods of a cycle becomes $v_S - \xi$. The third item of (69) is the contribution of p_S which is its length if it happens to be a class 1 partial loss period; in this case, the total contribution of the cycle is simply its full length.

THEOREM 4.2 *The sample derivative $Q_T'(\theta)$ is given by*

$$Q_T'(\theta) = \sum_{k=1}^{K} q_k'(\theta) \tag{71}$$

where K is the (random) number of cycles contained in $[0,T]$, including a possibly incomplete last one.

Proof: The result follows from (22) and Lemma 4.8, using the definitions in (69) and (70). ■

The expression in (71) provides the IPA estimator for the work metric defined in (4). Its implementation requires the same information as that for the loss metric with the addition of timers to measure the duration of the periods $p_i, i = 0,\ldots,S$.

5. IPA Estimator Unbiasedness

We now return to the general M-class case in order to prove the unbiasedness of the IPA derivatives $L_{m,T}'(\theta), m = 1,\ldots,M$ and $Q_T'(\theta)$. In general, the unbiasedness of an IPA derivative $\mathscr{L}'(\theta)$ is ensured by the following two conditions (see Rubinstein and Shapiro [20], Lemma A2, p. 70):

Condition 1. For every $\theta \in \Theta$, the sample derivative $\mathscr{L}'(\theta)$ exists w.p.1.

Condition 2. W.p.1, the random function $\mathscr{L}(\theta)$ is Lipschitz continuous throughout Θ, and the (generally random) Lipschitz constant has a finite first moment.

We have already discussed the mild technical conditions required to ensure the existence of $L_{m,T}'(\theta), m = 1,\ldots,M$ and $Q_T'(\theta)$. Consequently, establishing the unbiasedness of $L_{m,T}'(\theta), m = 1,\ldots,M$ and $Q_T'(\theta)$ as estimators of $dE[L_{m,T}(\theta)]/d\theta, m = 1,\ldots,M$ and $dE[Q_T(\theta)]/d\theta$, respectively, reduces to verifying the Lipschitz continuity of $L_{m,T}(\theta), m = 1,\ldots,M$ and $Q_T(\theta)$ with appropriate Lipschitz constants.

The buffer content over the nominal sample path is denoted by $x(\theta;t)$, while the buffer content in a perturbed sample path is denoted by $x(\theta+\Delta\theta;t)$ resulting when θ is replaced by $\theta+\Delta\theta$. We superimpose two such sample paths and denote the combined superimposed event times (e_1 and e_2) by $t_0 < t_1 < \cdots < t_{J(T)}$, where $t_0 = 0$, $t_{J(T)} = T$. Thus, $J(T)$ is the random number of all superimposed events (exogenous and endogenous) in $[0,T]$. The interval $[0,T]$ can, therefore, be divided into time intervals $I_j \equiv [t_{j-1}, t_j)$, $j = 1, \ldots, J(T)$; we shall use $I_j^o \equiv (t_{j-1}, t_j)$ to denote the corresponding open intervals. Recalling the definitions of Type I and Type II intervals in Section 2, in each I_j^o, the nominal sample path is in either a Type I or a Type II interval and the same is true for the perturbed sample path. Moreover, if I_j is a Type I interval, for $t \in I_j$ we have

$$\frac{dx(\cdot\,;t)}{dt^+} = 0 \tag{72}$$

while if I_j is a Type II interval and $T_m < x(\cdot\,;t) < T_{m+1}$ for $t \in I_j^o$, we have

$$\frac{dx(\cdot\,;t)}{dt^+} = A_{m+1}(t) \tag{73}$$

Then, under the assumption that $E[J(T)] < \infty$, we shall establish next that $L'_{m,T}(\theta), m = 1, \ldots, M$ and $Q'_T(\theta)$ are indeed unbiased estimators.

At this point it is worth recalling that $x(\theta;t)$ and $x(\theta + \Delta\theta;t)$ are continuous functions of t due to Assumption 1a. Next, we show that $0 \le \Delta x(t) \le \Delta\theta$ for all $t \in [0,T]$, where $\Delta x(t) = x(\theta + \Delta\theta;t) - x(\theta;t)$ and $\Delta\theta > 0$ (the case $\Delta\theta < 0$ is similarly handled). To do so, we first need the following result (Lemma 5.1). In order to maintain the notation as simple as possible, we define:

$$B_m(t) \equiv \sum_{\substack{n=n' \\ n' \ge m}}^{M} \alpha_n(t) - \beta(t) \quad \text{and} \quad C_m(t) \equiv \sum_{\substack{n=n' \\ n' \le m}}^{M} \alpha_n(t) - \beta(t)$$

Recalling the definition of $A_m(t)$ in (1), observe that

$$B_m(t) \le A_m(t) \le C_m(t)$$

LEMMA 5.1 *Consider an interval* $I_j = [t_{j-1}, t_j)$, *and assume that* $0 \le \Delta x(t_{j-1}) \le \Delta\theta$. *Then*

$$0 \le \Delta x(t) \le \Delta\theta \quad \textit{for all } t \in I_j \tag{74}$$

Proof: See Appendix. ■

We can now use this result to establish the same bound for $\Delta x(t)$ over all $t \in [0,T]$.

LEMMA 5.2 *For all* $t \in [0,T]$,

$$0 \le \Delta x(t) \le \Delta\theta \tag{75}$$

Proof: See Appendix. ■

Finally, we can establish the unbiasedness of $L'_{m,T}(\theta), m = 1, \ldots, M$ and $Q'_T(\theta)$.

THEOREM 5.1 *The IPA derivatives* $L'_{m,T}(\theta), m = 1, \ldots, M$ *and* $Q'_T(\theta)$ *are unbiased estimates of* $dE[L_{m,T}(\theta)]/d\theta, m = 1, \ldots, M$ *and* $dE[Q_T(\theta)]/d\theta$, *respectively. In other words,*

$$E\left[L'_{m,T}(\theta)\right] = \frac{dE[L_{m,T}(\theta)]}{d\theta}, \quad m = 1, \ldots, M \quad \textit{and} \quad E[Q'_T(\theta)] = \frac{dE[Q_T(\theta)]}{d\theta}$$

Proof: See Appendix. ■

6. Conclusions

We have considered in this paper a SFM of a queueing system with multiple customer classes subject to threshold-based buffer control. We have developed IPA estimators for class-specific loss metrics and for a work metric with respect to one of the threshold and shown them to be unbiased. We have also shown that in the two-class case it is possible to obtain closed-form expressions for the estimators (as opposed to evaluating them recursively using Theorem 3.1). The simplicity of the estimators derived and the fact they are not dependent on knowledge of the arrival or service processes makes them attractive for on-line control and optimization in applications such as manufacturing and communication networks. We emphasize again that the use of a SFM allows us to obtain the form of a particular gradient estimator, but the actual implementation is carried out using system data (e.g., observing events such as an actual buffer content crossing some threshold T_m) from the actual (discrete-event) system.

One of the limitations of IPA applied to SFMs to date is the absence of feedback mechanisms in the models analyzed, i.e., the assumption that arrival and service processes are not dependent on controllable parameters. Recent work (see Yu and Cassandras, 2003) has shown that it is possible to still obtain relatively simple and unbiased IPA estimators in such cases. This is a positive step towards the development of a more general theory for applying PA techniques to SFMs, but the level of generality currently available is limited to certain classes of models. The key elements of a general approach, however, would be typified by the main steps seen in this paper: (i) Decomposing a sample path through appropriately defined events that reflect changes in the time-driven dynamics of an underlying system; (ii) deriving sample derivatives (assuming they exist) for the event times and the state of the underlying system (in our case, the buffer content); (iii) translating these derivatives into derivatives of performance metrics of interest (in our case, loss and workload); and (iv) establishing the Lipschitz continuity of the performance metrics if that holds, which, under generally mild conditions, leads to proving unbiasedness of the IPA derivatives obtained in the previous step.

Appendix

Proof of Lemma 4.1: The proof is based on Theorem 3.1 applied to the two-class system as a special case, as well as the observation that all endogenous events occurring at v_i within a cycle must satisfy $x(\theta; v_i) = \theta$ (see also Figure 4). We begin with the first period $p_0 = [\xi, v_1)$ where $x(\theta; v_1) = \theta$, for which there are three possibilities:

1. $x(\theta; \xi) = 0$. Since ξ is an exogenous event, $\xi' = 0$, therefore by (36) and (34),

$$v_1' = G_0 = \frac{1}{A_{1,1}} \equiv \frac{A_{1,0}}{A_{1,1}} \tag{76}$$

2. $x(\theta; \xi) = b$. By (36) and (35) and in view of $\xi' = 0$, we get

$$v_1' = G_0 = \frac{1}{A_{2,1}} \equiv \frac{A_{2,0}}{A_{1,1}} \tag{77}$$

3. $x(\theta; \xi) = \theta$. By (36) and the facts that $G_0 = 0$ (see Theorem 3.1) and $\xi' = 0$, we get

$$v_1' = 0 \tag{78}$$

Next, consider any $p_i, 1 \leq i \leq S - 1$. Since we must have $x(\theta; v_i) = x(\theta; v_{i+1}) = \theta$, by Theorem 3.1 we get $G_i = 0$. We then have two possible cases:

1. $p_i \in V_1$. In this case, $x(\theta; t) > \theta$ for $t \in p_i^o$, and from (36) and (30) we get

$$v_i' = \frac{A_{2,i-1}}{A_{2,i}} v_{i-1}' \tag{79}$$

2. $p_i \in W_1$. In this case, $x(\theta; t) < \theta$ for $t \in p_i^o$, and from (36) and (31) we get

$$v_i' = \frac{A_{1,i-1}}{A_{1,i}} v_{i-1}' \tag{80}$$

The proof is completed by combining (76)–(80). ■

Proof of Lemma 4.2: For a cycle with $x(\theta; \xi) = \theta$, Lemma 4.1 asserts that all event time derivatives of interest are 0, so by (44) we immediately obtain (46). ■

Proof of Lemma 4.3: Let $l_i(\theta)$ be the class 1 loss over an interval p_i:

$$l_i(\theta) = \int_{v_i}^{v_{i+1}} \gamma_1(\theta; t) dt \tag{81}$$

Since $x(\theta; v_S) = \theta$, we have $p_S \in U_1$. Suppose $S > 1$ (the special case where $S = 1$ will easily follow). There are two possible cases regarding $x(\theta;\xi)$, that is, $x(\theta;\xi) = b$ or 0. First, suppose $x(\theta;\xi) = b$. Then, the periods $p_i, i = 0, \ldots, S-1$ belong to either W_1 or V_1 and appear alternately with the first one $p_0 = [\xi, v_1) \in V_1$. Thus, from (81) and in view of $\gamma_1(\theta; t) = \alpha_1(t)$ from (14) we get

$$l_0'(\theta) = \alpha_1(v_1)v_1'$$

since $v_0' = \xi' = 0$. In light of (7) the above equation can be rewritten as

$$l_0'(\theta) = A_{1,1}v_1' - A_{2,1}v_1' \tag{82}$$

Also from (81), for all other full loss periods $p_{2n}, n \geq 1$, we have

$$l_{2n}'(\theta) = \alpha_1(v_{2n+1})v_{2n+1}' - \alpha_1(v_{2n})v_{2n}' \tag{83}$$

By (79) in the proof of Lemma 4.2,

$$A_{2,2n+1}v_{2n+1}' - A_{2,2n}v_{2n}' = 0$$

Adding the left-hand-side above to the right-hand-side of (83) and recalling (7), we get

$$l_{2n}'(\theta) = A_{1,2n+1}v_{2n+1}' - A_{1,2n}v_{2n}' \tag{84}$$

For the loss over the partial loss period p_S, from (81) and $\gamma_1(\theta; t) = A_1(t)$ in (12), we have

$$l_S' = -A_{1,S}v_S' \tag{85}$$

In addition, we know that $p_{2n-1} \in W_1$ for $n \geq 1$, therefore $l_{2n-1}'(\theta) = 0$. Using (7) and (80) in the proof of Lemma 4.2, we have:

$$A_{1,2n-1}v_{2n-1}' = A_{1,2n}v_{2n}' \tag{86}$$

The cumulative contribution of all full loss periods is given by combining (82), (84), and (85). If S is odd (including $S = 1$), then $p_{S-1} \in V_1$ and the total contribution is

$$\lambda_1'(\theta) = \sum_{i=0}^{S} l_i'(\theta) = -A_{2,1}v_1' + \sum_{i=1}^{S}(-1)^{i+1}A_{1,i}v_i' - A_{1,S}v_S'$$

On the other hand, if S is even, then $p_{S-1} \in W_1$ and the total contribution is

$$\lambda_1'(\theta) = \sum_{i=0}^{S} l_i'(\theta) = -A_{2,1}v_1' + \sum_{i=1}^{S-1}(-1)^{i+1}A_{1,i}v_i' - A_{1,S}v_S'$$

The last two terms of the right-hand-side of the last two equations are equal to 0 by using (86), so for both cases we get

$$\lambda_1'(\theta) = -A_{2,1}v_1' \tag{87}$$

By Lemma 4.1, $v_1' = 1/A_{2,1}$, so the last equation yields $\lambda_1'(\theta) = -1$, which is precisely (47).

The second case is $x(\theta; \xi) = 0$, for which a similar argument also gives $\lambda_1'(\theta) = -1$, completing the proof of (47). ■

Proof of Lemmas 4.4 and 4.5: We combine the proofs of these two lemmas since they are similar in nature. First, note that if $S = 1$ the buffer content never reaches θ, therefore it is easy to see that any class 1 loss is unaffected by θ over this interval, which implies $\lambda_1'(\theta) = 0$. This establishes (48) and (54). Next, for $S > 1, p_1, \ldots, p_{S-1}$ belong alternately to V_1 and W_1. Depending on the value of $x(\theta; t)$ at ξ and v_S, there are four cases to consider:

Case 1: $x(\theta; \xi) = 0, x(\theta; v_S) = 0$. Since the buffer content never reaches b in this cycle, this case is identical to the infinite buffer case studied in Cassandras et al. (2003), and (48), (49) are precisely the results obtained in Lemma 3.3 of Cassandras et al. (2003).

Case 2: $x(\theta; \xi) = 0, x(\theta; v_S) = b$. In this case, there is at least one no loss and one full loss period before p_S and S must be even with p_{2n-1}, $1 \leq n \leq S/2$, all being full loss periods. It follows from (45) that the contribution of p_{S-1} and p_S together is

$$-\alpha_1(v_{S-1})v_{S-1}'$$

which we can rewrite, using (7), as

$$-A_{1,S-1}v_{S-1}' + A_{2,S-1}v_{S-1}' \tag{88}$$

Using (81) and (7), the loss derivative contribution of any other full loss period is

$$-A_{1,2n-1}v_{2n-1}' + A_{1,2n}v_{2n}', \quad 1 \leq n \leq \frac{S-2}{2} \tag{89}$$

Adding (89) for all $n = 1, \ldots, (S-2)/2$ and (88) together and then using (86) obtained in the proof of Lemma 4.3 gives

$$\lambda_1'(\theta) = -A_{1,1}v_1' + A_{2,S-1}v_{S-1}' \tag{90}$$

We can now replace v_1' and v_{S-1}' by (37) and (39) with $n = S - 2/2$ in Lemma 4.1, respectively, which immediately yields (51).

Case 3: $x(\theta; \xi) = b, x(\theta; v_S) = 0$. As in the previous case, S must be even and $p_{2n}, 0 \leq n \leq S - 2/2$, are all full loss periods. The loss derivative contribution of p_0 is given in (82). The contribution of any other full loss period is given by (84):

$$-A_{1,2n}v'_{2n} + A_{1,2n+1}v'_{2n+1}, \quad 1 \le n \le \frac{S-2}{2} \tag{91}$$

Adding (82) and (91) for all $n = 1, \ldots, (S-2)/2$ together and then using (86) gives

$$\lambda'_1(\theta) = -A_{2,1}v'_1 + A_{1,S-1}v'_{S-1}$$

Replacing v'_1 and v'_{S-1} by (40) and (42) with $n = (S-2)/2$ in Lemma 4.1, respectively, yields (53).

Case 4: $x(\theta;\xi) = b, x(\theta; v_S) = b$. In this case, S must be odd and $p_{2n}, 0 \le n \le (S-1)/2$, are all full loss periods. The loss derivative contribution of the first full loss period $[\xi, v_1)$ is given by (82). The contribution of the last full loss period $[v_{S-1}, v_S)$, together with the contribution of p_{S-1}, is given by (88). The loss derivative contribution of any other full loss period is as in (91):

$$-A_{1,2n}v'_{2n} + A_{1,2n+1}v'_{2n+1}, \quad 1 \le n \le \frac{S-3}{2} \tag{92}$$

Adding (82), (88), and (92) for all $n = 1, \ldots, (S-3)/2$ together and then using (86) we get

$$\lambda'_1(\theta) = -A_{2,1}v'_1 + A_{2,S-1}v'_{S-1} \tag{93}$$

Finally, replacing v'_1 and v'_{S-1} by (40) and (41) with $n = (S-1)/2$ in Lemma 4.1, respectively, yields (55). The proof is now complete. ■

Proof of Lemma 4.6: Let us focus on a cycle with $x(\theta;\xi) \ne \theta, x(\theta; v_S) \ne \theta$ and $S > 1$, since in all remaining cases $\lambda'_1(\theta) = -1$ or 0 from Lemmas 4.2 and 4.3. For such a cycle, at some $v_i, 1 \le i \le S-1$, the sample path crosses θ from either below or above. If at v_i the sample path crosses θ from below, this event corresponds to the start of a full loss period, therefore $A_{2,i} > 0$. Then, using (2), we have

$$A_{1,i} \ge A_{2,i} > 0$$

since $\alpha_1(v_i) \ge 0$. Therefore,

$$0 < \frac{A_{2,i}}{A_{1,i}} \le 1 \tag{94}$$

If, on the other hand, at v_i the sample path crosses θ from above, this event corresponds to the start of a No Loss period, implying that $A_{1,i} < 0$. Thus, recalling (2), we get

$$A_{2,i} \le A_{1,i} < 0$$

so that

$$0 < \frac{A_{1,i}}{A_{2,i}} \leq 1 \tag{95}$$

For Cases 1 and 2 in the proof of Lemma 4.4, the cycle starts with $x(\theta;\xi) = 0$, thus event times v_{2n-1} correspond to crossing θ from below, while event times $v_{2n}(\theta)$ correspond to crossing θ from above. For Cases 3 and 4 in the proof of Lemma 4.5, the cycle starts with $x(\theta;\xi) = b$, so event times v_{2n-1} correspond to crossing θ from above and event times v_{2n} correspond to crossing θ from below. By using (94) and (95) in (49)–(55) it is easy to see that $-1 < \lambda_1'(\theta) \leq 0$, which completes the proof. ■

Proof of Lemma 4.7: First, if $p_S \notin U_2$, then the first part of (61) immediately follows from (60). Next, consider a cycle with $p_S \in U_2$, so $x(\theta; v_S) = b$. If $x(\theta;\xi) = \theta$, we must have $x(\theta; v_{S-1}) = \theta$ and, by (43), $v_{S-1}' = 0$. Then by (32) we have

$$v_S' = -\frac{1}{A_{2,S}}$$

It follows from (60) that $\lambda_2'(\theta) = 1$, which establishes the second part of (61).

It remains to consider the case $x(\theta;\xi) \neq \theta$. If $S = 1$, we must have $x(\theta;\xi) = b$, which is Case 8 in Section 3.3, and by (31) we have $v_S' = 0$, since $v_0' = \xi' = 0$. It follows from (60) that

$$\lambda_2'(\theta) = 0 \tag{96}$$

This case also corresponds to (54) in Lemma 4.5, giving $\lambda_1'(\theta) = 0$, and we see that (61) is satisfied.

On the other hand, if $S > 1$, during the cycle the buffer content hits θ at least once; thus, p_{S-1} is a full loss period and the buffer content evolves from θ to b with

$$\int_{v_{S-1}}^{v_S} A_2(t)dt = b - \theta$$

Upon taking derivatives, we get

$$A_{2,S}v_S' - A_{2,S-1}v_{S-1}' = -1$$

Therefore, by (60) we get

$$\lambda_2'(\theta) = 1 - A_{2,S-1}v_{S-1}' \tag{97}$$

The cycle satisfies either $x(\theta;\xi) = 0$ or $x(\theta;\xi) = b$. In the former case, (90) in the proof of Lemma 4.4 applies and using Lemma 4.1 to get $A_{1,1}v_1' = 1$, we have

$$\lambda_1'(\theta) = -1 + A_{2,S-1}v_{S-1}'(\theta) = -\lambda_2'(\theta)$$

where the second equality follows from (97). In the latter case, (93) in the proof of Lemma 4.5 applies and using Lemma 4.1 to get $A_{2,1}v_1' = 1$, we get the exact same expression for $\lambda_1'(\theta)$ and we have

$$\lambda_1'(\theta) = -1 + A_{2,S-1}v_{S-1}' = -\lambda_2'(\theta)$$

which satisfies (61) and completes the proof. ■

Proof of Lemma 5.1: Recalling the definition of I_j, we know that there is no event occurring in I_j^o. Thus, within I_j^o, both the original sample path and the perturbed sample path are either at one of the thresholds or bounded between two adjacent thresholds. Next we consider all possible combinations of different types which I_j belongs to in nominal sample path and in the perturbed sample path. There are four cases.

Case 1: For both sample paths, I_j is in a Type I interval. By (72) we have $0 \leq \Delta x(t) = \Delta x(t_{j-1}) \leq \Delta\theta$ for $t \in I_j$.

Case 2: For the nominal sample path, I_j is in a Type I interval, but for the perturbed sample path, it is in a Type II interval. Assuming $x(\theta; t) = T_m$, we have

$$\begin{aligned} &\frac{dx(\theta; t)}{dt^+} = 0 \\ &A_m(t) > 0 \quad \text{and} \quad A_{m+1}(t) < 0 \end{aligned} \tag{98}$$

There are two possibilities regarding T_m:

1. If $T_m = \theta$, then $x(\theta; t) = \theta$ over I_j. By the assumption $0 \leq \Delta x(t_{j-1}) \leq \Delta\theta$, we have $\theta \leq x(\theta + \Delta\theta; t_{j-1}) \leq \theta + \Delta\theta$. Next we prove that $x(\theta + \Delta\theta; t) > \theta + \Delta\theta$ and $x(\theta + \Delta\theta; t) < \theta$ for $t \in I_j^o$ are both impossible. First, assuming $x(\theta + \Delta\theta; t) > \theta + \Delta\theta, t \in I_j^o$, we have

   $$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = B_{m+1}(t) \leq A_{m+1}(t) < 0$$

   but since $x(\theta + \Delta\theta;\ t_{j-1}) \leq \theta + \Delta\theta$, the above implies that

   $$x(\theta + \Delta\theta; t) \leq \theta + \Delta\theta, t \in I_j^o \tag{99}$$

   Next, suppose $x(\theta + \Delta\theta; t) < \theta, t \in I_j^o$. Since $\Delta\theta > 0$, we have $x(\theta + \Delta\theta; t) < \theta + \Delta\theta$, $t \in I_j^o$. It follows that

   $$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = C_m(t) \geq A_m(t) > 0$$

and since $x(\theta + \Delta\theta; t_{j-1}) \geq \theta$, we conclude that

$$x(\theta + \Delta\theta; t) \geq \theta, \quad t \in I_j^o \tag{100}$$

Combining (99) and (100) we have $\theta \leq x(\theta + \Delta\theta; t) \leq \theta + \Delta\theta, t \in I_j^o$. In view of $x(\theta; t) = \theta$ over I_j and the assumption $0 \leq \Delta x(t_{j-1}) \leq \Delta\theta$ we immediately get $0 \leq \Delta x(t) \leq \Delta\theta, t \in I_j$.

2. If $T_m \neq \theta$, we have $x(\theta; t) = T_m \neq \theta$. The perturbed sample path cannot cross T_m in I_j^o, therefore, I_j is *Type II* means the perturbed sample path is either strictly above or strictly below T_m throughout I_j^o. We consider each of these two cases next.

   a. If $x(\theta + \Delta\theta; t) > x(\theta; t) = T_m$ for $t \in I_j^o$, we have $\Delta x(t) > 0$. In addition, since $x(\theta + \Delta\theta; t) > T_m$ we must have

   $$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = B_{m+1}(t) \leq A_{m+1}(t) < 0 \tag{101}$$

   Combining (98) and (101) we have $\Delta x(t) < \Delta x(t_{j-1}) \leq \Delta\theta, t \in I_j^o$. Thus, we get $0 \leq \Delta x(t) \leq \Delta\theta, t \in I_j$.

   b. If $x(\theta + \Delta\theta; t) < x(\theta; t) = T_m$ for $t \in I_j^o$, we have $\Delta x(t) < 0$. Since we assume $\Delta x(t_{j-1}) \geq 0$, by the continuity of the sample path we must have $\Delta x(t_{j-1}) = 0$. In addition, since $x(\theta + \Delta\theta; t) < T_m$, we have

   $$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = C_m(t) \geq A_m(t) > 0 \tag{102}$$

   Combining (98) and (102) we have $\Delta x(t) > \Delta x(t_{j-1}) = 0$. This contradicts our previous conclusion that $\Delta x(t) < 0$. As a result, this case is impossible to occur.

Case 3: For the nominal sample path, I_j is in a Type II interval, but for the perturbed sample path, it is in a Type I interval. Assuming $x(\theta + \Delta\theta; t) = T_m$, we have

$$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = 0 \tag{103}$$
$$A_m(t) > 0 \quad \text{and} \quad A_{m+1}(t) < 0$$

There are two possibilities regarding T_m:

1. If $T_m = \theta$, then in the perturbed path we have $x(\theta + \Delta\theta; t) = \theta + \Delta\theta$ over I_j. By the assumption $0 \leq \Delta x(t_{j-1}) \leq \Delta\theta$, we have $\theta \leq x(\theta; t_{j-1}) \leq \theta + \Delta\theta$. Next, we prove that $x(\theta; t) > \theta + \Delta\theta$ and $x(\theta; t) < \theta$ for $t \in I_j^o$ are both impossible. First, suppose that $x(\theta; t) > \theta + \Delta\theta, t \in I_j^o$. Since $\Delta\theta > 0$, we have $x(\theta; t) > \theta, t \in I_j^o$. Thus,

$$\frac{dx(\theta;t)}{dt^+} = B_{m+1}(t) \le A_{m+1}(t) < 0$$

but since $x(\theta; t_{j-1}) \le \theta + \Delta\theta$, this implies that

$$x(\theta;t) \le \theta + \Delta\theta, \quad t \in I_j^o \tag{104}$$

Next, suppose $x(\theta;t) < \theta, t \in I_j^o$, in which case we must have

$$\frac{dx(\theta;t)}{dt^+} = C_m(t) \ge A_m(t) > 0$$

but since $x(\theta; t_{j-1}) \ge \theta$, we conclude that

$$x(\theta;t) \ge \theta, \quad t \in I_j^o \tag{105}$$

Combining (104) and (105) we have $\theta \le x(\theta;t) \le \theta + \Delta\theta, t \in I_j^o$. In view of $x(\theta + \Delta\theta; t) = \theta + \Delta\theta$ over I_j and the assumption $0 \le \Delta x(t_{j-1}) \le \Delta\theta$ we get $0 \le \Delta x(t) \le \Delta\theta, t \in I_j$.

2. If $T_m \ne \theta$, then in the perturbed sample path we have $x(\theta + \Delta\theta; t) = T_m \ne \theta + \Delta\theta$. The nominal sample path cannot cross T_m in I_j^o, therefore, I_j is Type II means the nominal sample path is either strictly above or strictly below T_m throughout I_j^o. We consider each of these two cases next.

   a. If $x(\theta;t) < x(\theta + \Delta\theta; t) = T_m$ for $t \in I_j^o$, then $\Delta x(t) > 0$. In addition, since $x(\theta;t) < T_m$ we must have

   $$\frac{dx(\theta;t)}{dt^+} = C_m(t) \ge A_m(t) > 0 \tag{106}$$

   Combining (103) and (106) we have $\Delta x(t) < \Delta x(t_{j-1}) \le \Delta\theta, t \in I_j^o$. Therefore we get $0 \le \Delta x(t) \le \Delta\theta, t \in I_j$.

   b. If $x(\theta;t) > x(\theta + \Delta\theta; t) = T_m$ for $t \in I_j^o$, we have $\Delta x(t) < 0$. In addition, by the assumption $\Delta x(t_{j-1}) \ge 0$ and the continuity of the sample path we must have $\Delta x(t_{j-1}) = 0$. Since $x(\theta;t) > T_m$ we must have

   $$\frac{dx(\theta;t)}{dt^+} = B_{m+1}(t) \le A_{m+1}(t) < 0 \tag{107}$$

   Combining (103) and (107) we have $\Delta x(t) > \Delta x(t_{j-1}) = 0$. This yields a contradiction. As a result, this case is impossible to occur.

Case 4: For both sample paths, I_j is in a Type II interval. Assume $T_m < x(\theta;t) < T_{m+1}, t \in I_j^o$. Then,

$$\frac{dx(\theta;t)}{dt^+} = A_{m+1}(t)$$

If $T_m < x(\theta + \Delta\theta; t) < T_{m+1}, t \in I_j^o$ too, then clearly the dynamics of both sample paths are identical and the result is trivial. Thus, let us consider the cases that the perturbed sample path is either above T_{m+1} or below T_m. Note that if either T_m or T_{m+1} is the threshold being perturbed, then its value is θ in the nominal and $\theta + \Delta\theta$ in the perturbed path. Thus, we have two cases to consider.

1. If $x(\theta + \Delta\theta; t) > T_{m+1}$ for $t \in I_j^o$, we have

$$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = B_{m+2}(t)$$

Then, since $A_{m+1}(t) \geq A_{m+2}(t) \geq B_{m+2}(t)$, we get

$$\frac{dx(\theta;t)}{dt^+} \geq \frac{dx(\theta + \Delta\theta; t)}{dt^+}$$

Therefore, $\Delta x(t) \leq \Delta x(t_{j-1}) \leq \Delta\theta, t \in I_j$. Next we prove that $\Delta x(t) \geq 0, t \in I_j$. If T_{m+1} is not the threshold being perturbed, then from $x(\theta;t) < T_{m+1}$ and $x(\theta + \Delta\theta; t) > T_{m+1}$ we immediately get $x(\theta;t) < x(\theta + \Delta\theta; t)$ for $t \in I_j^o$, so $\Delta x(t) > 0$. If T_{m+1} is the threshold which is perturbed, then we have $x(\theta;t) < \theta$ and $x(\theta + \Delta\theta; t) > \theta + \Delta\theta$, so that again we get $x(\theta;t) < x(\theta + \Delta\theta; t)$ for $t \in I_j^o$ in view of $\Delta\theta > 0$, so that $\Delta x(t) > 0$. Therefore, by combining this with $\Delta x(t) \leq \Delta\theta$, $t \in I_j$ and $\Delta x(t_{j-1}) \geq 0$ we get $0 \leq \Delta x(t) \leq \Delta\theta$, $t \in I_j$.

2. If $x(\theta + \Delta\theta; t) < T_m$ for $t \in I_j^o$, we have

$$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = C_m(t) \tag{108}$$

Then, from $A_{m+1}(t) \leq A_m(t) \leq C_m(t)$ we get

$$\frac{dx(\theta;t)}{dt^+} \leq \frac{dx(\theta + \Delta\theta; t)}{dt^+} \tag{109}$$

Therefore, $\Delta x(t) \geq \Delta x(t_{j-1}) \geq 0$, $t \in I_j$. Next, if T_m is not the threshold being perturbed, then $x(\theta + \Delta\theta; t) < T_m < x(\theta;t)$ for $t \in I_j^o$, which implies $\Delta x(t) < 0$. This, however, contradicts $\Delta x(t) \geq 0$, $t \in I_j$ which we just established, therefore this case is impossible to occur. Thus, T_m must be the threshold which is perturbed if this case arises, so we have $x(\theta;t) > \theta$, and $x(\theta + \Delta\theta; t) < \theta + \Delta\theta$ for $t \in I_j^o$, which implies $\Delta x(t) < \Delta\theta$ for $t \in I_j^o$. Therefore, by combining with $\Delta x(t) \geq 0, t \in I_j$ and $\Delta x(t_{j-1}) \leq \theta$ we get $0 \leq \Delta x(t) \leq \Delta\theta$, $t \in I_j$.

Since we have shown that $0 \le \Delta x(t) \le \Delta\theta,\ t \in I_j$ for all feasible cases, the proof is complete. ■

Proof of Lemma 5.2: The proof is by induction over all $I_j, j = 1, \ldots, J(T)$. We have $\Delta x(t_0) = 0$, therefore, using Lemma 5.1, we get $0 \le \Delta x(t) \le \Delta\theta$ for all $t \in I_1$. Next, we assume that

$$0 \le \Delta x(t) \le \Delta\theta, \quad \text{for all } t \in I_j \tag{110}$$

We have already shown that the result holds for I_1, so it remains to show that $0 \le \Delta x(t) \le \Delta\theta$ for all $t \in I_{j+1}$. Using the continuity of the sample path, from (110) we have $0 \le \Delta x(t_j) \le \Delta\theta$. In view of Lemma 5.1, we get $0 \le \Delta x(t) \le \Delta\theta$ for all $t \in I_{j+1}$ and complete the proof. ■

Proof of Theorem 5.1: We start with $L'_{m,T}(\theta)$ and recall that

$$L_{m,T}(\theta) = \int_0^T \gamma_m(\theta; t)dt$$

Using our definition of time intervals $I_j = [t_{j-1}, t_j), j = 1, \ldots, J(T)$ as before, we have

$$L_{m,T}(\theta) = \sum_{j=1}^{J(T)} \int_{t_{j-1}}^{t_j} \gamma_m(\theta; t)dt$$

so that

$$\Delta L_{m,T} = \sum_{j=1}^{J(T)} \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \tag{111}$$

where $\Delta L_{m,T} = L_{m,T}(\theta + \Delta\theta) - L_{m,T}(\theta)$, and $\Delta\gamma_m(t) = \gamma_m(\theta + \Delta\theta; t) - \gamma_m(\theta; t)$.

For any time interval I_j, the nominal and perturbed sample paths can each be (i) above T_m, (ii) at T_m, and (iii) below T_m. So there are nine cases to consider regarding the joint state of the sample paths in I_j:

Case 1: Both nominal and perturbed sample paths are below T_m. There is no class m loss, so we have

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = 0 \tag{112}$$

Case 2: Both nominal and perturbed sample paths are above T_m. Class m experiences full loss in both sample paths, so we also have

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = 0 \tag{113}$$

Case 3: Both nominal and perturbed sample paths are at T_m. We have $\gamma_m(\theta;t) = \gamma_m(\theta + \Delta\theta;t) = A_m(t)$, so that

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = 0 \tag{114}$$

Case 4: The nominal sample path is below T_m while the perturbed sample path is above T_m. Class m experiences no loss in the nominal sample path and full loss in the perturbed sample path, so we have $\gamma_m(\theta) = 0$ and $\gamma_m(\theta + \Delta\theta;t) = \alpha_m(t)$. Therefore

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = \int_{t_{j-1}}^{t_j} \alpha_m(t)dt \geq 0 \tag{115}$$

In this case, we also have

$$\frac{dx(\theta;t)}{dt^+} = C_m(t)$$
$$\frac{dx(\theta + \Delta\theta;t)}{dt^+} = B_{m+1}(t)$$

Therefore,

$$\frac{dx(\theta;t)}{dt^+} - \frac{dx(\theta + \Delta\theta;t)}{dt^+} = C_m(t) - B_{m+1}(t) \geq \alpha_m(t)$$

so that

$$\Delta x(t_{j-1}) - \Delta x(t_j) \geq \int_{t_{j-1}}^{t_j} \alpha_m(t)dt$$

and it follows that

$$\Delta\theta \geq \Delta x(t_{j-1}) \geq \Delta x(t_j) + \int_{t_{j-1}}^{t_j} \alpha_m(t)dt = \Delta x(t_j) + \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \geq \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \tag{116}$$

where the last inequality is due to $\Delta x(t_j) \geq 0$ from Lemma 5.2. Combining (115) and (116) we get

$$0 \leq \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \leq \Delta\theta \tag{117}$$

Case 5: The nominal sample path is above T_m while the perturbed sample path is below T_m. In view of $0 \leq \Delta x(t) \leq \Delta\theta$ for $t \in [0, T]$ in Lemma 5.2, the only possibility for this case is that T_m is the threshold being perturbed, that is, $x(\theta; t) > \theta$ and $x(\theta + \Delta\theta; t) < \theta + \Delta\theta$. In this case, class m experiences full loss in the nominal sample path and no loss in the perturbed sample path, so we have $\gamma_m(\theta) = \alpha_m(t)$ and $\gamma_m(\theta + \Delta\theta; t) = 0$. Therefore

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = -\int_{t_{j-1}}^{t_j} \alpha_m(t)dt \leq 0 \tag{118}$$

In addition, we have

$$\frac{dx(\theta; t)}{dt^+} = B_{m+1}(t)$$
$$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = C_m(t)$$

Therefore,

$$\frac{dx(\theta + \Delta\theta; t)}{dt^+} - \frac{dx(\theta; t)}{dt^+} = C_m(t) - B_{m+1}(t) \geq \alpha_m(t)$$

so that

$$\Delta x(t_j) - \Delta x(t_{j-1}) \geq \int_{t_{j-1}}^{t_j} \alpha_m(t)dt$$

therefore we get

$$\Delta\theta \geq \Delta x(t_j) \geq \Delta x(t_{j-1}) + \int_{t_{j-1}}^{t_j} \alpha_m(t)dt = \Delta x(t_{j-1}) - \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \geq -\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \tag{119}$$

where the last inequality is due to $\Delta x(t_{j-1}) \geq 0$ from Lemma 5.2. Combining (118) and (119) we get

$$-\Delta\theta \leq \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \leq 0 \tag{120}$$

Case 6: The nominal sample path is at T_m while the perturbed sample path is above T_m. Class m experiences partial loss in the nominal sample path and full loss in the perturbed sample path, so we have $\gamma_m(\theta) = A_m(t), \gamma_m(\theta + \Delta\theta; t) = \alpha_m(t)$ and $A_{m+1}(t) < 0$. Therefore

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = \int_{t_{j-1}}^{t_j} \alpha_m(t)dt - \int_{t_{j-1}}^{t_j} A_m(t)dt = -\int_{t_{j-1}}^{t_j} A_{m+1}(t)dt > 0 \tag{121}$$

In this case, we also have

$$\frac{dx(\theta + \Delta\theta; t)}{dt^+} = B_{m+1}(t)$$
$$\frac{dx(\theta; t)}{dt^+} = 0$$

so we get

$$\Delta x(t_j) - \Delta x(t_{j-1}) = \int_{t_{j-1}}^{t_j} B_{m+1}(t)dt$$

In view of

$$\int_{t_{j-1}}^{t_j} B_{m+1}(t)dt \leq \int_{t_{j-1}}^{t_j} A_{m+1}(t)dt$$

we get

$$\begin{aligned}\Delta\theta &\geq \Delta x(t_{j-1}) \geq \Delta x(t_{j-1}) - \Delta x(t_j)\\ &= -\int_{t_{j-1}}^{t_j} B_{m+1}(t)dt \geq -\int_{t_{j-1}}^{t_j} A_{m+1}(t)dt = \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt\end{aligned} \tag{122}$$

where the second inequality is due to $\Delta x(t_j) \geq 0$ from Lemma (5.2). Combining (121) and (122) we get

$$0 < \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \leq \Delta\theta \tag{123}$$

Case 7: The nominal sample path is at T_m while the perturbed sample path is below T_m. In view of $0 \leq \Delta x(t) \leq \Delta\theta$ for $t \in [0, T]$ in Lemma 5.2, the only possibility for this case is that T_m is the threshold being perturbed, that is, $x(\theta; t) = \theta$ while $\theta \leq x(\theta + \Delta\theta; t) < \theta + \Delta\theta$. In this case, class m experiences partial loss in the nominal sample path and no loss in the perturbed sample path, so we have $\gamma_m(\theta) = A_m(t), \gamma_m(\theta + \Delta\theta; t) = 0$ and $A_m(t) > 0$. Therefore

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = -\int_{t_{j-1}}^{t_j} A_m(t)dt < 0 \tag{124}$$

In this case we also have

$$\frac{dx(\theta+\Delta\theta;t)}{dt^+} = A_m(t)$$
$$\frac{dx(\theta;t)}{dt^+} = 0$$

so we get

$$\Delta x(t_j) - \Delta x(t_{j-1}) = \int_{t_{j-1}}^{t_j} A_m(t)dt$$

and it follows that

$$\Delta\theta \geq \Delta x(t_j) \geq \Delta x(t_j) - \Delta x(t_{j-1}) = \int_{t_{j-1}}^{t_j} A_m(t)dt = -\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \tag{125}$$

where the second inequality is due to $\Delta x(t_{j-1}) \geq 0$ from Lemma 5.2. Combining (124) and (125) we get

$$-\Delta\theta \leq \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt < 0 \tag{126}$$

Case 8: The nominal sample path is above T_m while the perturbed sample path is at T_m. In view of $0 \leq \Delta x(t) \leq \Delta\theta$ for $t \in [0, T]$ in Lemma 5.2, the only possibility for this case is that T_m is the threshold being perturbed, that is, $x(\theta+\Delta\theta;t) = \theta+\Delta\theta$ while $\theta < x(\theta;t) \leq \theta+\Delta\theta$. In this case, class m experiences full loss in the nominal sample path and partial loss in the perturbed sample path, so we have $\gamma_m(\theta) = \alpha_m(t), \gamma_m(\theta+\Delta\theta;t) = A_m(t)$ and $A_{m+1}(t) < 0$. Therefore

$$\begin{aligned}\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt &= \int_{t_{j-1}}^{t_j} A_m(t)dt - \int_{t_{j-1}}^{t_j} \alpha_m(t)dt \\ &= \int_{t_{j-1}}^{t_j} A_{m+1}(t)dt < 0 \end{aligned} \tag{127}$$

In this case, we also have

$$\frac{dx(\theta+\Delta\theta;t)}{dt^+} = 0$$
$$\frac{dx(\theta;t)}{dt^+} = A_{m+1}(t)$$

so we get

$$\Delta x(t_j) - \Delta x(t_{j-1}) = -\int_{t_{j-1}}^{t_j} A_{m+1}(t)dt$$

and it follows that

$$\Delta\theta \geq \Delta x(t_j) \geq \Delta x(t_j) - \Delta x(t_{j-1}) = -\int_{t_{j-1}}^{t_j} A_{m+1}(t)dt = -\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \qquad (128)$$

where the second inequality is due to $\Delta x(t_{j-1}) \geq 0$ from Lemma 5.2. Combining (127) and (128) we get

$$-\Delta\theta \leq \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt < 0 \qquad (129)$$

Case 9: The nominal sample path is below T_m while the perturbed sample path is at T_m. Class m experiences no loss in the nominal sample path and partial loss in the perturbed sample path, so we have $\gamma_m(\theta) = 0, \gamma_m(\theta+\Delta\theta;t) = A_m(t)$ and $A_m(t) > 0$. Therefore

$$\int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt = \int_{t_{j-1}}^{t_j} A_m(t)dt > 0 \qquad (130)$$

In this case, we also have

$$\frac{dx(\theta+\Delta\theta;t)}{dt^+} = 0$$
$$\frac{dx(\theta;t)}{dt^+} = C_m(t)$$

so we get

$$\Delta x(t_{j-1}) - \Delta x(t_j) = \int_{t_{j-1}}^{t_j} C_m(t)dt$$

therefore,

$$\Delta\theta \geq \Delta x(t_{j-1}) \geq \Delta x(t_{j-1}) - \Delta x(t_j) = \int_{t_{j-1}}^{t_j} C_m(t)dt \geq \int_{t_{j-1}}^{t_j} A_m(t)dt = \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \tag{131}$$

where the second inequality is due to $\Delta x(t_j) \geq 0$ from Lemma 5.2. Combining (130) and (131) we get

$$0 < \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \leq \Delta\theta \tag{132}$$

We may now combine the nine cases above, i.e., the inequalities (112)–(114), (117), (120), (123), (126), (129), (132) with (111) and obtain

$$\left|\Delta L_{m,T}\right| \leq \sum_{j=1}^{J(T)} \left| \int_{t_{j-1}}^{t_j} \Delta\gamma_m(t)dt \right| \leq J(T)|\Delta\theta|$$

Next, we consider $Q'_T(\theta)$ for which we can write

$$Q_T(\theta) = \int_0^T x(\theta; t)dt$$

therefore by Lemma 5.2 we have

$$|\Delta Q_T(\theta)| = \left| \int_0^T \Delta x(\theta; t)dt \right| \leq T|\Delta\theta|$$

This completes the proof. ■

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