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Optimal scheduling of parallel queues using stochastic flow models

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Abstract We consider a classic scheduling problem for optimally allocating a resource to multiple competing users and place it in the framework of Stochastic Flow Models (SFMs). We derive Infinitesimal Perturbation Analysis (IPA) gradient estimators for the average holding cost with respect to resource allocation parameters. These estimators are easily obtained from a sample path of the system without any knowledge of the underlying stochastic process characteristics. Exploiting monotonicity properties of these IPA estimators, we prove the optimality of the well-known $c\mu$-rule for an arbitrary finite number of queues and stochastic processes under non-idling policies and linear holding costs. Further, using the IPA derivative estimates obtained along with a gradient-based optimization algorithm, we find optimal solutions to similar problems with nonlinear holding costs extending current results in the literature.

Keywords Hybrid systems · Discrete-event systems · Stochastic flow models · Perturbation analysis · Scheduling algorithms

1 Introduction

The classic prototypical stochastic scheduling problem involves a single resource whose service capacity is to be optimally shared by N competing users. In a queueing

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theory framework, this problem is modeled as a system of N parallel queues, each with its own arrival process, connected to a single server. The server processes tasks from the nth queue with rate μ_n, $n = 1, \ldots, N$, and uses a policy to select the next queue to serve from. Each task requires a random amount of time to be processed. The server may preempt a task by interrupting its processing to serve a new task from some other queue. This model applies to a large group of applications in communication networks, manufacturing, and computer processing.

The usual objective in this scheduling problem is to minimize the overall average holding cost of the tasks in the system with c_n denoting the cost per unit waiting time in the nth queue. When the holding cost is a linear combination of the number of tasks in the competing queues, the well-known $c\mu$-rule has been shown, under certain conditions, to give the optimal allocation sequence. Following this rule, the queues are ordered according to the value of the product $c_n\mu_n$, from largest to smallest, and the server always selects a task from the first non-empty queue with the largest $c_n\mu_n$ value. The $c\mu$-rule is very attractive in that it is essentially static, except for the knowledge of whether a queue is empty or not. Thus, establishing its optimality in the most general possible setting is a goal that has been actively pursued through many years. The optimality of the $c\mu$-rule seems to have been first suggested in Smith (1956) under a deterministic and static setting, i.e., all tasks are present at time 0 with fixed service times. Relaxing these assumptions, Cox and Smith (1961) later proved the optimality of the $c\mu$-rule for a multi-class $M/G/1$ system. Using classical queueing models in a discrete time setting, the $c\mu$-rule was shown to be optimal for general arrival processes and geometrically distributed service times in Baras et al. (1985) and Buyukkoc et al. (1985). There have since been various attempts to extend these results. For example, it is shown in Hirayama et al. (1989) that for a discrete time $G/G/1$ model with a non-idling and non-preemptive server with decreasing failure rate the $c\mu$-rule is still optimal. Along a different direction, the scheduling problem above has been studied using a fluid flow abstraction in both a deterministic context (Avram et al. 1995; Chen and Yao 1993) and a stochastic setting where the optimality of the $c\mu$-rule can be obtained using the heavy traffic (fluid limit) arguments (Kingman 1961; Whitt 1968; Harrison 1986). A "generalized" $c\mu$-rule can then be shown to be asymptotically optimal (Mieghem 1995) not only for the linear but also for convex cost objectives.

In Kebarighotbi and Cassandras (2009), we studied the basic stochastic scheduling problem above using a *Stochastic Fluid* (or *Flow*) *Model* (SFM). Using this model, it was shown that the $c\mu$-rule is optimal in the two-queue case, extending previous results in the literature to a broader class of stochastic processes and without any heavy traffic conditions. Further, when the cost is nonlinear in the queue contents, it was shown that the $c\mu$-rule may no longer be optimal. Nevertheless, the techniques used were based on easily obtainable gradient estimators that can be used to find an optimal allocation policy.

Unlike a deterministic fluid model or a stochastic model that makes use of heavy traffic assumptions, an SFM treats the arrival and service *rates* as stochastic processes of arbitrary generality (except for mild technical conditions), even under light traffic. The value of SFMs (introduced in Cassandras et al. (2002)) lies not in deriving approximations of performance measures of the underlying discrete event system, but rather in studying sample paths from which one can derive structural properties and optimal policies by making use of *Infinitesimal Perturbation Analysis*

(IPA). IPA provides derivative estimates of performance measures (e.g., workload, loss) with respect to controllable parameters. These estimates are independent of the probability laws of the stochastic rate processes, therefore the actual values of these processes never enter the estimators, except (on occasion) for instantaneous rates at certain observable event times. Moreover, they require minimal information from the observed sample path. These properties, including the unbiasedness of the estimators, were established in Cassandras et al. (2002) for queueing systems with finite capacity and extended to serial networks (Sun et al. 2004), systems with feedback control mechanisms (Yu and Cassandras 2006), and some multi-class models (Sun et al. 2004; Cassandras et al. 2003; Panayiotou 2004).

This paper starts by extending the results in Kebarighotbi and Cassandras (2009) to establish the optimality of the $c\mu$-rule to $N > 2$ queues (which has proved to be a somewhat surprising challenge.) Its contributions include the following. First, because our results are independent of the stochastic nature of the arrival and service rate processes, they provide evidence of the generality of the $c\mu$-rule as an optimal policy. In fact, our analysis is based on an arbitrary sample path of the system, confirming prior conjectures that the optimality of the $c\mu$-rule is a property relying on the system and cost structure, not its stochastic characteristics. We should point out, however, that our analysis relies on showing that perturbing away from the $c\mu$-rule policy increases the average linear holding cost, thus, it is possible that there exist other optimal policies. This is not surprising and it arises in other proofs of the $c\mu$-rule as well (intuitively, one can see that more elaborate scheduling policies can indeed reproduce the cost accumulated under the $c\mu$-rule whose main appeal is its simplicity). Second, the analysis is based on explicit sample state and event time derivatives which can, therefore, be used to determine optimal schedules for different cost structures and be extended to more complex scheduling problems (e.g., systems with loss due to finite capacities or problems with nonlinear costs) where the $c\mu$-rule no longer applies. In fact, we have used IPA derivative estimates to find an optimal solution for a nonlinear cost metric which substantially outperforms the solution obtained through the $c\mu$-rule.

In Section 2, the basic scheduling problem is formulated in a SFM setting. In Section 3, using IPA we first prove the optimality of the $c\mu$-rule for two queues using a different approach than that of Kebarighotbi and Cassandras (2009) which subsequently allows the generalization to $N > 2$ queues by using this result as a base case for an induction argument. In Section 4 we provide the general IPA scheme for estimating cost gradients. We then use it in simulation examples to show how we can recover the $c\mu$-rule in the case of linear holding costs and, for nonlinear costs, to obtain scheduling policies that are optimal within the class of policies that allocate fractions of the service capacity to each queue.

2 Problem formulation

Consider a SFM comprised of N queues competing for a shared resource as shown in Fig. 1. We will be studying this system over a finite time interval $[0, T]$. User requests from different classes $n = 1, \ldots, N$ are abstracted into uncontrollable *inflows* $\{\alpha_n(t)\}$ capturing the instantaneous *rate* of arriving tasks and treated as random processes. The associated fluid content random processes are denoted by $\{x_n(t)\}$ with $x_n(t) \geq 0$.

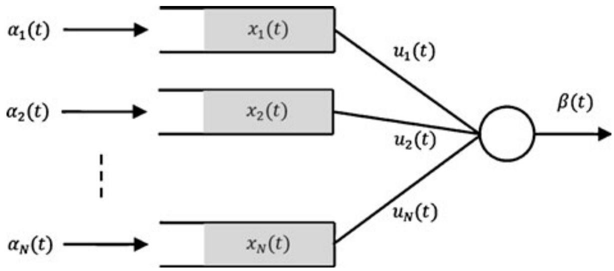


Fig. 1 Stochastic Flow Model (SFM) for the scheduling problem

At each time t, the rate at which the resource is processing the fluid from the queue n is denoted by $u_n(t) \in [0, \mu_n]$. Here, $\mu_n > 0$ denotes the maximum processing rate for queue n. The processing rates are subject to the capacity constraint

$$\sum_{n=1}^{N} \frac{u_n(t)}{\mu_n} \le 1, \quad \forall\, t \in [0, T]. \tag{1}$$

The *outflow* rate from the resource is denoted by $\{\beta(t)\}$ and is defined as $\beta(t) = \sum_{n=1}^{N} u_n(t)$ for all t. All random processes in the SFM are defined on a common probability space $(\Omega, \mathcal{F}, P)$. We define control functions $\theta_n(t) \in [0, 1]$ to represent the maximum fraction of μ_n at which the resource processes the fluid from the nth queue. Therefore, we have $u_n(t) \le \mu_n \theta_n(t)$ for all t.

Viewed as a stochastic hybrid system, each queue can only be in one of two discrete states: either $x_n(t) = 0$ over some *Empty Period* (EP) or $x_n(t) > 0$ over some *Non-Empty Period* (NEP). We assume that all the EPs and NEPs are close-left and open-right intervals. Given a sample path over $[0, T]$, we define Σ_n to be the set of all NEP start times and Γ_n to be the set of all NEP end times for queue $n = 1, \dots, N$. Regarding the controllability of event times in the SFM, they can be divided into two categories: (1) *Exogenous events* which are due to an uncontrollable discontinuity in some inflow rate and are assumed locally independent of $\theta_n(t)$, and (2) *Endogenous events* whose occurrence times can be controlled by $\theta_n(t)$ for some n. There are two further cases possible for an exogenous event: (a) an event that changes the discrete state of queue n by initiating a NEPn and (b) one due to a possible discontinuity in $\alpha_n(t)$ which leaves the discrete state intact. We denote the set of all event times in category (b) by Λ_n. Also, there are two possibilities for an endogenous event: (a) one that ends a NEP at some queue n and (b) one that starts a NEP at some time t but is not caused by any uncontrolled discontinuity in the inflow rates. We make the following mild technical assumptions.

Assumption 1 With probability 1, no two events can occur at the same time unless the occurrence of one causes that of the other.

Assumption 2 The inflow processes $\{\alpha_n(t)\}$ are piecewise continuous in $[0, T]$, with a finite number of discontinuities.

The queue contents $x_n(t;\theta_n(t))$, $n = 1, \ldots, N$ evolve according to the one-sided differential equations

$$\frac{dx_n(t;\theta_n(t))}{dt^+} = f_n(t;\theta_n(t)) = \alpha_n(t) - u_n(t;\theta_n(t)) \tag{2}$$

where, according to the definition of the rate processes, we can write for any $n = 1, \ldots, N$ and $t \in [0, T]$:

$$u_n(t;\theta_n(t)) = \begin{cases} \alpha_n(t) & \text{if } x_n(t;\theta_n(t)) = 0 \\ & \text{and } \alpha_n(t) \le \mu_n\theta_n(t) \\ \mu_n\theta_n(t) & \text{otherwise} \end{cases} \tag{3}$$

In what follows, we drop $\theta_n(t)$ from the arguments of the functions $u_n(t;\theta_n(t))$, $x_n(t;\theta_n(t))$ and $f_n(t;\theta_n(t))$ to keep the notation manageable. Notice that by Eqs. 2 and 3, we can write

$$f_n(t) = \begin{cases} 0 & \text{if } x_n(t;\theta_n(t)) = 0 \\ & \text{and } \alpha_n(t) \le \mu_n\theta_n(t) \\ \alpha_n(t) - \mu_n\theta_n(t) & \text{otherwise} \end{cases} \tag{4}$$

Let us consider a sample path $\omega \in \Omega$ generated under some fixed functions $\theta_n(t)$, $n = 1, \ldots, N$. We define $0 = t_0 < t_1 < \ldots < t_M = T$ to be the occurrence times of all events that either start or end NEPs over all queues in the interval $(0, T)$ with the addition of the points 0 and T. Notice that M generally depends on the functions $\theta_n(t)$. We further define:

$$\theta_{n,m}(t) = \theta_n(t), \quad t \in [t_m, t_{m+1}),\ m = 0, \ldots, M-1 \tag{5}$$

to be the control used over each interval $[t_m, t_{m+1})$ between any two successive events that start/end some NEP.

The cost objective for any sample path $\omega \in \Omega$ is the *total holding cost* and takes the following general nonlinear form:

$$Q(\theta,\omega) = \int_0^T \sum_{n=1}^N c_n g_n(x_n(t,\theta(t),\omega))dt \tag{6}$$

where $c_n > 0$ is a cost rate associated with the queue n and $g_n(\cdot)$ is a differentiable and generally nonlinear function in $x_n(t)$ over its domain. For any $n = 1, \ldots, N$ we assume that $g_n(0) = 0$ and $g_n(x_n(t)) > 0$ whenever $x_n(t) > 0$. Notice that when $g_n(x_n(t)) = x_n(t)$ for all $t \in [0, T]$, the objective becomes a linear total holding cost for which, as discussed in Section 1, it is shown that the $c\mu$-rule determines the optimal allocation policy under many settings. Regardless of the cost objective being linear or nonlinear, we let the queues be indexed according to their $c\mu$ products in descending order, i.e.,

$$c_1\mu_1 > c_2\mu_2 > \ldots > c_N\mu_N. \tag{7}$$

Motivated by the fact that our goal is the minimization of the total holding cost, we assume that all the resource capacity is used if at least one queue is non-empty. Thus,

$$\sum_{n=1}^N \theta_{n,m}(t) = 1,\ t \in [t_m, t_{m+1}) \quad \text{if } \exists n : x_n(t) > 0. \tag{8}$$

In this case we can write $\theta_{N,m}(t) = 1 - \sum_{n=1}^{N-1} \theta_{n,m}(t)$ and define the controllable vector:

$$\theta(t) = [v_0(t), \ldots, v_{M-1}(t)]$$

with $v_m(t) = [\theta_{1,m}(t), \ldots, \theta_{N-1,m}(t)]$.

Henceforth, we omit ω from the arguments of all processes with the understanding that our full analysis is carried out over some arbitrary but fixed sample path associated with it.

3 Infinitesimal perturbation analysis (IPA)

Let us rewrite Eq. 6 as

$$Q(\theta) = \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \sum_{n=1}^{N} c_n g_n(x_n(t)) dt.$$

Differentiating with respect to $\theta_{i,m}(t)$ for any pair of indices $i = 1, \ldots, N-1$ and $m = 0, \ldots, M-1$ gives

$$\frac{\partial Q(\theta)}{\partial \theta_{i,m}} = \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \sum_{n=1}^{N} c_n \frac{\partial g_n(x_n(t))}{\partial x_n} \frac{\partial x_n(t)}{\partial \theta_{i,m}} dt. \tag{9}$$

where $\frac{\partial g_n(x_n(t))}{\partial x_n}$ is a known function. By Eqs. 4 and 5 the control $\theta_{i,m}(t)$ can only affect the queue content evolutions for $t \geq t_m$, so we conclude that

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0 \quad \forall t < t_m. \tag{10}$$

However, the IPA derivative (Eq. 9) still requires evaluating the derivatives $\frac{\partial x_n(t)}{\partial \theta_{i,m}}$, $n = 1, \ldots, N$ for $t \geq t_m$. Integrating Eq. 4 for any n and $t \in [t_k, t_{k+1})$ with $k \geq m$ yields

$$x_n(t) = x_n(t_k) + \int_{t_k}^{t} f_n(\tau) d\tau. \tag{11}$$

Note that within (t_k, t_{k+1}) a number of events corresponding to discontinuities in arrival rate processes may occur. Thus, let $\tau_{k,1}, \ldots, \tau_{k,R_k}$, with $\tau_{k,l} \in \cup_{n=1}^{N} \Lambda_n$, be the associated event times and set $\tau_{k,0} = t_k$ and $\tau_{k,R_k+1} = t_{k+1}$ for convenience. Suppose $t \in [\tau_{k,l^*}, \tau_{k,l^*+1})$ for some $l^* \in \{0, \ldots, R_k\}$, so that

$$x_n(t) = x_n(t_k) + \sum_{l=0}^{l^*-1} \int_{\tau_{k,l}}^{\tau_{k,l+1}} f_n(\tau) d\tau + \int_{\tau_{k,l^*}}^{t} f_n(\tau) d\tau.$$

Next, define, for any function $h(t)$, the limits $h(t^+) = \lim_{\tau \downarrow t} h(\tau)$ and $h(t^-) = \lim_{\tau \uparrow t} h(\tau)$. Differentiating the term above with respect to $\theta_{i,m}(t)$ gives

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(t_k^-)}{\partial \theta_{i,m}} + \sum_{l=0}^{l^*} \frac{\partial \tau_{k,l}}{\partial \theta_{i,m}} \left[f_n\left(\tau_{k,l}^-\right) - f_n\left(\tau_{k,l}^+\right) \right] + \int_{\tau_{k,l^*}}^{t} \frac{\partial f_n(\tau)}{\partial \theta_{i,m}} d\tau.$$

All events occurring at $\tau_{k,l}$, $l = 1, \ldots, l^*$ are exogenous and independent of $\theta_{i,m}(t)$, hence $\frac{\partial \tau_{k,l}}{\partial \theta_{i,m}} = 0$, leaving only the term corresponding to $\tau_{k,0} = t_k$. Thus, we find

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(t_k^-)}{\partial \theta_{i,m}} + \frac{\partial t_k}{\partial \theta_{i,m}} \left[f_n\left(t_k^-\right) - f_n\left(t_k^+\right)\right] + \int_{t_k}^{t} \frac{\partial f_n(\tau)}{\partial \theta_{i,m}} d\tau. \quad (12)$$

Observe that in Eq. 12 the integrands $\frac{\partial f_n(\tau)}{\partial \theta_{i,m}}$ are readily obtained from Eq. 4 depending on information about $x_n(\tau)$ for $\tau \in [t_k, t)$. To use Eq. 12, we also need to evaluate the event time derivatives $\frac{\partial t_k}{\partial \theta_{i,m}}$. Whenever at t_k the associated event is exogenous, we have:

$$\frac{\partial t_k}{\partial \theta_{i,m}} = 0 \quad \text{if } t_k \text{ is exogenous.} \quad (13)$$

However, if $t_k \in \Gamma_n$ for some n, we have $x_n(t_k) = 0$. Differentiating both sides with respect to $\theta_{i,m}(t)$, we get

$$\frac{\partial x_n(t_k^-)}{\partial \theta_{i,m}} + \frac{\partial x_n(t)}{\partial t} \frac{\partial t}{\partial \theta_{i,m}}\Big|_{t=t_k^-} = 0$$

which results in

$$\frac{\partial t_k}{\partial \theta_{i,m}} = -\frac{\partial x_n(t_k^-)/\partial \theta_{i,m}}{f_n(t_k^-)} \quad \text{if } t_k \in \Gamma_n. \quad (14)$$

If, on the other hand, $t_k \in \Sigma_n$, the following result shows t_k does not affect the derivative (Eq. 12).

Lemma 1 *Let t_k be the start of a NEP for some queue n. Then, $\frac{\partial t_k}{\partial \theta_{i,m}}[f_l(t_k^-) - f_l(t_k^+)] = 0$ for all pairs (i, m) and $l = 1, 2, \ldots, N$.*

Proof See Appendix. □

In the next two sections we assume $g_n(x_n(t)) = x_n(t)$ for all $n = 1, \ldots, N$, thus limiting ourselves to the linear holding cost

$$Q(\theta) = \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \sum_{n=1}^{N} c_n x_n(t) dt. \quad (15)$$

Accordingly, we investigate the optimality of the $c\mu$-rule using the SFM and IPA framework extending the available results in the literature.

In this section, we will first limit ourselves to a SFM with two queues. This was analyzed by Kebarighotbi and Cassandras (2009) where the sample performance function $Q(\theta)$ was expressed in terms of a single parameter θ, defined as the fraction of resource capacity allocated to queue 1. An explicit IPA derivative $dQ/d\theta$ was derived and it was shown that if $c_1\mu_1 > c_2\mu_2$ then $dQ/d\theta < 0$. This result applies to any sample path, leading to the conclusion that $\theta^* = 1$ and the $c\mu$-rule is therefore optimal. In order to extend this result to an arbitrary number of queues, we use the setting proposed in the previous section, slightly different from the one in

Kebarighotbi and Cassandras (2009), which allows us to re-derive the IPA derivative and the optimality of the $c\mu$-rule in a simpler way. Our result will then be used in the following section as a base step for an inductive argument generalizing the optimality of the $c\mu$-rule to any $N > 2$ queues while also providing explicit IPA derivatives as performance gradient estimators in other problems where the $c\mu$-rule is no longer optimal. We prove the $c\mu$-rule based on a backward recursion in time in the spirit of a dynamic programming argument. Let us define

$$q(t) = \sum_{n=1}^{N} c_n x_n(t), \quad \forall t \in [0, T]$$

and introduce the following recursion for the cost in Eq. 15:

$$Q_M = 0 \tag{16a}$$

$$Q_m = h_{m+1}(v_0, \ldots, v_m) + Q_{m+1}, \tag{16b}$$

where

$$h_{m+1}(v_0, \ldots, v_m) = \int_{t_m}^{t_{m+1}} q(t, \theta(t))dt,$$

$$Q_0 = Q(\theta).$$

where, for any $m \in \{0, \ldots, M-1\}$, the function $h_{m+1}(\cdot)$ is dependent on the past controls $v_0, \ldots, v_m$. Additionally, the recursion defined by Eqs. 16a and 16b implies that the functions $Q_m(\cdot)$ are generally dependent on $v_0, \ldots, v_{M-1}$. In the sequel, we omit the control vectors from the arguments of the functions $h_m(\cdot)$, $Q_m(\cdot)$ and $Q(\cdot)$ for brevity. Noting that the control vector v_m can only affect the queue contents in the interval $[t_m, T)$, we can use Eqs. 16a and 16b to minimize $Q(\cdot)$ through the following (dynamic programming) recursion:

$$Q_M^* = 0 \tag{17a}$$

$$Q_m^* = \min_{v_m} \left\{ h_{m+1} + Q_{m+1}^* \right\}, \; m = 1, \ldots, M-1 \tag{17b}$$

where $Q_0^* = \min_{v_1, \ldots, v_{M-1}} Q(v_0, \ldots, v_{M-1})$. In simple terms, minimization of $Q(\cdot)$ boils down to recursively finding the optimal controls $v_{m+1}, \ldots, v_{M-1}$ for the interval $[t_{m+1}, T)$ and the best v_m for the interval $[t_m, t_{m+1})$ assuming that the optimal controls for the interval $[t_{m+1}, T)$ have already been implemented. Notice that when $N = 2$, $v_m(t) = \theta_{1,m}(t)$ for any $m = 0, \ldots, M-1$.

3.1 IPA for two-queue system and the $c\mu$-rule

For any interval $[t_m, t_{m+1})$ let N_m be the number of non-empty queues in it. Using IPA, we establish the optimality of the $c\mu$-rule for two queues through the following theorem.

Theorem 1 *The $c\mu$-rule minimizes the cost (Eq. 15) for a two-queue system with dynamics (Eq. 4), controls (Eq. 5) and constraint 8.*

Proof We start with the last interval $[t_{M-1}, t_M)$. We then use the recursions 17a and 17b to extend the optimality of the $c\mu$-rule to the whole sample path. Considering the interval $[t_{M-1}, t_M)$, the following four cases are possible:

Case 1 If $N_{M-1} = 0$ by Eq. 3, it is trivially concluded that any pair $(\theta_{1,M-1}(t), \theta_{2,M-1}(t))$ keeping the queue contents at zero is optimal. In other words, if $\theta_{1,M-1}(t) \geq \frac{\alpha_1(t)}{\mu_1}$ and $\theta_{2,M-1}(t) \geq \frac{\alpha_2(t)}{\mu_2}$, the allocation parameters $\theta_{1,M-1}(t)$ and $\theta_{2,M-1}(t)$ are optimal for any $t \in [t_{M-1}, t_M)$. The pair

$$\theta^*_{1,M-1}(t) = \frac{\alpha_1(t)}{\mu_1}, \quad \theta^*_{2,M-1}(t) = \frac{\alpha_2(t)}{\mu_2}$$

for any $t \in [t_{M-1}, t_M)$ is a special case in this set of solutions. This is consistent with the $c\mu$-rule.

Case 2 If $N_{M-1} = 1$ and $x_1(t) > 0$ for all $t \in [t_{M-1}, t_M)$, by Eq. 8, we have a one-dimensional problem with control $\theta_{1,M-1}(t)$. Note that, by Eq. 10 with $i = 1$ and $m = M-1$, $\frac{\partial x_n(t^-_{M-1})}{\partial \theta_{1,M-1}} = 0$ for $n = 1, 2$. Moreover, at t_{M-1} only two events are possible: (*i*) If $t_{M-1} \in \Sigma_1$, by Lemma 1 we have $\frac{\partial t_{M-1}}{\partial \theta_{1,M-1}}[f_n(t^-_{M-1}) - f_n(t^+_{M-1})] = 0$ for $n = 1, 2$, and (*ii*) If $t_{M-1} \in \Gamma_2$, since $\frac{\partial x_n(t^-_{M-1})}{\partial \theta_{1,M-1}} = 0$ for $n = 2$, it follows from Eq. 14 that $\frac{\partial t_{M-1}}{\partial \theta_{1,M-1}} = 0$. Also, from Eq. 4, we get $f_2(\tau) = 0$ for all $\tau \in [t_{M-1}, t_M)$ which gives $\frac{\partial f_2(\tau)}{\partial \theta_{1,M-1}} = 0$. On the other hand, $f_1(\tau) = \alpha_1(\tau) - \mu_1\theta_{1,M-1}(\tau)$, yielding $\frac{\partial f_1(\tau)}{\partial \theta_{1,M-1}} = -\mu_1$. Inserting these results into Eq. 12 for any $t \in [t_{M-1}, t_M)$ yields $\frac{\partial x_1(t)}{\partial \theta_{1,M-1}} = -\mu_1(t - t_{M-1})$ and $\frac{\partial x_2(t)}{\partial \theta_{1,M-1}} = 0$. It then follows from the derivative (Eq. 9) with $g_n(x_n(t)) = x_n(t)$ that

$$\frac{\partial Q(\theta)}{\partial \theta_{1,M-1}} = -c_1\mu_1\frac{(t_M - t_{M-1})^2}{2} < 0$$

thus, increasing $\theta_{1,M-1}(t)$ has a decreasing effect on the linear holding cost.

Case 3 If $N_{M-1} = 1$ but $x_2(t) > 0$ for all $t \in [t_{M-1}, t_M)$, using Eq. 4 for any $\tau \in [t_{M-1}, t)$ we find $f_1(\tau) = 0$ and $f_2(\tau) = \alpha_2(\tau) - \mu_2(1 - \theta_{1,M-1}(\tau))$, giving $\frac{\partial f_1(\tau)}{\partial \theta_{1,M-1}} = 0$ and $\frac{\partial f_2(\tau)}{\partial \theta_{1,M-1}} = \mu_1$. Similar to the previous case, we also find that $\frac{\partial t_{M-1}}{\partial \theta_{1,M-1}}[f_n(t^-_{M-1}) - f(t^+_{M-1})] = 0$. Using these results in Eq. 12 for any $t \in [t_{M-1}, t_M)$ gives $\frac{\partial x_1(t)}{\partial \theta_{1,M-1}} = 0$ and $\frac{\partial x_2(t)}{\partial \theta_{1,M-1}} = \mu_2(t - t_{M-1})$. Application of Eq. 9 gives the final result:

$$\frac{\partial Q(\theta)}{\partial \theta_{1,M-1}} = c_2\mu_2\frac{(t_M - t_{M-1})^2}{2} > 0,$$

hence, decreasing $\theta_{1,M-1}(t)$ has a decreasing effect on the holding cost.

Case 4 If $N_{M-1} = 2$ (both queues nonempty), then using Eqs. 3, 4 and 8 we find that $f_1(\tau) = \alpha_1(\tau) - \mu_1\theta_{1,M-1}(\tau)$ and $f_2(\tau) = \alpha_2(\tau) - \mu_2(1 - \theta_{1,M-1}(\tau))$ for any $\tau \in [t_{M-1}, t_M)$. Therefore, $\frac{\partial f_1(\tau)}{\partial \theta_{1,M-1}} = -\mu_1$ and $\frac{\partial f_2(\tau)}{\partial \theta_{1,M-1}} = \mu_2$. Moreover, if $N_{M-1} = 2$ we must have $t_{M-1} \in \Sigma_1$ or Σ_2 which, using Lemma 1, in either case implies that $\frac{\partial t_{M-1}}{\partial \theta_{1,M-1}}[f_n(t^-_{M-1}) - f_n(t^+_{M-1})] = 0$ for $n = 1, 2$. Furthermore, by Eq. 10 with $i = 1$ and

$m = M-1$, we get $\frac{\partial x_n(t_{M-1})}{\partial \theta_{1,M-1}} = 0$. Combining these results in Eq. 12 gives $\frac{\partial x_1(t)}{\partial \theta_{1,M-1}} = -\mu_1 (T - t_{M-1})$ and $\frac{\partial x_2(t)}{\partial \theta_{1,M-1}} = \mu_2(t - t_{M-1})$. It follows from Eq. 9 for $m = M-1$ that

$$\frac{\partial Q(\theta)}{\partial \theta_{1,M-1}} = -\left(c_1\mu_1 - c_2\mu_2\right)\frac{(t_M - t_{M-1})^2}{2} < 0$$

hence, increasing $\theta_{1,M-1}(t)$ has a decreasing effect on the cost.

Collecting all above results for all four cases, we obtain the following optimal allocation policy, proving the $c\mu$-rule for $t \in [t_{M-1}, T)$:

$$\theta^*_{1,M-1}(t) = \begin{cases} \frac{\alpha_1(t)}{\mu_1} & \text{if } x_1(t) = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$\theta^*_{2,M-1}(t) = 1 - \theta^*_{1,M-1}(t)$$

Next, we show that if the $c\mu$-rule is optimal and is applied to the interval $[t_{m+1}, T)$, it is also optimal for the interval $[t_m, T)$. The case of $N_m = 0$ is trivial and excluded. When $N_m = 1$ and $x_1(t) > 0$, applying the same analysis as we did for t_{M-1} results in $\frac{\partial x_1(t)}{\partial \theta_{1,m}} = -\mu_1(t - t_m) < 0$ and $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = 0$ for any $t \in [t_m, t_{m+1})$. At t_{m+1} the following two events are possible:

First, $t_{m+1} \in \Gamma_1$, i.e., the end of a NEP1. In this case $f_n(\tau) = 0$ for all $\tau \in [t_{m+1}, t_{m+2})$ yielding $\frac{\partial f_n(\tau)}{\partial \theta_{1,m}} = 0$. Using Eqs. 14 and 12 for $t \in [t_{m+1}, t_{m+2})$ gives

$$\frac{\partial x_1(t)}{\partial \theta_{1,m}} = \frac{\partial x_1(t^-_{m+1})}{\partial \theta_{1,m}} + \frac{\frac{-\partial x_1(t^-_{m+1})}{\partial \theta_{1,m}}}{f_1(t^-_{m+1})}\left[f_1(t^-_{m+1}) - 0\right] = 0$$

$$\frac{\partial x_2(t)}{\partial \theta_{1,m}} = \frac{\partial x_2(t^-_{m+1})}{\partial \theta_{1,m}} + \frac{\partial t_{m+1}}{\partial \theta_{1,m}}[0 - 0] = 0$$

Using Eq. 9, we conclude that $\frac{\partial Q(\theta)}{\partial \theta_{1,m}} = -c_1\mu_1\frac{(t_{m+1}-t_m)^2}{2} < 0$.

Second, if $t_{m+1} \in \Sigma_2$ (start of a NEP2) by Lemma 1 we get $\frac{\partial t_{m+1}}{\partial \theta_{1,m}}[f_n(t^-_{m+1}) - f_n(t^+_{m+1})] = 0$ for $n = 1, 2$. Moreover, noticing that $\theta_{1,m}(t)$ only affects the dynamics inside $[t_m, t_{m+1})$, we have $\frac{\partial f_n(\tau)}{\partial \theta_{1,m}} = 0$ for $\tau > t_{m+1}$. Using these results in Eq. 12 gives $\frac{\partial x_1(t)}{\partial \theta_{1,m}} = \frac{\partial x_1(t^-_{m+1})}{\partial \theta_{1,m}} = -\mu_1(t_{m+1} - t_m)$ and $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = \frac{\partial x_2(t^-_{m+1})}{\partial \theta_{1,m}} = 0$ for any $t \in [t_{m+1}, t_{m+2})$. Next, if $t_{m+2} \in \Gamma_2$, we get $x_2(t) = 0$ for $t \in [t_{m+2}, t_{m+3})$ which yields $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = 0$ on the same interval. Since we have shown $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = 0$ in $[t_{m+1}, t_{m+2})$, we conclude $\frac{\partial x_2(t^-_{m+2})}{\partial \theta_{1,m}} = 0$. Using this in Eq. 14 gives $\frac{\partial t_{m+2}}{\partial \theta_{1,m}} = 0$. Also, because $f_1(\tau)$ is independent of $\theta_{1,m}(\tau)$ for $\tau > t_{m+1}$, we get $\frac{\partial f_1(\tau)}{\partial \theta_{1,m}} = 0$. Using these in Eq. 12 with $t_k = t_{m+2}$ gives $\frac{\partial x_1(t)}{\partial \theta_{1,m}} = -\mu_1(t_{m+1} - t_m)$ for any $t \in [t_{m+2}, t_{m+3})$. If, on the other hand, we have $t_{m+2} \in \Gamma_1$ (end of a NEP1), we get into a state where both queues are empty. In this case simple calculations reveal $\frac{\partial x_n(t)}{\partial \theta_{1,m}} = 0$ for $t \in [t_{m+2}, t_{m+3})$ and $n = 1, 2$. Since $\frac{\partial f_n(\tau)}{\partial \theta_{i,m}} = 0$, for the rest of the sample path no perturbation can be generated with respect to $\theta_{n,m}$. Thus, repeating the same analysis on all subsequent intervals, i.e. considering the possible events at t_{m+i}, for $i > 2$ reveals that $\frac{\partial x_2(t)}{\partial \theta_{1,m}}$ remains zero for the rest of the sample path and $\frac{\partial x_1(t)}{\partial \theta_{1,m}}$ remains constant until the first time queue 1 becomes empty at some point $t < T$ and remains zero afterwards. If queue 1 never becomes empty $\frac{\partial x_1(t)}{\partial \theta_{1,m}}$ stays constant for all $t \in [t_{m+1}, T)$. Hence, after applying Eq. 9, we get

$$\frac{\partial Q(\theta)}{\partial \theta_{1,m}} = -c_1\mu_1\left[\frac{(t_{m+1}-t_m)^2}{2} + (t_{m+1}-t_m)(\eta - t_{m+1})\right] < 0$$

where η is either the first time queue 1 goes empty after t_{m+1} or $\eta = T$ if it never happens. The derivative found is negative, so we get $\theta^*_{1,m}(t) = 1$, obeying the $c\mu$-rule.

If $N_m = 1$ and $x_2(t) > 0$, following the same procedure as in the interval $[t_{M-1}, t_M)$ we find that $\frac{\partial x_1(t)}{\partial \theta_{1,m}} = 0$ and $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = \mu_2(t - t_m)$ for any $t \in [t_m, t_{m+1})$. At t_{m+1} either $t_{m+1} \in \Gamma_2$ (NEP2 ends) or $t_{m+1} \in \Sigma_1$ (NEP1 starts). In the former case, proceeding exactly as in the previous case, we get $\frac{\partial x_n(t)}{\partial \theta_{1,m}} = 0$ for any $t \in [t_{m+1}, T)$ which, after applying Eq. 9, gives $\frac{\partial Q(\theta)}{\partial \theta_{1,m}} = c_2\mu_2\frac{(t_{m+1}-t_m)^2}{2}$. Then, with the same analysis as in interval $[t_{M-1}, t_M)$, we find that $a_1(t) = \frac{\alpha_1(t)}{\mu_1}$ is an attractive upper bound for $\theta_{2,m}(t)$. If $t_{m+1} \in \Sigma_1$, with exactly the same procedure in the case with $N_m = 1$ and $x_1(t) > 0$, we get $\frac{\partial x_n(t)}{\partial \theta_{1,m}} = \frac{\partial x_n(t^-_{m+1})}{\partial \theta_{1,m}} = \mu_2(t_{m+1} - t_m)$ for any $t \in [t_{m+1}, t_{m+2})$. Following the same procedure as in the previous case we get

$$\frac{\partial Q(\theta)}{\partial \theta_{1,m}} = c_2\mu_2\left[\frac{(t_{m+1}-t_m)^2}{2} + (t_{m+1}-t_m)(\eta - t_{m+1})\right] > 0$$

where η is either the first time queue 2 becomes empty or $\eta = T$ if it never happens. This is a positive term, hence $\theta^*_{1,m}(t) = \frac{\alpha_1(t)}{\mu_1}$ and $\theta^*_{2,m}(t) = 1 - \theta^*_{1,m}(t)$ for all $t \in [t_m, t_{m+1})$ which obeys the $c\mu$-rule.

Finally, if $N_m = 2$, we find $f_1(\tau) = \alpha_1(\tau) - \mu_1\theta_{1,m}(\tau)$ and $f_2(\tau) = \alpha_2(\tau) - \mu_2(1 - \theta_{1,m}(\tau))$ for any $\tau \in [t_m, t_{m+1})$. Therefore, $\frac{\partial f_1(\tau)}{\partial \theta_{1,m}} = -\mu_1$ and $\frac{\partial f_2(\tau)}{\partial \theta_{2,m}} = \mu_2$. Since $N_m = 2$, the event at t_m is such that $t_m \in \Sigma_1$ or Σ_2 and application of Lemma 1 implies $\frac{\partial t_m}{\partial \theta_{1,m}}[f_n(t^-_m) - f_n(t^+_m)] = 0$ for $n = 1, 2$.. Moreover, by Eq. 10 with $i = 1$ we get $\frac{\partial x_n(t^-_m)}{\partial \theta_{1,m}} = 0$. Using these results in Eq. 12 for $t \in [t_m, t_{m+1})$, we obtain $\frac{\partial x_1(t)}{\partial \theta_{1,m}} = -\mu_1(t - t_m)$ and $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = \mu_2(t - t_m)$ for any $t \in [t_m, t_{m+1})$. At t_{m+1}, we either have $t_{m+1} \in \Gamma_1$ or Γ_2. In the former case, applying Eq. 12 for $[t_{m+1}, t_{m+2})$ gives $\frac{\partial x_1(t)}{\partial \theta_{1,m}} = 0$. Regarding $x_2(t)$, application of Eq. 4 at t^-_{m+1} results in $f_2(t^-_{m+1}) = \alpha_2(t^-_{m+1}) - \mu_2(1 - \theta_{1,m}(t^-_{m+1}))$. After t_{m+1} the $c\mu$-rule applies (by the induction hypothesis) and we have $f_2(t^+_{m+1}) = \alpha_2(t^+_{m+1}) - \mu_2(1 - \frac{\alpha_1(t^+_{m+1})}{\mu_1})$. Therefore, $f_2(t^-_{m+1}) - f_2(t^+_{m+1}) = -\frac{\mu_2}{\mu_1}[\alpha_1(t^+_{m+1}) - \theta_{1,m}(t^+_{m+1})\mu_1]$. Since, by Assumption 1, $\alpha_1(t)$ is continuous at t_{m+1}, we get $f_2(t^-_{m+1}) - f_2(t^+_{m+1}) = -\frac{\mu_2}{\mu_1} f_1(t^-_{m+1})$. Moreover, by the definition of $\theta_{1,m}(\tau)$, we also have $\frac{\partial f_n(\tau)}{\partial \theta_{1,m}} = 0$ for all $\tau \in [t_{m+1}, t_{m+2})$, $n = 1, 2$. Using these results in Eqs. 14 and 12 for $n = 2$ and $t \in [t_{m+1}, t_{m+2})$, we get

$$\frac{\partial x_2(t)}{\partial \theta_{1,m}} = \frac{\partial x_2(t^-_{m+1})}{\partial \theta_{1,m}} + \frac{-\frac{\partial x_1(t^-_{m+1})}{\partial \theta_{1,m}}}{f_1(t^-_{m+1})}\left(-\frac{\mu_2}{\mu_1} f_1\left(t^-_{m+1}\right)\right).$$

If we insert the derivatives $\frac{\partial x_1(t^-_{m+1})}{\partial \theta_{1,m}} = -\mu_1(t_{m+1} - t_m)$ and $\frac{\partial x_2(t^-_{m+1})}{\partial \theta_{1,m}} = \mu_2(t_{m+1} - t_m)$ in the above equation, we find $\frac{\partial x_2(t)}{\partial \theta_{1,m}} = 0$ for all $t \in [t_{m+1}, t_{m+2})$. Using the same analysis in all subsequent intervals reveals that the derivatives obtained remain zero for the rest of the sample path. Therefore, application of Eq. 9 gives

$$\frac{\partial Q(\theta)}{\partial \theta_{1,m}} = -(c_1\mu_1 - c_2\mu_2)\frac{(t_{m+1}-t_m)^2}{2} < 0.$$

The last case pertaining to $t_{m+1} \in \Gamma_2$ is similar and the analysis results in the same expression for $\frac{\partial Q(\theta)}{\partial \theta_{1,m}}$, so it is omitted. Therefore, when $N_m = 2$ we conclude that $\theta^*_{1,m}(t) = 1$ and $\theta^*_{2,m}(t) = 0$. Gathering all the results, one can summarize the optimal allocations in the following equations for all $t \in [0, T]$:

$$\theta^*_1(t) = \begin{cases} 1 & \text{if } x_1(t) > 0 \\ \frac{\alpha_1(t)}{\mu_1} & \text{otherwise} \end{cases}, \quad \theta^*_2(t) = \begin{cases} 0 & \text{if } x_1(t) > 0 \\ 1 - \frac{\alpha_1(t)}{\mu_1} & \text{if } x_1(t) = 0, x_2(t) > 0 \\ \frac{\alpha_2(t)}{\mu_2} & \text{otherwise} \end{cases} \tag{18}$$

This is precisely the $c\mu$-rule, so the proof is complete. □

3.2 IPA for $N > 2$ queues and the $c\mu$-rule

Outline of the analysis Considering the result of Theorem 1 ($N = 2$) as the base step, we set up the induction hypothesis by assuming that the $c\mu$-rule is optimal for arbitrary $K = 2, \ldots, N-1$ queues. In the inductive step, we combine Eqs. 17a and 17b with the induction hypothesis to show that when N queues are present and the $c\mu$-rule is optimal and is implemented in the interval $[t_{m+1}, T)$, it is also optimal for the interval $[t_m, T)$. We do this by showing that deviating from the $c\mu$-rule in the interval $[t_m, t_{m+1})$ increases the cost $Q(u_0, \ldots, u_{M-1})$. Specifically, we perturb $\theta_{1,m}(t)$ away from its value under the $c\mu$-rule by $\delta_{1,m}(t) > 0$ and $\theta_{n,m}(t)$ by $\delta_{n,m}(t) \geq 0$ for $n > 1$ such that no event order change results at and after t_{m+1}. Moreover, this perturbation is such that Eq. 8 is preserved. Notice that under the $c\mu$-rule no two queues can end their NEPs at the same time, thus, we can always find such arbitrary perturbations $\delta_{n,m}(t)$. Next, stepping backwards in time, the induction hypothesis applies to all intervals with fewer than N non-empty queues. Thus, we need only examine those intervals where all N queues are non-empty. To this end, let us consider an interval $[t_m, t_{m+1})$ with $N_m = N$ and let $\eta_{n,m}$ be the first time at which the nth queue becomes empty after t_m. We assume that the newly added queue has the lowest $c\mu$ product, that is, $c_N\mu_N < c_n\mu_n$ for any $n \neq N$. This is simply for convenience and does not restrict generality because, knowing the products $c_n\mu_n$ of all the queues, one can do this after sorting them out instead of randomly adding a queue.

To apply the above mentioned idea, we will introduce a series of lemmas all of which apply to the following setting: Consider an interval $[t_m, t_{m+1})$ with $N_m = N$. Also, assume that the $c\mu$-rule is applied for $t \geq t_{m+1}$. Moreover, let $\theta_{1,m}(t)$ be perturbed as explained above. Thus, as shown in Fig. 2, queues are served according to their $c\mu$ value highest to lowest and NEPs end at $\eta_{1,m}, \eta_{2,m}, \ldots, \eta_{N,m}$ such that

$$t_{m+1} = \eta_{1,m} < \eta_{2,m} < \ldots < \eta_{N,m}. \tag{19}$$

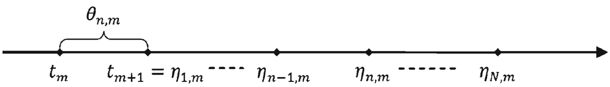


Fig. 2 Time line of events in and after the interval $[t_m, t_{m+1})$

Now let us define for any event time t_k, $k = 1, \ldots, M-1$:

$$\Delta\theta_n(t_k) = \theta_n\left(t_k^+\right) - \theta_n\left(t_k^-\right) = \theta_{n,k}\left(t_k^+\right) - \theta_{n,k-1}\left(t_k^-\right). \tag{20}$$

representing the amount by which a control changes when this event occurs and affecting the queue content derivatives in Eq. 12. The first lemma gives explicit expressions for this quantity at the event times $\eta_{1,m}, \ldots, \eta_{N,m}$.

Lemma 2

(a)

$$\Delta\theta_n(\eta_{1,m}) = \begin{cases} \frac{\alpha_1(\eta_{1,m})}{\mu_1} - \theta_1(\eta_{1,m}^-) & \text{if } n = 1 \\ 1 - \frac{\alpha_1(\eta_{1,m})}{\mu_1} - \theta_2(\eta_{1,m}^-) & \text{if } n = 2 \\ -\theta_n(\eta_{1,m}^-) & \text{if } n > 2. \end{cases}$$

(b) *For* $n = 3, \ldots, N$, $\Delta\theta_n(\eta_{n,m}) = -\left(1 - \sum_{k=1}^{n} \frac{\alpha_k(\eta_{n,m})}{\mu_k}\right)$, $\Delta\theta_n(\eta_{n-1,m}) = 1 - \sum_{k=1}^{n-1} \frac{\alpha_k(\eta_{n,m})}{\mu_k}$ *and* $\Delta\theta_n(\eta_{k,m}) = 0$ *for all* $k \neq n, n-1$.

Proof See Appendix. □

In the next three lemmas we derive convenient expressions for the queue content derivatives with respect to a parameter $\theta_{i,m}$ inside the intervals $[t_m, \eta_{1,m})$, $[\eta_{1,m}, \eta_{N,m})$ and $[\eta_{N,m}, T)$, respectively.

Lemma 3 *For all* $t \in [t_m, \eta_{1,m})$ *with* $N_m = N$,

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \begin{cases} -\mu_i(t - t_m) & \text{if } n = i \\ \mu_N(t - t_m) & \text{if } n = N \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

for any $n = 1, \ldots, N$, $i = 1, \ldots, N-1$.

Proof See Appendix. □

An immediate corollary of Lemma 3 which will be used in the sequel is that for $i = 1, \ldots, N-1$ and $t \in [t_m, \eta_{1,m})$:

$$\frac{\partial x_i(t)}{\partial \theta_{i,m}} = \frac{\mu_i}{\mu_1} \frac{\partial x_1(t)}{\partial \theta_{1,m}} \tag{22}$$

$$\frac{\partial x_N(t)}{\partial \theta_{i,m}} = -\frac{\mu_N}{\mu_1} \frac{\partial x_1(t)}{\partial \theta_{1,m}} \tag{23}$$

Lemma 4 *For any* $t \in [\eta_{j,m}, \eta_{j+1,m})$, $j = 1, \ldots, N-1$,

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} + \sum_{k=1}^{j} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \mu_n \Delta\theta_n(\eta_{k,m}) \tag{24}$$

for any $n = 1, \ldots, N$, $i = 1, \ldots, N-1$.

Proof See Appendix. □

Lemma 5 *For any $t \in [\eta_{N,m}, T)$, we have $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$, $n = 1, \ldots, N$, $i = 1, \ldots, N-1$.*

Proof See Appendix. □

Observe that the evaluation of Eq. 24 involves some event time derivatives. Lemmas 6 and 7 allow us to evaluate derivatives of the form $\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}}$ depending on whether $n < i$ or $n \geq i$.

Lemma 6 *For any $i = 2, \ldots, N-1$,*

$$\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}} = 0, \quad n < i \tag{25}$$

Proof See Appendix. □

In what follows, we will make use of the following definition for any k and m,

$$r_{k,m} = \frac{-\mu_k}{f_k(\eta_{k,m}^-)} > 0 \tag{26}$$

where the inequality is due to the fact that at $t = \eta_{k,m}$ queue k becomes empty, therefore, recalling Eqs. 2 and 3, $f_k(\eta_{k,m}^-) = \alpha_k(\eta_{k,m}^-) - \mu_k \theta_{k,m}(\eta_{k,m}^-) < 0$.

The following lemma gives necessary tools to determine $\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}}$ when $n \geq i$.

Lemma 7 *For $n = 1, \ldots, N-1$ and $i \leq n$,*

$$\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}} = \frac{r_{i,m}}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} \prod_{k=i+1}^{n} r_{k,m} \Delta\theta_k(\eta_{k-1,m}), \tag{27}$$

where by convention, when $n = i$ we define the product term in the above equation as 1.

Proof See Appendix. □

We are now in the position to prove the main theorem of this section.

Theorem 2 *The $c\mu$-rule minimizes the cost (Eq. 15) for a system of N parallel queues with dynamics (Eq. 4), control parameters (Eq. 5) and the constraint 8.*

Proof As mentioned in the outline of the analysis, we use an induction argument along with a backward recursion in time through Eqs. 17a and 17b to prove the theorem. We first consider the last interval.

Last interval $[t_{M-1}, t_M)$: If $N_{M-1} < N$, by the induction hypothesis the $c\mu$-rule is already optimal. If $N_m = N$, we have

$$\frac{\partial q(t)}{\partial \theta_{i,M-1}} = \sum_{n=1}^{N} c_n \frac{\partial x_n(t)}{\partial \theta_{i,M-1}}, \ i = 1, \ldots, N-1.$$

Using Lemma 3 for each $i = 1, \ldots, N-1$ we get

$$\frac{\partial q(t)}{\partial \theta_{i,M-1}} = -(c_i\mu_i - c_N\mu_N)(t - t_{M-1})$$

Since $c_i\mu_i > c_N\mu_N$ for all $i = 1, \ldots, N-1$, we conclude that

$$\frac{\partial q(t)}{\partial \theta_{i,M-1}} < 0, \text{ if } t \in [t_{M-1}, t_M),\ i = 1, \ldots, N-1. \tag{28}$$

Moreover, according to the $c\mu$ order in Eq. 7, we can write $(c_i\mu_i - c_N\mu_N) > (c_j\mu_j - c_N\mu_N)$, $1 \le i < j \le N-1$. Hence, for $t \in [t_{M-1}, t_M)$ we have

$$\frac{\partial q(t)}{\partial \theta_{i,M-1}} < \frac{\partial q(t)}{\partial \theta_{j,M-1}}, \text{ if } 1 \le i < j \le N-1. \tag{29}$$

Since, $\frac{\partial Q_{M-1}}{\partial \theta_{i,M-1}} = \int_{t_{M-1}}^{t_M} \frac{\partial q(t)}{\partial \theta_{j,M-1}} dt$, the inequalities 28 and 29 pertaining to $q(t)$ are inherited by $Q_{M-1}(\theta)$ and we get $\frac{\partial Q_{M-1}(\theta)}{\partial \theta_{1,M-1}} < \frac{\partial Q_{M-1}(\theta)}{\partial \theta_{i,M-1}} < 0$ for $i = 2, \ldots, N-1$. Therefore, increasing $\theta_{1,m}$ has the greatest minimizing effect on $Q(\theta)$, i.e., $v^*_{M-1}(t) = [1, 0, \ldots, 0]$, $t \in [t_{M-1}, T)$.

Arbitrary interval $[t_m, t_{m+1}), m \le M-2$*:* Suppose the $c\mu$-rule is optimal and already applied to the interval $[t_{m+1}, t_M)$. We will show that it is also optimal in the interval $[t_m, t_M)$ for any $m \le M-2$. If $N_m < N$ we can apply the induction hypothesis to the interval $[t_m, t_{m+1})$ and prove the optimality of the $c\mu$-rule over $[t_m, T)$. If $N_m = N$, we prove the theorem by first following the same procedure as for $[t_{M-1}, t_M)$ and apply Lemma 3 to show for any $t \in [t_m, t_{m+1})$

$$\frac{\partial q(t)}{\partial \theta_{i,m}} < \frac{\partial q(t)}{\partial \theta_{l,m}} < 0, \text{ if } 1 \le i < l \le N-1.$$

Next, we show that for any $t \in [\eta_{j,m}, \eta_{j+1,m})$

$$\frac{\partial q(t)}{\partial \theta_{1,m}} \le \frac{\partial q(t)}{\partial \theta_{2,m}} = \ldots = \frac{\partial q(t)}{\partial \theta_{j+1,m}} < \ldots < \frac{\partial q(t)}{\partial \theta_{N-1,m}} < 0 \tag{30}$$

with $j = 1, \ldots, N-2$. This relation covers the interval $[\eta_{1,m}, \eta_{N,m})$. Note that by Eq. 10 and Lemma 5, we need not consider the rest of the sample path. Using these relations in Eq. 9 establisehs that $\frac{\partial Q(\theta)}{\partial \theta_{1,m}} < \frac{\partial Q(\theta)}{\partial \theta_{i,m}} < 0$ for any $i = 2, \ldots, N-1$ and proves the theorem. We omit the analysis for the interval $[t_m, \eta_{1,m})$ due to its similarity to the one for $[t_{M-1}, t_M)$ and focus on the interval $[\eta_{1,m}, \eta_{N,m})$.

Let us obtain an expression for $\frac{\partial q(t)}{\partial \theta_{i,m}}$ over the interval $[\eta_{j,m}, \eta_{j+1,m})$ where $j \in \{1, \ldots, N-1\}$ is fixed. If $x_n(t) = 0$ for all $n \le j$ and $t \in [\eta_{j,m}, \eta_{j+1,m})$, we see that $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$ which gives

$$\frac{\partial q(t)}{\partial \theta_{i,m}} = \sum_{n=j+1}^{N} c_n \frac{\partial x_n(t)}{\partial \theta_{i,m}}, \quad t \in [\eta_{j,m}, \eta_{j+1,m}). \tag{31}$$

If there is an interval $[t_{l^*}, t_l) \subset [\eta_{j,m}, \eta_{j+1,m})$ over which $x_n(t) > 0$ with $n \le j$, we show that $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$ and Eq. 31 is still valid. This is the case where a queue n depleted

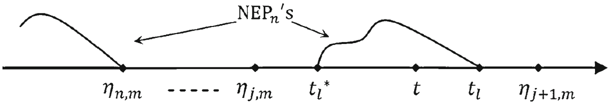


Fig. 3 A re-start of a NEPn, $n \leq j$ in interval $[\eta_{j,m}, \eta_{j+1,m})$

by or at $\eta_{n,m}$ with $n \leq j$ has become non-empty again (as illustrated in Fig. 3). We only consider one such interval, thereby having $t_{l^*} = t_{l-1}$, since the extension to multiple intervals is straightforward. By the $c\mu$-rule, this queue should make full use of the resource at t_{l-1} and return it to the $(j+1)$th queue after it becomes empty again at t_l. It suffices to show that $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$ over $[t_{l-1}, t_l)$ because $x_n(t) = 0$ right before and after this interval. Since t_{l-1} is the start of a NEPn, Lemma 1 gives $\frac{\partial t_{l-1}}{\partial \theta_{i,m}}[f_n(t_{l-1}^-) - f_n(t_{l-1}^+)] = 0$. Moreover, $x_n(t) = 0$ for $t \in [\eta_{j,m}, t_{l-1})$ and $\frac{\partial x_n(t_{l-1}^-)}{\partial \theta_{i,m}} = 0$. In addition, because $t_l \geq \eta_{1,m}$, by Eqs. 4 and 5, $f_n(\tau)$ is not a function of $\theta_{i,m}$ for any $\tau \in [t_{l-1}, t_l)$. Therefore, $\frac{\partial f_n(\tau)}{\partial \theta_{i,m}} = 0$. Using these results in Eq. 12 with $t_k = t_{l-1}$ and $t = t_l$ we find that when $n \leq j$, $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$ for any $t \in [t_{l-1}, t_l)$, hence, $t \in [\eta_{j,m}, \eta_{j+1,m})$ and Eq. 31 still holds.

By Lemma 6 we have $\frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} = 0$ for $k \leq i$. Using this in Lemma 4, i.e., Eq. 24, and inserting the result in Eq. 31 gives:

$$\frac{\partial q(t)}{\partial \theta_{i,m}} = \sum_{n=j+1}^{N} c_n \left[\frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} + \sum_{k=i}^{j} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \mu_n \Delta\theta_n(\eta_{k,m}) \right]. \tag{32}$$

There are now two cases regarding the range of i:

- $i > j$: In this case, the inner sum in Eq. 32 vanishes. Moreover, by Lemma 3, $\frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} = 0$ unless we have either $n = i$ or $n = N$. Using these facts in Eq. 32 yields:

$$\frac{\partial q(t)}{\partial \theta_{i,m}} = c_i \frac{\partial x_i(\eta_{1,m}^-)}{\partial \theta_{i,m}} + c_N \frac{\partial x_N(\eta_{1,m}^-)}{\partial \theta_{i,m}}.$$

Invoking Eqs. 22 and 23 this further reduces to

$$\frac{\partial q(t)}{\partial \theta_{i,m}} = \frac{1}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} (c_i \mu_i - c_N \mu_N). \tag{33}$$

By Lemma 3, $\frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} < 0$. In view of Eq. 7, it follows that for $t \in [\eta_{j,m}, \eta_{j+1,m})$:

$$\frac{\partial q(t)}{\partial \theta_{j+1,m}} < \frac{\partial q(t)}{\partial \theta_{j+2,m}} < \ldots < \frac{\partial q(t)}{\partial \theta_{N-1,m}} < 0. \tag{34}$$

- $i \le j$: We consider two subcases:

  (a) $j \ge 2$: In this case, since the sum in Eq. 32 is such that $n \ge j+1$, the condition $n = i$ cannot be true. Using Lemma 3, we find that the first sum in Eq. 32 becomes

$$\sum_{n=j+1}^{N} c_n \frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} = c_N \frac{\partial x_N(\eta_{1,m}^-)}{\partial \theta_{i,m}}.$$

Concerning the inner sum in Eq. 32 we can use part (b) of Lemma 2 to see that $\Delta\theta_n(\eta_{k,m}) = 0$ except for when $n = j+1$ and $k = j$ where we have $\Delta\theta_{j+1}(\eta_{j,m}) = 1 - \sum_{k=1}^{j} \frac{\alpha_k(\eta_{j,m})}{\mu_k}$. Using these results in Eq. 32 we conclude that, for $t \in [\eta_{j,m}, \eta_{j+1,m})$,

$$\frac{\partial q(t)}{\partial \theta_{i,m}} = c_{j+1}\mu_{j+1} \frac{\partial \eta_{j,m}}{\partial \theta_{i,m}} \Delta\theta_{j+1}(\eta_{j,m}) + c_N \frac{\partial x_N(\eta_{1,m}^-)}{\partial \theta_{i,m}}. \tag{35}$$

By Eq. 23, we have $\frac{\partial x_N(\eta_{1,m}^-)}{\partial \theta_{i,m}} = -\frac{\mu_N}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}}$. The derivative $\frac{\partial \eta_{j,m}}{\partial \theta_{i,m}}$ can be calculated by Lemma 7 for $j \ge i$. For convenience, we also define:

$$A_{i,j} = \prod_{k=i+1}^{j} r_{k,m} \Delta\theta_k(\eta_{k-1,m})$$

Combining these expressions in (35) we find for any $t \in [\eta_{j,m}, \eta_{j+1,m})$,

$$\frac{\partial q(t;\theta)}{\partial \theta_{i,m}} = \frac{1}{\mu_1} \frac{\partial x_1(\eta_{1,m})}{\partial \theta_{1,m}} \left[c_{j+1}\mu_{j+1} r_{i,m} A_{i,j} \Delta\theta_{j+1}\left(\eta_{j,m}\right) - c_N \mu_N \right] \tag{36}$$

By Lemma 3 we have $\frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} < 0$. We shall next show that $r_{i,m} A_{i,j} \Delta\theta_{j+1}(\eta_{j,m}) = 1$ for $1 < i \le j$ and $r_{i,m} A_{i,j} \Delta\theta_{j+1}(\eta_{j,m}) \ge 1$ for $1 = i \le j$ which yields

$$\frac{\partial q(t)}{\partial \theta_{1,m}} \le \frac{\partial q(t)}{\partial \theta_{2,m}} = \ldots = \frac{\partial q(t)}{\partial \theta_{j-1,m}} = \frac{\partial q(t)}{\partial \theta_{j,m}} < 0 \tag{37}$$

for any $t \in [\eta_{j,m}, \eta_{j+1,m})$, $j = 2, \ldots, N-1$. To accomplish this, we show that,

$$r_{1,m} \Delta\theta_2(\eta_{1,m}) \ge 1, \tag{38}$$

$$r_{k-1,m} \Delta\theta_k(\eta_{k-1,m}) = 1 \qquad \text{if } k = 3, \ldots, j. \tag{39}$$

Using part (a) of Lemma 2 and (26) with $k = 1$ we get

$$\Delta\theta_2(\eta_{1,m}) r_{1,m} = \frac{-\mu_1\left[1 - \theta_2(\eta_{1,m}^-) - \frac{\alpha_1(\eta_{1,m})}{\mu_1}\right]}{\alpha_1(\eta_{1,m}) - \mu_1\theta_1(\eta_{1,m}^-)}$$

and by factoring out $-\mu_1$ in the denominator we find

$$\Delta\theta_2(\eta_{1,m}) r_{1,m} = \frac{1 - \theta_2(\eta_{1,m}^-) - \frac{\alpha_1(\eta_{1,m})}{\mu_1}}{\theta_1(\eta_{1,m}^-) - \frac{\alpha_1(\eta_{1,m})}{\mu_1}}.$$

As discussed in the outline of the analysis, in $[t_m, \eta_{1,m})$ we deviate from the $c\mu$-rule by perturbing $\theta_{n,m}(t)$ by $\delta_{n,m}(t)$ while preserving Eq. 8. It follows that $\theta_1(\eta_{1,m}^-) \le 1 - \theta_2(\eta_{1,m}^-)$, hence, Eq. 38 is proven. □

Next, recalling Eq. 4, we have

$$f_{k-1}\left(\eta_{k-1,m}^{-}\right)=\alpha_{k-1}\left(\eta_{k-1,m}^{-}\right)-\mu_{k-1}\left[1-\sum_{l=1}^{k-2}\frac{\alpha_l(\eta_{k-1,m})}{\mu_l}\right]$$

or

$$f_{k-1}\left(\eta_{k-1,m}^{-}\right)=-\mu_{k-1}\left[1-\sum_{l=1}^{k-1}\frac{\alpha_l(\eta_{k-1,m})}{\mu_l}\right]$$

By part (b) of Lemma 2, for $k=i+1,\ldots,j$ we have $\Delta\theta_k(\eta_{k-1,m})=1-\sum_{l=1}^{k-1}\frac{\alpha_l(\eta_{k-1,m})}{\mu_l}$ and by Eq. 26, we have $r_{k-1,m}=\frac{-\mu_{k-1}}{f_{k-1}(\eta_{k-1,m}^{-})}$. Therefore, the equation above yields

$$r_{k-1,m}\Delta\theta_k(\eta_{k-1,m})=1$$

which verifies Eq. 39.

Using Eq. 39 when $i>1$ in the definition of $A_{i,j}$, it is easy to check that $r_{i,m}A_{i,j}=r_{j,m}$. Therefore, invoking Eq. 39 again, we get $r_{i,m}A_{i,j}\Delta\theta_{j+1}(\eta_{j,m})=r_{j,m}\Delta\theta_{j+1}(\eta_{j,m})=1$. Applying this to Eq. 36 for $i>1$, gives

$$\frac{\partial q(t)}{\partial\theta_{i,m}}=\frac{\partial x_1(\eta_{1,m})}{\mu_1\partial\theta_{1,m}}[c_{j+1}\mu_{j+1}-c_N\mu_N],\ \forall t\in[\eta_{j,m},\eta_{j,m+1}).$$

which is independent of i. This proves the equalities in Eq. 37. In a similar way, we find that when $i=1$ (Eq. 38) yields $r_{i,m}A_{i,j}\Delta\theta_{j+1}(\eta_{j,m})\geq 1$. Using this in Eq. 36 proves the inequality in Eq. 37. Combining Eqs. 34 and 37, for any $t\in[\eta_{j,m},\eta_{j+1,m})$ with $j\geq 2$ we have

$$\frac{\partial q(t)}{\partial\theta_{1,m}}\leq\frac{\partial q(t)}{\partial\theta_{2,m}}=\ldots=\frac{\partial q(t)}{\partial\theta_{j+1,m}}<\ldots<\frac{\partial q(t)}{\partial\theta_{N-1,m}}<0.$$

We complete the proof by showing that the above relationship is also true for $j=1$.
(b) $j=1$: Since we are considering a subcase of $i\leq j$, we have $i=j=1$. Using this in Eq. 32 results in

$$\frac{\partial q(t)}{\partial\theta_{1,m}}=\sum_{n=2}^{N}c_n\left[\frac{\partial x_n(\eta_{1,m}^{-})}{\partial\theta_{1,m}}+\frac{\partial\eta_{1,m}}{\partial\theta_{1,m}}\mu_n\Delta\theta_n(\eta_{1,m})\right].$$

According to Lemma 3, $\frac{\partial x_n(\eta_{1,m}^{-})}{\partial\theta_{1,m}}\neq 0$ only when $n=1$ or $n=N$. However, since the summation in the above equation has its lower limit at $n=2$, the case $n=1$ is not possible and it follows that

$$\frac{\partial q(t)}{\partial\theta_{1,m}}=c_N\frac{\partial x_N(\eta_{1,m}^{-})}{\partial\theta_{1,m}}+\frac{\partial\eta_{1,m}}{\partial\theta_{1,m}}\sum_{n=2}^{N}c_n\mu_n\Delta\theta_n(\eta_{1,m}).$$

Applying Eq. 23 with $t=\eta_{1,m}^{-}$ and Eq. 14 with $t_k=\eta_{1,m}$, this reduces to

$$\frac{\partial q(t)}{\partial\theta_{1,m}}=\frac{1}{\mu_1}\frac{\partial x_1(\eta_{1,m}^{-})}{\partial\theta_{1,m}}\left[\frac{\sum_{n=2}^{N}c_n\mu_n\Delta\theta_n(\eta_{1,m})}{\theta_1(\eta_{1,m}^{-})-\frac{\alpha_1(\eta_{1,m}^{-})}{\mu_1}}-c_N\mu_N\right].\tag{40}$$

By Lemma 3, $\frac{\partial x_1(\eta_{1,m}^-)}{\partial\theta_{1,m}} < 0$, therefore it remains to establish the negativity of the above derivative by showing that the bracketed term is positive. We accomplish this by finding a lower bound $L_m < \frac{\sum_{n=2}^{N} c_n\mu_n\Delta\theta_n(\eta_{1,m})}{\theta_1(\eta_{1,m}^-)-\frac{\alpha_1(\eta_{1,m}^-)}{\mu_1}}$ such that $L_m - c_N\mu_N > 0$. Using part (a) of Lemma 2, we have $\Delta\theta_2(\eta_{1,m}) > 0$ and $\Delta\theta_n(\eta_{1,m}) < 0$ for $n > 2$. Using the fact that $c_2\mu_2 > c_n\mu_n$ for $n > 2$, we find a lower bound $L_m = c_2\mu_2\frac{\sum_{n=2}^{N}\Delta\theta_n(\eta_{1,m})}{\theta_1(\eta_{1,m}^-)-\frac{\alpha_1(\eta_{1,m}^-)}{\mu_1}}$. Applying part (a) of Lemma 2 to replace $\Delta\theta_n(\eta_{1,m})$, $n \geq 2$, gives:

$$L_m = c_2\mu_2\frac{1-\sum_{n=2}^{N}\theta_n(\eta_{1,m}^-)-\frac{\alpha_1(\eta_{1,m})}{\mu_1}}{\theta_1(\eta_{1,m}^-)-\frac{\alpha_1(\eta_{1,m}^-)}{\mu_1}}.$$

According to Eq. 8 we have $\theta_1(\eta_{1,m}^-) = 1-\sum_{n=2}^{N}\theta_n(\eta_{1,m}^-)$, thus, we get $L_m = c_2\mu_2$. Finally, noting that $c_2\mu_2 > c_N\mu_N$ reveals that the bracketed term in Eq. 40 is indeed positive, therefore, $\frac{\partial q(t)}{\partial\theta_{1,m}} < 0$. To complete the proof of this part it remains to show that $\frac{\partial q(t)}{\partial\theta_{1,m}} < \frac{\partial q(t)}{\partial\theta_{i,m}}$ for all $t \in [\eta_{1,m}, \eta_{2,m})$ and $i = 2, \ldots, N-1$. Since in this case $j = 1 < i$, Eq. 34 holds and it suffices to show that $\frac{\partial q(t)}{\partial\theta_{1,m}} < \frac{\partial q(t)}{\partial\theta_{2,m}}$ over $[\eta_{1,m}, \eta_{2,m})$. Using $j = 1$ in Eq. 33 we get $\frac{\partial q(t)}{\partial\theta_{2,m}} = \frac{\partial x_1(\eta_{1,m}^-)}{\mu_1\partial\theta_{1,m}}[c_2\mu_2 - c_N\mu_N]$. Recalling that $L_m = c_2\mu_2$, it is then easy to see that $\frac{\partial q(t)}{\partial\theta_{1,m}} < \frac{1}{\mu_1}\frac{\partial x_1(\eta_{1,m}^-)}{\partial\theta_{1,m}}[L_m - c_N\mu_N] = \frac{\partial q(t)}{\partial\theta_{2,m}}$.

This completes the proof of Eq. 30 and it follows that increasing $\theta_{1,m}(t)$ has the most decreasing effect on the holding cost so that $\theta_{1,m}^*(t) = 1$. This proves the optimality of the $c\mu$-rule and the theorem. □

4 IPA in a general scheduling setting

In the previous section, our goal was to use IPA for the purpose of proving the $c\mu$-rule when the total holding cost is a linear function of queue contents. In doing so, we took advantage of the fact that the $c\mu$-rule was a given "target policy" whose optimality we established. In this section, we turn our attention to the use of IPA as a derivative estimation method which, combined with standard gradient-based optimization schemes, can lead to scheduling policies in more general settings where the holding cost is nonlinear and where no target scheduling policy is specified a priori. We proceed by redefining the control vector $\theta_n(t)$ in a manner more suitable for this goal. Using this modified setting, we first show that the IPA derivative estimates, combined with a gradient-based algorithm, allow us to naturally recover the $c\mu$-rule. Next, we use the same approach to find the best set of control parameters in a general scheduling setting seeking to minimize a nonlinear holding cost metric.

The control policy we use in this section is based on a *state partition* with states being the queue contents $x_n(t)$, $n = 1, \ldots, N$. Considering the system of Fig. 1, we view each queue as in either an *empty* or *non-empty* discrete state (or *mode*). Since we have N queues, there are 2^N modes or discrete states possible for this system. We denote the mode of the system at time $t \in [0, T]$ by $M(t) \in \{0, \ldots, 2^N - 1\}$. When $x_n(t) > 0$ for some n, we set $\theta_{n,M} \in [0, 1]$ to be the fraction of service received by queue n when the system is operating in mode M, i.e. $u_n(t) = \mu_n\theta_{m,M}$. On the other hand, when $x_n(t) = 0$, we let $u_n(t) = \alpha_n(t)$ so that $\frac{dx_n(t)}{dt^+} = 0$ is satisfied. We define the set

$$Z_n = \{M : x_n(t) = 0 \text{ when } M(t) = M\}$$

to be the collection of mode indices associated with the case where queue n is empty. Thus, the queue content dynamics in Eq. 4 can be rewritten as

$$\frac{dx_n(t)}{dt^+} = f_n(t) = \begin{cases} 0 & \text{if } M(t) \in Z_n \\ \alpha_n(t) - \mu_n \theta_{n,M(t)} & \text{otherwise} \end{cases} \tag{41}$$

Observe that we choose the control parameters such that the resource capacity constraint 1 is always satisfied. However, rather than imposing the maximum utility constraint 8 in Section 2, we use a more general constraint set:

$$\sum_{n=1}^{N} \theta_{n,M} = 1 \text{ only when } x_n(t) > 0, \forall n = 1, \dots, N. \tag{42}$$

and, otherwise:

$$\sum_{\substack{n=1 \\ n:M \notin Z_n}}^{N} \theta_{n,M} + \sum_{\substack{n=1 \\ n:M \in Z_n}}^{N} \frac{\alpha_n(t)}{\mu_n} \leq 1 \tag{43}$$

In this way, we make it possible to include a large class of idling policies where some of the resource capacity may remain unutilized.

Similar to the setting in Section 2, let us define events as either the starts or ends of NEPs in the queues and let t_k be the occurrence time of the kth such event. Notice that $M(t)$ remains constant inside $[t_k, t_{k+1})$ for any k. Similar to Eq. 9 we can differentiate the cost and write

$$\frac{\partial Q(\theta_{i,M})}{\partial \theta_{i,M}} = \sum_{k=0}^{M-1} \int_{t_k}^{t_{k+1}} \sum_{n=1}^{N} c_n \frac{\partial x_n(t)}{\partial \theta_{i,M}} \frac{\partial g_n(x_n(t))}{\partial x_n} dt, \quad \forall i = 1, \dots, N \text{ and } \forall M \notin O_i. \tag{44}$$

where the functions $g_n(\cdot)$ are known, therefore, $\frac{\partial g_n(x_n(t))}{\partial x_n}$ is also known for any $x_n(t)$. Since in the underlying DES $x_n(t)$ is actually an integer value $X_n(t) = 0, 1, \dots$, we can pre-compute the values $\frac{\partial g_n(0)}{\partial x_n}$, $\frac{\partial g_n(1)}{\partial x_n}$, … and use them to estimate $\frac{\partial g_n(x_n(t))}{\partial x_n}$ in Eq. 44 for any $t \in [0, T]$. On the other hand, for any $t \in [t_k, t_{k+1})$, the state derivatives $\frac{\partial x_n(t)}{\partial \theta_{i,M}}$ in Eq. 44 can be obtained by differentiating Eq. 11 with respect to any $\theta_{i,M}$, $M \notin Z_i$. This is done in the same manner through which Eq. 12 was obtained resulting in:

$$\frac{\partial x_n(t)}{\partial \theta_{i,M}} = \frac{\partial x_n(t_k^-)}{\partial \theta_{i,M}} + \frac{\partial t_k}{\partial \theta_{i,M}} \left[f_n\left(t_k^-\right) - f_n\left(t_k^+\right) \right] + \int_{t_k}^{t_{k+1}} \frac{\partial f_n(\tau)}{\partial \theta_{i,M}} d\tau \tag{45}$$

where we assume that $\frac{\partial x_n(0^-)}{\partial \theta_{i,M}} = 0$, i.e., the IPA derivatives are reset at the start of each sample path. It is also easy to check that Lemma 1 is still in effect so the start of NEPs do not affect the IPA derivatives. Finally, at times when a NEP ends, similar to Eq. 14 we find

$$\frac{\partial t_k}{\partial \theta_{i,M}} = -\frac{\partial x_n(t_k^-)/\partial \theta_{i,M}}{f_n(t_k^-)} \quad \text{if } t_k \in \Gamma_n. \tag{46}$$

It is now easy to see that Eqs. 45 and 46 constitute the IPA derivative estimation process. Proceeding on an event-by-event basis, these two equations allow us to recursively obtain the state derivatives required in Eq. 44 to evaluate the sample cost derivatives used in a gradient-based optimization algorithm. We omit proofs that these derivatives are unbiased estimates of the derivatives of the average holding cost $E[Q(\theta_{i,M})]$.

5 Numerical examples

5.1 IPA and online optimization for average linear holding cost

In what follows, we consider a system consisting of two queues ($N = 2$; applying a similar approach to $N > 2$ is straightforward) and provide simulation results in which IPA is used in conjunction with a simple gradient-based algorithm to minimize a linear holding cost metric:

$$Q(\theta) = \frac{1}{T}\int_0^T [c_1 x_1(t) + c_2 x_2(t)]\, dt \tag{47}$$

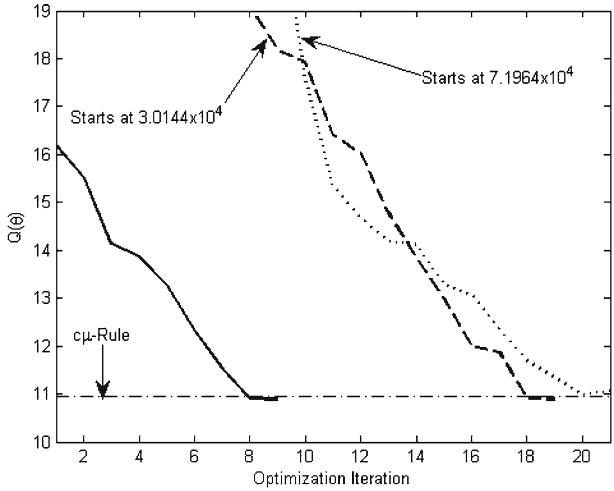
with $c_1 = 2$ and $c_2 = 5$. The stochastic service processes for the queue 1 and 2 are modeled by exponential distributions with average rates $\mu_1 = 3$ and $\mu_2 = 1$, respectively, yielding the $c\mu$ product relation $c_1\mu_1 > c_2\mu_2$. The arrival processes are Poisson with average rates $\bar{\alpha_1} = 1.2$ and $\bar{\alpha_2} = 0.4$. Notice that since the arrival rates satisfy the capacity constraint $\frac{\bar{\alpha_1}}{\mu_1} + \frac{\bar{\alpha_2}}{\mu_2} = 0.8 < 1$, the system is deemed to be stabilizable with a proper choice of resource allocation parameters. For this system, we define the modes according to the following table:

Mode(M)	Condition
0	$x_1(t) = 0, x_2(t) = 0$
1	$x_1(t) > 0, x_2(t) = 0$
2	$x_1(t) = 0, x_2(t) > 0$
3	$x_1(t) > 0, x_2(t) > 0$

Associated with mode 1, we define $\theta_{1,1}$ to be the fraction of the service received by queue 1. Since queue 2 is empty in this mode, we also get $\frac{u_2(t)}{\mu_2} = \frac{\alpha_2(t)}{\mu_2} \le 1 - \theta_{1,1}$. Similarly, we let $\theta_{2,2}$ be the fraction of the service received by queue 2 when $M(t) = 2$ and let $\frac{u_1(t)}{\mu_1} = \frac{\alpha_1(t)}{\mu_1} \le 1 - \theta_{2,2}$. Also, when $M(t) = 3$, we let $\theta_{1,3}$ and $\theta_{2,3} = 1 - \theta_{1,3}$ be the service fractions received by queue 1 and queue 2, respectively.

We next show that utilizing the resulting IPA derivatives evaluated through Eqs. 45, 46 and 44 in a gradient descent optimization algorithm recovers the $c\mu$-rule as the optimal resource allocation policy. We have averaged the estimated values over eight different intervals of length $T = 80{,}000$. As shown in Figs. 4 and 5, we have put the obtained estimates to test by using them in a gradient optimization algorithm. Three initial conditions are considered for the allocation parameters: (a) $\theta_{1,1} = \theta_{2,2} = \theta_{1,3} = 0.77$; (b) $\theta_{1,1} = 0.47$, $\theta_{2,2} = 0.66$ and $\theta_{1,3} = 0.51$; (c) $\theta_{1,1} = 0.23$, $\theta_{2,2} = 0.36$ and $\theta_{1,3} = 0.17$. Notice that the system starting with the initial conditions (b) or (c) is at first unstable which incurs a very high cost not shown in Fig. 4. Moreover, in case (a) $\theta_{1,1}$ and $\theta_{2,2}$ evolve very close to each other making it difficult to differentiate them in the figure. One can see that, regardless of the starting point, the

Fig. 4 Average linear holding cost vs. optimization iteration obtained for three different initial conditions: **a** Solid, **b** Dashed and **c** Dotted



optimization algorithm recovers the $c\mu$-rule. The obtained optimal solution matches that of the $c\mu$-rule directly applied to the underlying DES (black dash-dot line). Notice that whether the initial conditions are non-idling or not the optimal solution turns out to be non-idling.

5.2 IPA and online optimization for average non-linear holding cost

In this section, we apply the same IPA estimation approach to a nonlinear holding cost

$$Q(\theta) = \frac{1}{T}\int_0^T \left[c_1 x_1(t) + c_2 x_2^2(t)\right] dt \tag{48}$$

in which we have $g_1(x_1(t)) = x_1(t)$ and $g_2(x_2(t)) = x_2^2(t)$, therefore, $\frac{\partial g_2(x_2(t))}{\partial x_2} = 2x_2(t)$ for all $t \in [0, T]$ and show that $c\mu$-rule is no longer optimal. Although using IPA derivatives we can find an optimal solution to this problem, we emphasize that,

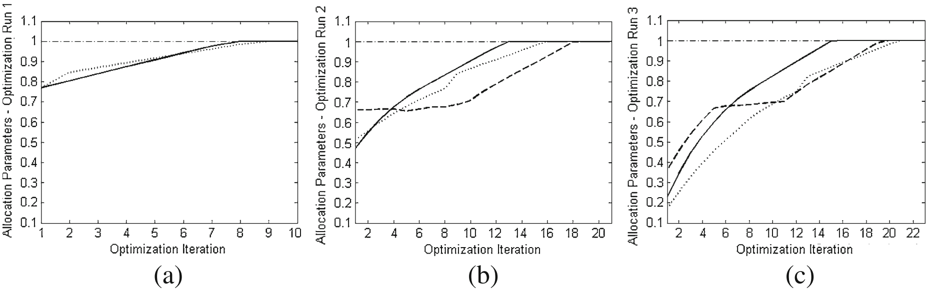


Fig. 5 Allocation parameters vs. optimization iteration for three different initial conditions. ($\theta_{1,1}$: *Solid*, $\theta_{2,2}$: *Dash*, $\theta_{1,3}$: *Dot*)

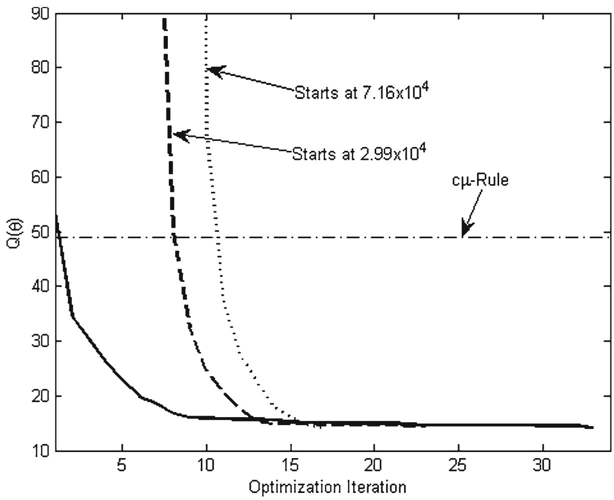


Fig. 6 Section 5.2—average nonlinear holding cost vs. optimization iteration obtained for three different initial conditions: **a** Solid, **b** Dashed and **c** Dotted

in general, the solution will be a local optimum. When carrying out the numerical examples we treat the values of the allocation parameters as the probability that the resource is given to their associated queue. For example, if $\theta_{1,1} = 0.5$ we use a randomization technique which, with probability 0.5, chooses queue 1 to serve from.

Figures 6 and 7 summarize the results by showing the convergence of the online optimization algorithm to a solution far lower than the benchmark obtained from applying the $c\mu$-rule to minimize the cost (Eq. 48). This can be further amplified when the system on hand is unstable, i.e., when

$$\frac{\bar{\alpha_1}}{\mu_1} + \frac{\bar{\alpha_2}}{\mu_2} > 1.$$

Notice that $\theta^*_{1,1} = 1$ and $\theta^*_{2,2} = 1$ indicate that the optimal policy should be non-idling.

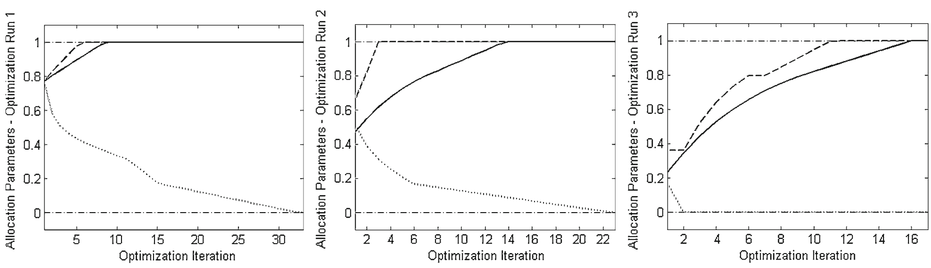


Fig. 7 Allocation parameters vs. optimization iteration for three different initial conditions. ($\theta_{1,1}$: *Solid*, $\theta_{2,2}$: *Dashed*, $\theta_{1,3}$: *Dotted*)

The above experiments unveil the ability of the IPA estimates along with simple online optimization algorithms to recover the optimal solutions for nonlinear problems which are very difficult to find theoretically. It should be pointed out that when the number of queues grows larger this approach can become cumbersome. In fact the number of parameters to be optimized for a problem with N queues can be found to be:

$$D = \sum_{n=1}^{N} n \binom{N}{n} - 1 = N2^{N-1} - 1.$$

Nevertheless, we usually do not face dimensions high enough to make the optimization a difficult task. Finally, we should mention that, as in all gradient-based optimization, the optimal solutions obtained are generally local. However, finding a local optimal solution can be the best to serve solving a highly nonlinear problem and as Section 5.2 suggest can outperform the heuristic solutions.

6 Conclusions

We have considered a classic scheduling problem with a single resource shared by N competing queues in the context of SFMs. By means of the IPA methodology, we have derived explicit sample derivatives of the cost function with respect to a controllable set of parameters in the scheduling policy. In case of the linear total holding cost objective, exploiting the monotonicity of these sample derivatives we have proved the optimality of the well-known $c\mu$-rule not only for the two-queue case (as in earlier work Kebarighotbi and Cassandras 2009) but for an arbitrary finite number of queues and stochastic processes under non-idling policies. The generality of our results confirms the validity of the $c\mu$-rule without having to make explicit distributional assumptions on the random processes involved or resort to heavy traffic analysis. It is worth pointing out, however, that the optimality of the $c\mu$-rule for an arbitrary number of queues does not exclude the existence of other optimal policies.

The use of SFMs and IPA opens up a spectrum of possibilities for studying complex stochastic scheduling problems without having to resort to explicit probabilistic models. As a first step, we have shown that IPA derivatives can be used to determine optimal scheduling policies for the case where the holding costs are nonlinear and the $c\mu$-rule is no longer optimal. The same approach can also be applied to settings with finite queue capacities and different performance metrics.

Appendix: Proofs of the Lemmas

Proof of Lemma 1 If the start of a NEPn is due to a positive jump in $\alpha_n(t)$ at t_k, it is an exogenous event and by Eq. 13 the Lemma's statement immediately follows. If $\alpha_n(t)$ is continuous at t_k, we have $\alpha_n(t_k^-) = \alpha_n(t_k^+) = \mu_n\theta_{n,m}(t_k)$. Then, using Eq. 4, we see that $f_n(t_k^-) - f_n(t_k^+) = 0$. By Assumption 1 no other inflow rate can be discontinuous at t_k. Therefore, $\alpha_l(t_k^-) = \alpha_l(t_k^+)$ for all $l \neq n$. Looking at Eq. 3, we find that $u_l(t_k^-) = u_l(t_k^+)$ for $l \neq n$, so that from Eq. 4 we get $f_l(t_k^-) - f_l(t_k^+) = 0$. Therefore, in the case of an endogenous event at t_k we find that $[f_l(t_k^-) - f_n(t_l^+)] = 0$ for all $l = 1, \ldots, N$ and the proof is complete. □

Proof of Lemma 2 Starting with part (a) for $n = 1$, by the definition of $\eta_{1,m}$ we have $\theta_1(\eta_{1,m}^+) = \frac{\alpha_1(\eta_{1,m})}{\mu_1}$. Consequently, $\Delta\theta_1(\eta_{1,m}) = \frac{\alpha_1(\eta_{1,m})}{\mu_1} - \theta_1(\eta_{1,m}^-)$. At $\eta_{1,m}$ the system switches to the $c\mu$-rule, so the total residual resource capacity is allocated to queue 2, i.e., $\theta_2(\eta_{1,m}^+) = 1 - \frac{\alpha_1(\eta_{1,m})}{\mu_1}$, $\theta_n(\eta_{1,m}^+) = 0$ for $n > 2$. Applying Eq. 20, part (a) of the lemma is proved.

For part (b), let us first prove $\Delta\theta_n(\eta_{k,m}) = 0$ for all $k \neq n, n-1$. If $n < k$ then by Eq. 19 queue n is empty before $\eta_{k,m}$. By Assumption 1, $\alpha_n(t)$ is continuous at $\eta_{k,m}$, so Eq. 3 gives $\theta_n(\eta_{k,m}^-) = \theta_n(\eta_{k,m}^+) = \frac{\alpha_n(\eta_{k,m})}{\mu_k}$ which in turn results in $\Delta\theta_n(\eta_{k,m}) = 0$ for $n < k$. On the other hand, when $n > k+1$, according to the $c\mu$-rule, no resource is given to queue n before or after $\eta_{k,m}$ and we get $\theta_n(\eta_{k,m}^-) = \theta_n(\eta_{k,m}^+) = 0$, thereby, yielding $\Delta\theta_n(\eta_{k,m}) = 0$ for $n > k+1$. Combining these two results establishes $\Delta\theta_n(\eta_{k,m}) = 0$ for all $k \neq n, n-1$. Next, since $n > 2$, the $c\mu$-rule applies to the interval before and after $\eta_{n-1,m}$. At $\eta_{n-1,m}^-$ according to the $c\mu$-rule we have $\theta_n(\eta_{n-1,m}^-) = 0$. Right after $\eta_{n-1,m}$, queue n takes over the total residual resource capacity, which gives $\theta_n(\eta_{n-1,m}^+) = 1 - \sum_{k=1}^{n-1} \frac{\alpha_k(\eta_{n-1,m})}{\mu_k}$ and $\Delta\theta_n(\eta_{n-1,m}) = 1 - \sum_{k=1}^{n-1} \frac{\alpha_k(\eta_{n-1,m})}{\mu_k}$. Finally, by the $c\mu$-rule, we have $\theta_n(\eta_{n,m}^-) = 1 - \sum_{k=1}^{n-1} \frac{\alpha_k(\eta_{n,m})}{\mu_k}$. After $\eta_{n,m}$, queue n enters an EP and, according to Eq. 3, $\theta_n(\eta_{n,m}^+) = \frac{\alpha_n(\eta_{n,m})}{\mu_n}$. It follows that $\Delta\theta_n(\eta_{n,m}) = -\big(1 - \sum_{k=1}^{n} \frac{\alpha_k(\eta_{n,m})}{\mu_k}\big)$. This completes part (b) of the proof. □

Proof of Lemma 3 Consider Eq. 12 with $t_k = t_m$. By Eq. 10 we have $\frac{\partial x_n(t_m^-)}{\partial\theta_{i,m}} = 0$ for any i. Moreover, t_m is the start of a NEP because $N_m = N$, i.e., all queues are non-empty in $[t_m, t_{m+1})$. Therefore, by Lemma 1, we have $\frac{\partial t_m}{\partial\theta_{i,m}}[f_n(t_m^-) - f_n(t_m^+)] = 0$ for all $i = 1, \ldots, N-1$. Finally, by Eqs. 3, 4, and 8 only $f_i(\tau)$ and $f_N(\tau)$ are dependent on $\theta_{i,m}(t)$ in the interval $[t_m, \eta_{1,m})$. Specifically, we have $f_i(\tau) = \alpha_i(t) - \mu_i\theta_{i,m}(\tau)$ and $f_N(\tau) = \alpha_N(\tau) - \mu_N(1 - \sum_{n=1}^{N-1}\theta_{n,m}(\tau))$ resulting in $\frac{\partial f_i(\tau)}{\partial\theta_{i,m}} = -\mu_i$ and $\frac{\partial f_N(t)}{\partial\theta_{i,m}} = \mu_N$. The proof is complete by applying Eq. 12. □

Proof of Lemma 4 Let us define the set $A = \{\eta_{1,m}, \ldots, \eta_{N,m}\}$. In view of Eq. 12, we first show that $\frac{\partial t_l}{\partial\theta_{i,m}}[f_n(t_l^-) - f_n(t_l^+)] = 0$ for any $l \geq m+1$ such that $t_l \notin A$. There are only two possibilities for this:

Case 1 $t_l \in \Sigma_p$, $p = 1, \ldots, k$ and $k \leq j$. In this case, queue p which was emptied at some $\eta_{p,m} \leq \eta_{k,m}$ becomes non-empty again, i.e., t_l is the start of an NEP. It then follows from Lemma 1 that $\frac{\partial t_l}{\partial\theta_{i,m}}[f_n(t_l^-) - f_n(t_l^+)] = 0$, for all $i = 1, \ldots, N-1$.

Case 2 $t_l \in \Gamma_p$, $p = 1, \ldots, k$ and $k \leq j$. This case is contingent upon the previous one, since it corresponds to the end of a NEP for a queue p with the NEP starting at some time $t_{l^*} < t_l$ such that $t_{l^*} \in (\eta_{k,m}, \eta_{k+1,m})$ (as illustrated in Fig. 8) for some $k \leq j$. We limit ourselves to only one such event occurring in $(\eta_{k,m}, \eta_{k+1,m})$, thereby having $t_{l^*} = t_{l-1}$, since the extension to more events of this kind is straightforward. Since the $c\mu$-rule applies for $t \geq \eta_{1,m}$, the resource switches all its available capacity from queue $k+1$ to queue p in the interval $[t_{l-1}, t_l)$. Using Eq. 14, we will show that $\frac{\partial t_l}{\partial\theta_{i,m}} = 0$. Since t_{l-1} is the start of a NEPp, by Lemma 1 we have $\frac{\partial t_{l-1}}{\partial\theta_{i,m}}[f_n(t_{l-1}^-) - f_n(t_{l-1}^+)] = 0$.

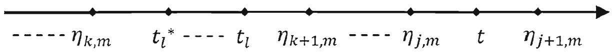


Fig. 8 Lemma 4—a re-start of a NEP of a higher priority queue in an interval $[\eta_{k,m}, \eta_{k+1,m})$

Moreover, $x_p(t) = 0$ for $t \in [\eta_{k,m}, t_{l-1})$ and $\frac{\partial x_p(t_{l-1}^-)}{\partial \theta_{i,m}} = 0$. In addition, by Eqs. 4 and 5, for any $\tau \in [t_{l-1}, t_l)$, $f_p(\tau)$ is not a function of $\theta_{i,m}$ and we have $\frac{\partial f_p(\tau)}{\partial \theta_{i,m}} = 0$. Utilizing these results in Eq. 12 with $t_k = t_{l-1}$ and $n = p$, we get

$$\frac{\partial x_p(t_l^-)}{\partial \theta_{i,m}} = \frac{\partial x_p(t_{l-1}^-)}{\partial \theta_{i,m}} = 0.$$

Using this in Eq. 14 gives $\frac{\partial t_l}{\partial \theta_{i,m}} = 0$.

We can now directly focus on proving Eq. 24. Let $t \in [\eta_{j,m}, \eta_{j+1,m})$ and suppose the last NEP start or end event prior to t occurs at $t_l \geq t_{m+1}$. Clearly, $\frac{\partial f_n(\tau)}{\partial \theta_{i,m}} = 0$ for any $\tau \in [t_l, t)$ and any n. Therefore, applying Eq. 12 at t_l gives

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(t_l^-)}{\partial \theta_{i,m}} + \frac{\partial t_l}{\partial \theta_{i,m}} \left[f_n\left(t_l^-\right) - f_n\left(t_l^+\right) \right], \ t \in [t_l, t).$$

Based on our analysis of the two cases above, if $t_l \notin A$ we get $\frac{\partial t_l}{\partial \theta_{i,m}}[f_n(t_l^-) - f_n(t_l^+)] = 0$ which yields $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(t_l^-)}{\partial \theta_{i,m}}$. If $t_l \in A$, since $t \in [\eta_{j,m}, \eta_{j+1,m})$, it follows that $t_l \equiv \eta_{j,m}$. In this case, by Assumption 1, $\alpha_n(t)$ is continuous at $\eta_{j,m}$ and $f_n(\eta_{j,m}^-) - f_n(\eta_{j,m}^+) = \mu_n \Delta\theta_n(\eta_{j,m})$. Regarding $\frac{\partial x_n(t_l^-)}{\partial \theta_{i,m}}$ in the equation above, we look at the interval $[t_{l-1}, t_l)$ and apply the same analysis at t_{l-1}. It is now clear that by doing this recursively backwards in time we either encounter an event at $t_r < t_{l-1}$ with $\frac{\partial t_r}{\partial \theta_{i,m}}[f_n(t_r^-) - f_n(t_r^+)] = 0$ or some $\eta_{k,m}$ with $k \leq j$, in which case a term $\mu_n \Delta\theta_n(\eta_{k,m})$ is added on. The recursion ends at $\eta_{1,m}$ where Eq. 24 is recovered and the lemma is proved. □

Proof of Lemma 5 The case where $\eta_{N,m} > T$ is trivial since after T the derivatives vanish. When $\eta_{N,m} \leq T$, recall that all queues become empty right after $\eta_{N,m}$, i.e., $x_n(\eta_{N,m}) = 0$ for all $n = 1, \ldots, N$ and $\frac{\partial x_n(\eta_{N,m})}{\partial \theta_{i,m}} = 0$. Since $\eta_{N,m} > t_{m+1}$, all functions $f_n(\tau)$ in Eq. 4 become independent of $\theta_{i,m}$ for every $\tau \in [\eta_{N,m}, T)$. As a result, all $x_n(t)$ are independent of $\theta_{i,m}$ for the same interval. Let us consider the event time derivatives $\frac{\partial t_k}{\partial \theta_{i,m}}$ for $t_k > \eta_{N,m}$. If $t_k \in \Lambda_n$ or Σ_n, Eq. 13 or Lemma 1 gives $\frac{\partial t_k}{\partial \theta_{i,m}}[f_n(t_k^-) - f_n(t_k^+)] = 0$ for any n and i. If $t_k \in \Gamma_n$ for some n, its associated NEP must have started at some t_l such that after $\eta_{N,m} < t_l < t_k$. Since, $x_n(t) = 0$ for $t \in [\eta_{n,m}, t_l)$, we get $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$. Moreover, by Lemma 1 we have $\frac{\partial t_l}{\partial \theta_{i,m}}[f_n(t_l^-) - f(t_l^+)] = 0$. Also, by the definition of $\theta_{i,m}(t)$, $f_n(\tau)$ is independent of $\theta_{i,m}(t)$ over the interval $[t_l, t_k)$. Starting from t_l and using all these observations in Eq. 12, we find that $\frac{\partial x_n(t)}{\partial \theta_{i,m}} = 0$ over the associated NEPn. Specifically, we find $\frac{\partial x_n(t_k^-)}{\partial \theta_{i,m}} = 0$. Using this in Eq. 14 implies that $\frac{\partial t_k}{\partial \theta_{i,m}} = 0$ for all $t_k > \eta_{N,m}$ such that $t_k \in \Gamma_n$ for some n. Since the event time derivatives do not contribute a nonzero value for any k such that $t_k > \eta_{N,m}$ and since $f_n(\tau)$ is independent of $\theta_{i,m}$ for all $\tau \in [\eta_{N,m}, T)$, the lemma's statement is easily concluded by Eq. 12. □

Proof of Lemma 6 By Lemma 3 for $n < i$ we get

$$\frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} = 0. \tag{49}$$

Invoking Eq. 14 with $t_k = \eta_{1,m}$ immediately proves Eq. 25 for $n = 1$, i.e. $\frac{\partial \eta_{1,m}}{\partial \theta_{i,m}} = 0.$. Considering $n \geq 2$, using Lemma 4 for interval $[\eta_{n-1,m}, \eta_{n,m})$ and $t = \eta_{n,m}^-$ we get

$$\frac{\partial x_n(\eta_{n,m}^-)}{\partial \theta_{i,m}} = \frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} + \sum_{k=1}^{n-1} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \mu_n \Delta\theta_n(\eta_{k,m}). \tag{50}$$

By Eq. 49 the first term in the above equation is zero. Assuming $i > 2$, evaluating Eq. 50 for $n = 2$ gives

$$\frac{\partial x_2(\eta_{2,m}^-)}{\partial \theta_{i,m}} = \frac{\partial \eta_{1,m}}{\partial \theta_{i,m}} \mu_2 \Delta\theta_2(\eta_{1,m}) = 0.$$

Inserting this result in Eq. 14 for $t_k = \eta_{2,m}$ implies that $\frac{\partial \eta_{2,m}}{\partial \theta_{i,m}} = 0$. Repeating the same process for $n = 3, \ldots, i-1$ completes the proof of the lemma. □

Proof of Lemma 7 We proceed by considering the cases $i = n$ and $i < n$ separately.

- $i = n$: When $n = 1$, it is easy to verify the Lemma's claim since, according to Eq. 14, we have

  $$\frac{\partial \eta_{1,m}}{\partial \theta_{1,m}} = \frac{-\frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}}}{f_1(\eta_{1,m}^-)}$$

  Multiplying both numerator and denominator by μ_1 and using Eq. 26 gives

  $$\frac{\partial \eta_{1,m}}{\partial \theta_{1,m}} = \frac{r_{1,m}}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}}.$$

  When $n > 1$ we use Lemma 4 with $j = n-1$ to get, for any $t \in [\eta_{n-1,m}, \eta_{n,m})$,

  $$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} + \sum_{k=1}^{n-1} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \mu_n \Delta\theta_n(\eta_{k,m}).$$

  Because $i = n$ in each term of the sum above, we have $k < i$, hence, by Lemma 6, $\frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} = 0$ for $k = 1, \ldots, i-1$ and we conclude that

  $$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} = \frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{n,m}}.$$

  By Eq. 22 we have

  $$\frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{n,m}} = \frac{\mu_n}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}}. \tag{51}$$

  Using this expression in Eq. 14 yields

  $$\frac{\partial \eta_{n,m}}{\partial \theta_{n,m}} = \frac{-\mu_n}{f_n(\eta_{n,m}^-)} \frac{1}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} = \frac{r_{n,m}}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} \tag{52}$$

  which completes the proof of the lemma for $n = i$.

- $i < n$: By Lemma 3, because $n \neq i$ we have $\frac{\partial x_n(\eta_{1,m}^-)}{\partial \theta_{i,m}} = 0$. Therefore, invoking Lemma 4, for any $t \in [\eta_{n-1,m}, \eta_{n,m})$ we have:

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \sum_{k=1}^{n-1} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \mu_n \Delta\theta_n(\eta_{k,m}).$$

However, by Lemma 6, $\frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} = 0$ when $k < i$. Therefore, the above equation reduces to

$$\frac{\partial x_n(t)}{\partial \theta_{i,m}} = \sum_{k=i}^{n-1} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \mu_n \Delta\theta_n(\eta_{k,m}). \tag{53}$$

Using this expression in Eq. 14 yields the event time derivative

$$\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}} = r_{n,m} \sum_{k=i}^{n-1} \frac{\partial \eta_{k,m}}{\partial \theta_{i,m}} \Delta\theta_n(\eta_{k,m}). \tag{54}$$

According to part (b) of Lemma 2, $\Delta\theta_n(\eta_{k,m})$ can only be nonzero when $k = n - 1$. Thus, the above expression reduces to

$$\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}} = r_{n,m} \frac{\partial \eta_{n-1,m}}{\partial \theta_{i,m}} \Delta\theta_n(\eta_{k,m}). \tag{55}$$

This is clearly a recursive equation in n. Starting with $n = i + 1$ and proceeding forward in time we find that

$$\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}} = \frac{\partial \eta_{i,m}}{\partial \theta_{i,m}} \prod_{k=i+1}^{n} r_{k,m} \Delta\theta_k(\eta_{k-1,m}).$$

Now using Eq. 52 with n replaced by i to find $\frac{\partial \eta_{i,m}}{\partial \theta_{i,m}}$ and inserting the result in the above equation gives the final result as

$$\frac{\partial \eta_{n,m}}{\partial \theta_{i,m}} = \frac{r_{i,m}}{\mu_1} \frac{\partial x_1(\eta_{1,m}^-)}{\partial \theta_{1,m}} \prod_{k=i+1}^{n} r_{k,m} \Delta\theta_k(\eta_{k-1,m}) \tag{56}$$

which completes the proof. □

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