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Optimal routing and charging of energy-limited vehicles in traffic networks

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SUMMARY

We study the problem of routing vehicles with energy constraints through a network where there are at least some charging nodes. We seek to minimize the total elapsed time for vehicles to reach their destinations by determining routes, as well as recharging amounts when the vehicles do not have adequate energy for the entire journey. For a single vehicle, we formulate a mixed-integer nonlinear programming problem and derive properties of the optimal solution allowing it to be decomposed into two simpler problems. For a multi-vehicle problem, where traffic congestion effects are included, we seek to optimize a system-wide objective and formulate the problem by grouping vehicles into 'subflows'. We also provide an alternative flow optimization formulation leading to a computationally simpler problem solution with minimal loss in accuracy. Because the problem size increases with the number of subflows, a proper selection of this number is essential to render the problem computationally manageable and reflects a trade-off between proximity to optimality and computational effort needed to solve the problem. We propose a criterion and procedure leading to an appropriate choice of the number of subflows. We also quantify the 'price of anarchy' for this problem and compare user-optimal to system-optimal performance. Finally, when the system consists of both electric vehicles (EVs) and non-electric vehicles, we formulate a system-centric optimization problem for optimal routing of both non-electric vehicles and EVs along with an optimal policy for charging EVs along the way if needed. Numerical results are included to illustrate these approaches. Copyright © 2015 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The increasing presence of battery-powered vehicles (BPVs), such as electric vehicles (EVs), in traffic networks has created new challenges for integrating BPVs into such networks while also maintaining high overall traffic performance levels. More generally, when the entities in a network are characterized by physical attributes exhibiting a dynamic behavior, this behavior can play an important role in the routing decisions thus giving rise to novel issues in classical network routing problems [1]. In the case of BPVs, the key physical attribute is *energy*, and there are four BPV characteristics that are crucial in routing problems: limited cruising range, long charge times, sparse coverage of charging stations, and the BPV energy recuperation ability [2], which can be exploited.

In recent years, the vehicle routing literature has been enriched by work aiming to accommodate the aforementioned BPV characteristics. For example, by incorporating the recuperation ability of EVs (which leads to negative energy consumption on some paths), extensions to general shortest

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path algorithms are proposed in [2] that address the energy-optimal routing problem. The energy requirements in this problem are modeled as constraints, and the proposed algorithms are evaluated in a prototypical navigation system. Extensions provided in [3] employ a generalization of Johnson's potential shifting technique to make Dijkstra's algorithm applicable to the negative link cost shortest path problem so as to improve the results and allow for route planning of EVs in large networks. This work, however, does not consider the presence of charging stations modeled as nodes in the network. Charging times are incorporated into a multi-constrained optimal path planning problem in [4], which aims to minimize the length of an EV's route and meet constraints on total traveling time, total time delay due to signals, total recharging time, and total recharging cost; a particle swarm optimization algorithm is then used to find a suboptimal solution. In this formulation, however, recharging times are simply treated as parameters and not as controllable variables. In [5], algorithms for several routing problems are proposed, including a single vehicle routing problem with inhomogeneously priced refueling stations for which a dynamic programming-based algorithm is proposed to find a least cost path from source to destination. In [6], the same problem is revisited, assuming that the recharging cost is a nonlinear function of the battery charging level and a dynamic programming algorithm is proposed to find a minimum cost path for an EV. More recently, an EV routing problem with time windows and recharging stations was proposed in [7], where an EV's energy constraint is first introduced into vehicle routing problems, and recharging times depend on the battery charge of the vehicle upon arrival at the station. Controlling recharging times is circumvented by simply forcing vehicles to be always fully recharged. In [8], an integer programming optimization problem was formulated to simultaneously find optimal routes and charging station locations for commercial electric vehicles. In [9], a heuristic algorithm is proposed to find the energy-optimal routing for EVs taking into account the energy recuperation, battery capacity limitations, and dynamic energy cost imposed by the vehicle properties. Combinatorial optimization methods for different aspects of EV management such as energy-efficient routing and facility location problems are studied in [10]. In the unmanned autonomous vehicle (UAV) literature, Sunder and Rathinam [11] consider a UAV routing problem with refueling constraints. In this problem, given a set of targets and depots, the goal is to find an optimal path such that each target is visited by the UAV at least once, while the fuel constraint is never violated. A Mixed-Integer Nonlinear Programming (MINLP) formulation is proposed with a heuristic algorithm to determine feasible solutions.

In this paper, our objective is to study a vehicle total traveling time minimization problem (including both the time on paths and at charging stations) as introduced in [12], where an energy constraint is considered so that the vehicle is not allowed to run out of power before reaching its destination. We view this as a network routing problem where vehicles control not only their routes but also times to recharge at various nodes in the network. Our contributions are twofold. First, for the single energy-aware vehicle routing problem, formulated as a MINLP, we show that there are properties of the optimal solution and the energy dynamics allowing us to decompose the original problem into two simpler problems, provided charging speeds are homogeneous (but charging prices need not be the same). Thus, we separately determine route selection through a Linear Programming (LP) problem and then recharge amounts through another LP or simple optimal control problem. Because we do not impose full recharging constraints, the solutions obtained are more general than, for example, in [7] and recover full recharging when this is optimal. Second, we study a multi-vehicle energy-aware routing problem, where a traffic flow model is used to incorporate congestion effects. This system-wide optimization problem appears to have not yet attracted much attention. By grouping vehicles into 'subflows', we are once again able to decompose the problem into route selection and recharge amount determination, although we can no longer reduce the former problem to an LP. Moreover, we provide an alternative flow-based formulation such that each subflow is not required to follow a single end-to-end path but may be split into an optimally determined set of paths. This formulation reduces the computational complexity of the MINLP problem by orders of magnitude with numerical results showing little or no loss in optimality. We further study the 'price of anarchy' for the multi-vehicle routing problem so as to determine the difference in performance between selfish routing and system-optimal routing. Finally, we address the issue of selecting the number of subflows, seeking to keep it as small as possible.

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The structure of the paper is as follows. In Section 2, we introduce and address the single-vehicle routing problem and identify properties, which lead to its decomposition. In Section 3, the multi-vehicle routing problem is formulated, first as an MINLP and then as an alternative flow optimization problem. We also investigate the price of anarchy for this problem and provide simulation examples illustrating our approach and giving insights on the relationship between recharging speed and optimal routes. In Section 4 we define a criterion and a systematic procedure for the proper selection of the number of subflows. In Section 5, the multi-vehicle routing problem is revisited when the inflow to the network contains both EVs and non-EVs. Finally, conclusions and further research directions are outlined in Section 6.

2. SINGLE VEHICLE ROUTING

We consider a traffic network modeled as a directed graph $G = (\mathcal{N}, \mathcal{A})$ with $\mathcal{N} = \{1, \dots, n\}$ and $|\mathcal{A}| = m$ (Figure 1). Node $i \in \mathcal{N}/\{n\}$ represents a charging station, and $(i, j) \in \mathcal{A}$ is an arc (link) connecting node i to j. We assume for simplicity that all nodes have a charging capability, although this is not necessary (we can always reduce a graph with some non-charging nodes into one including only those that can charge BPVs). We also define $I(i)$ and $O(i)$ to be the set of start nodes (respectively, end nodes) of arcs that are incoming to (respectively, outgoing from) node i; that is, $I(i) = \{j \in \mathcal{N} | (j, i) \in \mathcal{A}\}$ and $O(i) = \{j \in \mathcal{N} | (i, j) \in \mathcal{A}\}$.

We are first interested in a single-origin-single-destination vehicle routing problem. Nodes 1 and n, respectively, are defined to be the origin and destination. For each arc $(i, j) \in \mathcal{A}$, there are two cost parameters: the required traveling time τ_{ij} and the required energy consumption e_{ij} on this arc. Note that $\tau_{ij} > 0$ (if nodes i and j are not connected, then $\tau_{ij} = \infty$), whereas e_{ij} is allowed to be negative due to a BPV's potential energy recuperation effect [2]. Letting the vehicle's charge capacity be B, we assume that $e_{ij} < B$ for all $(i, j) \in \mathcal{A}$. Because we are considering a single vehicle's behavior, we assume that it will not affect the overall network's traffic state; therefore, τ_{ij} and e_{ij} are assumed to be fixed depending on given traffic conditions at the time the single-vehicle routing problem is solved. Clearly, this cannot apply to the multi-vehicle case in the next section, where the decisions of multiple vehicle routes affect traffic conditions thus influencing traveling times and energy consumption. Because the BPV has limited battery energy, it may not be able to reach the destination without recharging. Thus, recharging amounts at charging nodes $i \in \mathcal{N}$ are also decision variables.

We denote the selection of arc (i, j) and energy recharging amount at node i by $x_{ij} \in \{0, 1\}$, $i, j \in \mathcal{N}$ and $r_i \geqslant 0, i \in \mathcal{N}/\{n\}$, respectively. Moreover, because we take into account the vehicle's energy constraints, we use E_i to represent the vehicle's residual battery energy at node i. Then, for all E_j, $j \in O(i)$, we have

$$E_j = \begin{cases} E_i + r_i - e_{ij} & \text{if } x_{ij} = 1 \\ 0 & \text{otherwise} \end{cases}$$

which can also be expressed as

$$E_j = \sum_{i \in I(j)} (E_i + r_i - e_{ij}) x_{ij}, \quad x_{ij} \in \{0, 1\}$$

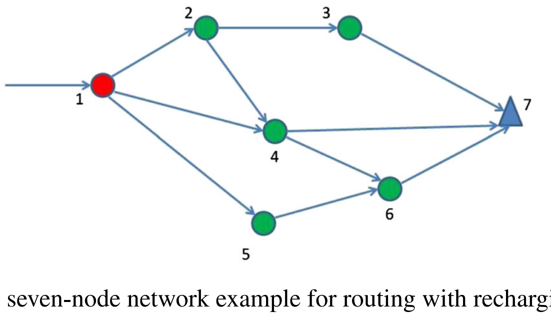

Figure 1. A seven-node network example for routing with recharging nodes.

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The problem objective is to determine a path from 1 to n, as well as recharging amounts, so as to minimize the total elapsed time for the vehicle to reach the destination. Figure 1 is a sample network for this vehicle routing problem. We formulate an MINLP problem as follows:

$$\min_{x_{ij},r_i,\, i,j\in\mathcal{N}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\tau_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} r_i g x_{ij} \tag{1}$$

$$s.t. \quad \sum_{j\in O(i)} x_{ij} - \sum_{j\in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N} \tag{2}$$

$$b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1, n \tag{3}$$

$$E_j = \sum_{i\in I(j)} (E_i + r_i - e_{ij})x_{ij},\ \text{for } j = 2, \ldots, n \tag{4}$$

$$0 \leqslant E_i \leqslant B, \quad E_1 \text{ given},\ \text{for each } i \in \mathcal{N} \tag{5}$$

$$x_{ij} \in \{0, 1\}, \quad r_i \geqslant 0 \tag{6}$$

where g is the charging time per energy unit, that is, the reciprocal of a fixed charging rate. The constraints (2)–(3) stand for the flow conservation, which implies that only one path starting from node i can be selected, that is, $\sum_{j\in O(i)} x_{ij} \leqslant 1$ [13]. It is easy to check that this also implies $x_{ij} \leqslant 1$ for all i, j because $b_1 = 1$, $I(1) = \varnothing$. Constraint (4) represents the vehicle's energy dynamics. Finally, (5) indicates that the vehicle cannot run out of energy before reaching a node or exceed a given capacity B. All other parameters are predetermined according to the network topology. We point out that in this formulation, we assume homogeneous charging nodes (i.e., g is fixed). The more general case of inhomogeneous charging nodes is addressed in [14] and is the subject of continuing research.

2.1. Properties

Rather than directly tackling the MINLP problems (1)–(6), we derive some key properties that will enable us to simplify the solution procedure. The main difficulty in this problem lies in the coupling of the decision variables, x_{ij} and r_i, in (4). The following lemma will enable us to exclude r_i from the objective function by showing that the difference between the total recharging energy and the total energy consumption while traveling is given only by the difference between the vehicle's residual energy at the destination and at the origin.

Lemma 1
Given (1)–(6),

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(r_i x_{ij} - e_{ij}x_{ij}) = E_n - E_1 \tag{7}$$

Proof
From (4), we sum up both sides to obtain

$$\sum_{j=2}^{n} E_j - \sum_{j=2}^{n}\sum_{i\in I(j)} E_i x_{ij} = \sum_{j=2}^{n}\sum_{i\in I(j)} (r_i - e_{ij})x_{ij} \tag{8}$$

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Moreover, we can write

$$\sum_{j=2}^{n} \sum_{i \in I(j)} E_i x_{ij} = \sum_{i \in I(2)} E_i x_{i2} + \cdots + \sum_{i \in I(n)} E_i x_{in}$$

representing the sum of E_i on the selected path from node 1 to n, excluding E_n. On the other hand, from (4), we have $E_i = 0$ for any node i not selected on the path. Therefore, $\sum_{j=2}^{n} E_j$ is the sum of E_i on the selected path from node 1 to n, excluding E_1. It follows that

$$\sum_{j=2}^{n} E_j - \sum_{j=2}^{n} \sum_{i \in I(j)} E_i x_{ij} = E_n - E_1 \tag{9}$$

Returning to (8), we use (9) and observe that all terms in the double sum $\sum_{i=1}^{n} \sum_{j=1}^{n} (r_i - e_{ij}) x_{ij}$ are zero except for those with $i \in I(j)$, we obtain

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (r_i - e_{ij}) x_{ij} = \sum_{j=2}^{n} \sum_{i \in I(j)} (r_i - e_{ij}) x_{ij} = \sum_{j=2}^{n} E_j - \sum_{j=2}^{n} \sum_{i \in I(j)} E_i x_{ij} = E_n - E_1$$

which proves the lemma. □

In view of Lemma 1, we can replace $\sum_{i=1}^{n} \sum_{j=1}^{n} r_i g x_{ij}$ in (1) by $(E_n - E_1)g + \sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} g x_{ij}$ and eliminate the presence of r_i, $i = 2, \ldots, n-1$, from the objective function. Note that E_1 is given, leaving us only with the task of determining the value of E_n. Now, let us investigate the recharging energy amounts r_i^*, $i = 1, \ldots, n-1$, in an optimal policy. There are two possible cases: (i) $\sum_i r_i^* > 0$, that is, the vehicle has to get recharged at least once; and (ii) $\sum_i r_i^* = 0$, that is, $r_i^* = 0$ for all i, and the vehicle has adequate energy to reach the destination without recharging. For case (i), we establish the following lemma.

Lemma 2
If $\sum_i r_i^* > 0$ in the optimal routing policy, then $E_n^* = 0$.

Proof
We use a contradiction argument. Assume that we have already achieved an optimal route, where $E_n^* > 0$ and the objective function is $J^* = \sum_{i \in P} (\tau_{i,i+1} + r_i^* g)$ for an optimal path denoted by P. Without loss of generality, we re-index nodes so that we may write $P = \{1, \ldots, n\}$. Then, each $i \in P$ such that $i < n$ on this optimal path satisfies

$$E_{i+1}^* = E_i^* + r_i^* - e_{i,i+1} \tag{10}$$

Consider first the case where $r_{n-1}^* > 0$. Let us perturb the current policy as follows: $r_{n-1}^{'} = r_{n-1}^* - \Delta$, and $r_i^{'} = r_i^*$ for all $i < n-1$, where $\Delta > 0$. Then, from (10), we have

$$E_n^* = E_1 + \sum_{i=1}^{n-1} \left(r_i^* - e_{i,i+1} \right)$$

Under the perturbed policy,

$$E_n^{'} = E_1 + \sum_{i=1}^{n-1} \left(r_i^{'} - e_{i,i+1} \right) = E_1 + \sum_{i=1}^{n-1} \left(r_i^* - e_{i,i+1} \right) - \Delta = E_n^* - \Delta$$
$$E_i^{'} = E_i^*, \text{ for all } i < n$$

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and, correspondingly,

$$J' = \sum_{i=1}^{n-1} \left(\tau_{i,i+1} + r_i' g\right) = \sum_{i=1}^{n-1} \left(\tau_{i,i+1} + r_i^* g\right) - \Delta g = J^* - \Delta g$$

Because $E_n^* > 0$, we may select $\Delta > 0$ sufficiently small so that $E_n' > 0$, and the perturbed policy is still feasible. However, $J' = J^* - \Delta g < J^*$, which leads to a contradiction to the assumption that the original path was optimal.

Next, consider the case where $r_{n-1}^* = 0$. Then, because of $E_n^* > 0$ and $e_{i,i+1} > 0$ for all $i \in P$, we can always find some $j \in P$, $j < n$ such that $E_j^* > 0$, $r_{j-1}^* > 0$, and $r_k^* = 0$ for $k \geqslant j$. Thus, still, because of (10), we have

$$E_j^* = E_n^* + \sum_{k=j}^{n-1} e_{k,k+1} > 0$$

At this time, because $r_{j-1}^* > 0$, the argument is similar to the case $r_{n-1}^* > 0$, leading again to the same contradiction argument, and the lemma is proved. $\square$

Turning our attention to case (ii) where $r_i^* = 0$ for all $i \in \{1, \ldots, n\}$, observe that problem (1) can be transformed into

$$\min_{x_{ij},\ i,j \in \mathcal{N}} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \tau_{ij} x_{ij} \tag{11}$$

$$s.t. \quad \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N}$$

$$b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1, n$$

$$E_j = \sum_{i \in I(j)} (E_i - e_{ij}) x_{ij},\ \text{for } j = 2, \ldots, n \tag{12}$$

$$0 \leqslant E_i \leqslant B, \quad E_0 \text{ given},\ \text{for each } i \in \mathcal{N}$$

$$x_{ij} \in \{0, 1\} \tag{13}$$

In this case, constraint (12) gives

$$\sum_{j=2}^{n} E_j - \sum_{j=2}^{n} \sum_{i \in I(j)} E_i = -\sum_{j=2}^{n} \sum_{i \in I(j)} e_{ij} x_{ij}$$

Using (9) and $E_i \geqslant 0$, we have

$$E_n = E_1 - \sum_{j=2}^{n} \sum_{i \in I(j)} e_{ij} x_{ij} \geqslant 0$$

and it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_{ij} x_{ij} \leqslant E_1 \tag{14}$$

With (14) in place of (12), the determination of x_{ij}^* boils down to an integer *linear* programming problem in which only variables x_{ij}, $i, j \in \mathcal{N}$, are involved, a much simpler problem.

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We are normally interested in case (i), where some recharging decisions must be made, so let us assume that the vehicle's initial energy is not large enough to reach the destination. Then, in view of Lemmas 1 and 2, we have the following theorem.

Theorem 1
If $\sum_i r_i^* > 0$ in the optimal policy, then $x_{ij}^*, i, j \in \mathcal{N}$, in the original problem (1) can be determined by solving an LP problem

$$\begin{aligned} &\min_{x_{ij},\ i,j\in\mathcal{N}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(\tau_{ij}+e_{ij}g)x_{ij} \\ s.t. \quad &\sum_{j\in O(i)} x_{ij} - \sum_{j\in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N} \\ &b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1, n \\ &0 \leqslant x_{ij} \leqslant 1 \end{aligned} \tag{15}$$

Proof
Given Lemmas 1 and 2, we know that the optimal solution satisfies $\sum_i \sum_j r_i^* x_{ij}^* = \sum_i \sum_j e_{ij} x_{ij}^* - E_1$. Consequently, we can change the objective (1) to the form in the succeeding text without affecting optimality

$$\min_{x_{ij},\ i,j\in\mathcal{N}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(\tau_{ij}+e_{ij}g)x_{ij} - E_1 g$$

Because r_i no longer appears in the objective function and is only contained in the energy dynamics (4), we can choose any r_i satisfying the constraints (4)–(5) without affecting the optimal objective function value. Therefore, x_{ij}^* can be determined by the following problem:

$$\begin{aligned} &\min_{x_{ij},\ i,j\in\mathcal{N}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}(\tau_{ij}+e_{ij}g)x_{ij} - E_1 g \\ s.t. \quad &\sum_{j\in O(i)} x_{ij} - \sum_{j\in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N} \\ &b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1, n \\ &x_{ij} \in \{0, 1\} \end{aligned}$$

which is a typical shortest path problem formulation. Moreover, by the property of minimum cost flow problems [15], the aforementioned integer programming problem is equivalent to the LP problem with the integer restriction of x_{ij} relaxed. Finally, because E_1 is given, the problem reduces to (15), which proves the theorem. □

Note that using Theorem 1, an optimal path is determined by solving an LP problem, and because this is a convex optimization problem [16], the solution is the global optimum.

2.2. *Determination of optimal recharging amounts r_i^**

Once we determine the optimal route, P, in (15), it is relatively easy to find a feasible solution for r_i, $i \in P$, to satisfy the constraint (4), which is obviously non-unique in general. Then, we can introduce a second objective into the problem, that is, the minimization of charging costs on the selected path, because charging prices normally vary over stations. As before, we re-index nodes and define $P = \{1, \ldots, n\}$. We denote the charging price at node i by p_i. Once an optimal route is determined, we seek to control the energy recharging amounts r_i to minimize the total charging cost dependent on p_i, $i \in \mathcal{N}/\{n\}$. This can be formulated as a multi-stage optimal control problem

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$$\begin{aligned} \min_{r_i,\, i\in P} \quad & \sum_{i\in P} p_i r_i \\ s.t. \quad & E_{i+1} = E_i + r_i - e_{i,i+1} \\ & 0 \leqslant E_i \leqslant B, \quad E_1 \text{ given} \\ & r_i \geqslant 0 \text{ for all } i \in \mathcal{N} \end{aligned} \tag{16}$$

This is a simple two-point boundary-value problem and can be easily solved by discrete-time optimal control approaches [17] or treating it as an LP problem where E_i and r_i are both decision variables.

Finally, we note that Theorem 1 holds under the assumption that charging nodes are homogeneous in terms of charging speeds (i.e., the charging rate $1/g$ is fixed). However, our analysis allows for inhomogeneous charging prices. The case of node-dependent charging rates is addressed in [14] and can be shown to still allow a decomposition of the MINLP, although we can no longer generally obtain an LP. Thus, alternative techniques need to be explored in order to reduce the computational complexity of the problem.

It is important to ensure that a solution to the overall problem is computationally efficient, because it may have to be repeatedly obtained during the course of a vehicle's trip: Although we treat the state variable E_i as deterministic, in reality, there is noise in the process that may force a re-evaluation of routing and charging at each node when E_{i+1} is observed and may satisfy $E_{i+1} = E_i + r_i - e_{i,i+1} + w_{ij}$, where w_{ij} is a random variable. In this case, one can re-solve the optimal routing and charging problem for the vehicle with new initial conditions at node $i+1$, which is possible as long as we only have to deal with the simple problems (15) and (16).

3. MULTIPLE VEHICLE ROUTING

The results obtained for the single vehicle routing problem pave the way for the investigation of multi-vehicle routing, where we seek to optimize a system-wide objective by routing vehicles through the same network topology. The main technical difficulty in this case is that we need to consider the influence of traffic congestion on both traveling time and energy consumption. A second difficulty is that of implementing an optimal routing policy. In the case of a centrally controlled system, this can be accomplished through appropriately communicated commands. In the case of a traffic network with individual drivers, implementation requires signaling mechanisms and possibly incentive structures to enforce desired routes assigned to vehicles, bringing up a number of additional research issues. In the sequel, we limit ourselves to resolving the first difficulty, leaving implementation challenges as part of ongoing research.

If we proceed as in the single vehicle case, that is, determining a path selection through x_{ij}^k, $i, j \in \mathcal{N}$, and recharging amounts r_i^k, $i \in \mathcal{N}/\{n\}$ for all vehicles $k = 1, \ldots, K$, for some K, then the dimensionality of the solution space is prohibitive. Moreover, the inclusion of traffic congestion effects introduces additional nonlinearities in the dependence of the travel time τ_{ij} and energy consumption e_{ij} on the traffic flow through arc (i, j), which now depends on $x_{ij}^1, \ldots, x_{ij}^K$. Instead, we will proceed by grouping subsets of vehicles into N 'subflows', where N may be selected to render the problem manageable (Section 4).

Let all vehicles enter the network at the origin node 1, and let R denote the rate of vehicles arriving at this node. Viewing vehicles as defining a *flow*, we divide them into N *subflows* (we will discuss the effect of N in Section 3.3), each of which may be selected so as to include the same type of homogeneous vehicles (e.g., vehicles with the same initial energy). Thus, all vehicles in the same subflow follow the same routing and recharging decisions so that we only consider energy recharging at the subflow level rather than individual vehicles. Note that asymptotically, as $N \to \infty$, we can recover routing at the individual vehicle level.

Clearly, not all vehicles in our system are EVs, in which case these can be treated as uncontrollable interfering traffic and are accommodated in our analysis as long as their flow rates are

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known. For simplicity, we will assume here that every arriving vehicle is an EV and joins a subflow. However, we will show in Section 5 how the problem can be solved by optimizing over both EVs and non-EVs.

Our objective is to determine optimal routes and energy recharging amounts for each subflow of vehicles so as to minimize the total elapsed time of these vehicle flows traveling from the origin to the destination. The decision variables consist of $x_{ij}^k \in \{0,1\}$ for all arcs (i,j) and subflows $k = 1,\ldots,N$, as well as charging amounts r_i^k for all nodes $i = 1,\ldots,n-1$ and $k = 1,\ldots,N$. Given traffic congestion effects, the time and energy consumption on each arc depends on the values of x_{ij}^k and the fraction of the total flow rate R associated with each subflow k; the simplest such flow allocation (which we will adopt) is one where each subflow is assigned R/N. Let $\mathbf{x_{ij}} = \left(x_{ij}^1,\ldots,x_{ij}^N\right)^T$ and $\mathbf{r_i} = \left(r_i^1,\ldots,r_i^N\right)^T$. Then, we denote the traveling time (delay) a vehicle will experience through link (i,j) by some nonlinear function $\tau_{ij}(\mathbf{x_{ij}})$. The corresponding energy consumption of the kth vehicle subflow through link (i,j) is a nonlinear function denoted by $e_{ij}^k(\mathbf{x_{ij}})$. As already mentioned, $\tau_{ij}(\mathbf{x_{ij}})$ and $e_{ij}^k(\mathbf{x_{ij}})$ can also incorporate the influence of uncontrollable (non-EV) vehicle flows, which can be treated as parameters in these functions (we discuss this further in Section 5). Similar to the single vehicle case, we use E_i^k to represent the residual energy of subflow k at node i given by the aggregated residual energy of all vehicles in the subflow. If the subflow does not go through node i, then $E_i^k = 0$. The problem formulation is as follows:

$$\min_{\mathbf{x_{ij}},\mathbf{r_i},\, i,j\in\mathcal{N}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\left(\tau_{ij}(\mathbf{x_{ij}})x_{ij}^k\frac{R}{N} + r_i^k g x_{ij}^k\right) \tag{17}$$

$$s.t. \text{ for each } k \in \{1,\ldots,N\}:$$

$$\sum_{j\in O(i)} x_{ij}^k - \sum_{j\in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N} \tag{18}$$

$$b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1,n \tag{19}$$

$$E_j^k = \sum_{i\in I(j)}\left(E_i^k + r_i^k - e_{ij}^k(\mathbf{x_{ij}})\right)x_{ij}^k, \quad j = 2,\ldots,n \tag{20}$$

$$E_1^k \text{ is given}, \quad E_i^k \geqslant 0, \quad \text{for each } i \in \mathcal{N} \tag{21}$$

$$x_{ij}^k \in \{0,1\}, \quad r_i^k \geqslant 0 \tag{22}$$

Obviously, this MINLP problem is difficult to solve. However, as in the single-vehicle case, we are able to establish some properties that will allow us to simplify it.

3.1. Properties

Even though the term $\tau_{ij}(\mathbf{x_{ij}})$ in the objective function is no longer linear in general, for each subflow k, the constraints (18)–(22) are still similar to the single-vehicle case. Consequently, we can derive similar useful properties for this problem in the form of the following two lemmas.

Lemma 3
For each subflow $k = 1,\ldots,N$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(r_i^k - e_{ij}^k(\mathbf{x_{ij}})\right)x_{ij}^k = E_n^k - E_1^k \tag{23}$$

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Proof
From (20), we sum up both sides of the equation as follows:

$$\begin{aligned}
&\text{For each } k: \\
&\sum_{j=2}^{n} E_j^k = \sum_{j=2}^{n} \sum_{i \in I(j)} \left(E_i^k + r_i^k - e_{ij}^k(\mathbf{x_{ij}}) \right) x_{ij}^k \\
&\Longrightarrow \sum_{j=2}^{n} E_j^k - \sum_{j=2}^{n} \sum_{i \in I(j)} E_i^k x_{ij}^k = \sum_{j=2}^{n} \sum_{i \in I(j)} \left(r_i^k - e_{ij}^k(\mathbf{x_{ij}}) \right) x_{ij}^k
\end{aligned} \tag{24}$$

Moreover, $\sum_{j=2}^{n} \sum_{i \in I(j)} E_i^k x_{ij}^k = \sum_{i \in I(j)} E_i^k \sum_{j=2}^{n} x_{ij}^k$ representing the sum of E_i^k on the selected path from node 1 to n. On the other hand, from (20), we have $E_i^k = 0$ for node i not selected on the route. Therefore,

$$\sum_{j=2}^{n} E_j^k - \sum_{j=2}^{n} \sum_{i \in I(j)} E_i^k x_{ij}^k = E_n^k - E_1^k$$

Back to (24),

$$\begin{aligned}
\sum_i \sum_j \left(r_i^k - e_{ij}^k(\mathbf{x_{ij}}) \right) x_{ij}^k &= \sum_{j=2}^{n} \sum_{i \in I(j)} \left(r_i^k - e_{ij}^k(\mathbf{x_{ij}}) \right) x_{ij}^k \\
&= \sum_{j=2}^{n} E_j^k - \sum_{j=2}^{n} \sum_{i \in I(j)} E_i^k x_{ij}^k = E_n^k - E_1^k
\end{aligned}$$

which proves the result. □

Similar to Lemma 2, we can determine E_n^{k*} when $\sum_i r_i^{k*} > 0$ by Lemma 4

Lemma 4
If $\sum_{i=1}^{n} r_i^{k*} > 0$ in the optimal routing policy, then $E_n^{k*} = 0$ for all $k = 1, \ldots, N$.

Proof
Assume that we have already achieved the optimal routes for these k vehicle subflows such that $E_n^{k*} > 0$ and the contribution of kth subflow to the objective function value

$$J_k^* = \sum_{i=1}^{n} \sum_{j=1}^{n} \tau_{ij}(\mathbf{x_{ij}}) x_{ij}^{k*} \frac{R}{N} + \sum_{i=1}^{n} \sum_{j=1}^{n} r_i^{k*} g x_{ij}^{k*}$$

Because only the second part of the objective function is dependent on r_i^k, we only need to concentrate on the value of $\sum_{i=1}^{n} \sum_{j=1}^{n} r_i^{k*} g x_{ij}^{k*}$. Then each $i < n$ on this route satisfies

$$E_{i+1}^{k*} = E_i^{k*} + r_i^{k*} - e_{i,i+1} \tag{25}$$

where $e_{i,i+1}$ is the value of $e_{ij}^k(\mathbf{x_{ij}})$ on the determined route by x_{ij}^{k*} for all k. Now, if $r_{n-1}^{k*} > 0$, then let us perturb the current policy by

$$\begin{aligned}
r_{n-1}^{k'} &= r_{n-1}^{k*} - \Delta \\
r_i^{k'} &= r_i^{k*}, \text{ for all } i < n-1
\end{aligned}$$

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where $\Delta > 0$. Then according to (25), under the perturbed policy,

$$\begin{aligned} E_n^{k'} &= E_n^{k*} - \Delta \\ E_i^{k'} &= E_i^{k*}, \text{ for all } i < n \end{aligned}$$

and correspondingly, $J_k' = J_k^* - \Delta g$. Because $E_n^{k*} > 0$, then as long as we make Δ small enough such that $E_n^{k'} > 0$, the perturbed policy is still feasible. However, J_k' is smaller than J_k^*, which draws a contradiction to the assumption. Now, if $r_{n-1}^{k*} = 0$, then because of $E_n^{k*} > 0$ and $e_{i,i+1} > 0$ for all i, we can always find some $j < n$ such that $E_j^{k*} > 0$, $r_{j-1}^{k*} > 0$, and $r_l^{k*} = 0$ for $l > j$. Thus, still owing to (25), we have

$$E_j^{k*} = E_n^{k*} + \sum_{l=j}^{n-1} e_{l,l+1} > 0$$

At this time, because $r_{j-1}^{k*} > 0$, the argument is similar to the case $r_{n-1}^{k*} > 0$, in which the lemma can be justified by the contradiction argument. Consequently, the lemma is proven. $\square$

In view of Lemma 3, we can replace $\sum_{i=1}^{n}\sum_{j=1}^{n} r_i^k g x_{ij}^k$ in (17) by $\left(E_n^k - E_1^k\right) g + \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}^k(\mathbf{x_{ij}}) g x_{ij}$ and eliminate, for all $k = 1, \ldots, N$, the presence of r_i^k, $i = 1, \ldots, n-1$, from the objective function similar to the single-vehicle case. Because E_1^k is given, this leaves only the task of determining the value of E_n^k. There are two possible cases: (i) $\sum_i r_i^{k*} > 0$, that is, the kth vehicle subflow has to get recharged at least once; and (ii) $\sum_i r_i^{k*} = 0$, that is, $r_i^{k*} = 0$ for all i, and the kth vehicle subflow has adequate energy to reach the destination without recharging.

Similar to the derivation of (14), case (ii) results in a new constraint $\sum_i \sum_j e_{ij}^k(\mathbf{x_{ij}}) x_{ij}^k \leqslant E_1^k$ for subflow k. However, because $e_{ij}^k(\mathbf{x_{ij}})$ now depends on all $x_{ij}^1, \ldots, x_{ij}^N$, problem (17)–(22) with all $r_i^k = 0$ are not as simple to solve as was the case with (11)–(13). Let us instead concentrate on the more interesting case (i) for which Lemma 4 applies and we have $E_n^{k*} = 0$. Therefore, along with Lemma 3, we have for each $k = 1, \ldots, N$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_i^k x_{ij}^k = \sum_{i=1}^{n}\sum_{j=1}^{n} e_{ij}^k(\mathbf{x_{ij}}) x_{ij}^k - E_1^k$$

Then, proceeding as in Theorem 1, we can replace the original objective function (17) and obtain the following new problem formulation to determine x_{ij}^{k*} for all $i, j \in \mathcal{N}$ and $k = 1, \ldots, N$

$$\begin{gathered} \min_{\mathbf{x_{ij}},\, i,j\in\mathcal{N}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\left(\tau_{ij}(\mathbf{x_{ij}}) x_{ij}^k \frac{R}{N} + e_{ij}^k(\mathbf{x_{ij}}) g x_{ij}^k\right) \\ s.t. \text{ for each } k \in \{1, \ldots, N\}: \\ \sum_{j\in O(i)} x_{ij}^k - \sum_{j\in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N} \\ b_1 = 1,\ b_n = -1,\ b_i = 0, \text{ for } i \neq 1, n \\ x_{ij}^k \in \{0, 1\} \end{gathered} \tag{26}$$

Because the objective function is no longer necessarily linear in x_{ij}^k, (26) cannot be further simplified into an LP problem as in Theorem 1. The computational effort required to solve this problem heavily depends on the dimensionality of the network and the number of subflows. Nonetheless, from the transformed formulation mentioned earlier, we are still able to separate the determination of routing variables x_{ij}^k from recharging amounts r_i^k. Similar to the single-vehicle case, once the routes are

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determined, we can obtain any r_i^k satisfying the energy constraints (20)–(21) such that $E_n^k = 0$ thus preserving the optimality of the objective value. To further determine r_i^{k*}, we can introduce a second level optimization problem similar to the single-vehicle case in (16). Next, we will present an alternative formulation for the original problem (17)–(22), which lead to a computationally simpler solution approach.

3.2. *Flow control formulation*

We begin by relaxing the binary variables in (22) and letting $0 \leqslant x_{ij}^k \leqslant 1$. Thus, we switch our attention from determining a single path for any subflow k to several possible paths by treating x_{ij}^k as the normalized vehicle flow on arc (i, j) for the kth subflow. This is in line with many network routing algorithms in which fractions x_{ij} of entities are routed from a node i to a neighboring node j using appropriate schemes ensuring that, in the long term, the fraction of entities routed on (i, j) is indeed x_{ij} [18]. Following this relaxation, the objective function in (17) is changed to

$$\min_{\mathbf{x_{ij}}, \mathbf{r_i},\ i,j \in \mathcal{N}} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{N} \tau_{ij}(\mathbf{x_{ij}}) x_{ij}^k \frac{R}{N} + \sum_{i=1}^{n} \sum_{k=1}^{N} r_i^k g$$

Moreover, the energy constraint (20) needs to be adjusted accordingly. Let E_{ij}^k represent the fraction of residual energy of subflow k associated with the x_{ij}^k portion of the vehicle flow exiting node i. Therefore, the constraint (21) becomes $E_{ij}^k \geqslant 0$. We can now capture the relationship between the energy associated with subflow k and the vehicle flow as follows:

$$\left[\sum_{h \in I(i)} \left(E_{hi}^k - e_{hi}^k(\mathbf{x_{ij}}) \right) + r_i^k \right] \frac{x_{ij}^k}{\sum_{h \in I(i)} x_{hi}^k} = E_{ij}^k \tag{27}$$

$$\frac{E_{ij}^k}{\sum_{j \in O(i)} E_{ij}^k} = \frac{x_{ij}^k}{\sum_{j \in O(i)} x_{ij}^k} \tag{28}$$

In (27), the energy values of different vehicle flows entering node i are aggregated, and the energy corresponding to each portion exiting a node, $E_{ij}^k,\ j \in O(i)$, is proportional to the corresponding fraction of vehicle flows, as expressed in (28). Clearly, this aggregation of energy leads to an approximation, because one specific vehicle flow may need to be recharged in order to reach the next node in its path, whereas another might have enough energy without being recharged. This approximation foregoes controlling recharging amounts at the individual vehicle level and leads to approximate solutions of the original problem (17)–(22). Several numerically based comparisons are provided in the next section showing little or no loss of optimality relative to the solution of (17).

Adopting this formulation with $x_{ij}^k \in [0, 1]$ instead of $x_{ij}^k \in \{0, 1\}$, we obtain the following simpler nonlinear programming problem (NLP):

$$\min_{\mathbf{x_{ij}}, \mathbf{r_i},\ i,j \in \mathcal{N}} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{N} \tau_{ij}(\mathbf{x_{ij}}) x_{ij}^k \frac{R}{N} + \sum_{i=1}^{n} \sum_{k=1}^{N} r_i^k g \tag{29}$$
$$s.t. \text{ for each } k \in \{1, \ldots, N\}:$$

$$\sum_{j \in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N}$$
$$b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1, n \tag{30}$$

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$$\left[\sum_{h\in I(i)}\left(E_{hi}^{k}-e_{hi}^{k}(\mathbf{x_{ij}})\right)+r_i^k\right]\frac{x_{ij}^k}{\sum_{h\in I(i)}x_{hi}^k}=E_{ij}^k \tag{31}$$

$$\frac{E_{ij}^k}{\sum_{j\in O(i)}E_{ij}^k}=\frac{x_{ij}^k}{\sum_{j\in O(i)}x_{ij}^k} \tag{32}$$

$$E_{ij}^k\geqslant 0, \tag{33}$$

$$0\leqslant x_{ij}^k\leqslant 1,\quad r_i^k\geqslant 0 \tag{34}$$

As in our previous analysis, we are able to eliminate $\mathbf{r_i}$ from the objective function in (29) as follows.

Lemma 5
For each subflow $k=1,\dots,N$,

$$\sum_{i=1}^{n}r_i^k=\sum_{i=1}^{n}\sum_{j=1}^{n}e_{ij}^k(\mathbf{x_{ij}})+\sum_{i\in I(n)}E_{in}^k-\sum_{i\in O(1)}E_{1i}^k$$

Proof
Summing (31) over all $i=1,\dots,n$ gives

$$\sum_{i=1}^{n}r_i^k=\sum_{i=1}^{n}\sum_{j=1}^{n}e_{ij}^k(\mathbf{x_{ij}})+\sum_{i=1}^{n}\sum_{j\in O(i)}E_{ij}^k-\sum_{i=1}^{n}\sum_{h\in I(i)}E_{hi}^k$$

and using (30), (32), we obtain

$$\sum_{i=1}^{n}r_i^k=\sum_{i=1}^{n}\sum_{j=1}^{n}e_{ij}^k(\mathbf{x_{ij}})+\sum_{i\in I(n)}E_{in}^k-\sum_{i\in O(1)}E_{1i}^k$$

which proves the lemma. □

Similar to Lemma 3, we can easily see that if $\sum_i r_i^{k*}>0$ under an optimal routing policy, then $\sum_{i\in I(n)}E_{in}^{k*}=0$. In addition, $\sum_{i\in O(1)}E_{1i}^k=E_1^k$, which is given. We can now transform the objective function (29) into (35) and determine the optimal routes x_{ij}^{k*} by solving the following NLP:

$$\begin{gathered}\min_{\substack{\mathbf{x_{ij}}\\ i,j\in\mathcal{N}}}\quad\sum_{k=1}^{N}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\tau_{ij}(\mathbf{x_{ij}})x_{ij}^k\frac{R}{N}+e_{ij}^k(\mathbf{x_{ij}})g\right]-E_1^k\right)\\ s.t.\text{ for each }k\in\{1,\dots,N\}:\\ \sum_{j\in O(i)}x_{ij}^k-\sum_{j\in I(i)}x_{ji}^k=b_i,\quad\text{for each }i\in\mathcal{N}\\ b_1=1,\ b_n=-1,\ b_i=0,\ \text{for }i\neq 1,n\\ 0\leqslant x_{ij}^k\leqslant 1\end{gathered} \tag{35}$$

Note that in the aforementioned formulation, the nonlinearity appears in the objective function due to the traffic congestion effect on traveling time and energy consumption. Thus, if $\tau_{ij}(\mathbf{x_{ij}})x_{ij}^k\dfrac{R}{N}$ and

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$e_{ij}^k(\mathbf{x_{ij}})$ are convex functions, the NLP is a convex optimization problem, and the global optimum can be found generally fast. Once we find the optimal routes, the values of r_i^k, $i = 1, \ldots, n$, $k = 1, \ldots, N$, can be determined so as to satisfy the energy constraints (31)–(33), and they are obviously not unique. We may then proceed with a second level optimization problem to determine optimal values similar to Section 2.2.

3.3. *Objective function selection*

The selection of $\tau_{ij}(\mathbf{x_{ij}})$ in either (26) or (35) is based on models originating in the traffic engineering literature. For the rest of this paper, we use a commonly used relationship between speed and density of a vehicle flow as in [19–21]

$$v(k(t)) = v_f \left(1 - \left(\frac{k(t)}{k_{jam}}\right)^p\right)^q \tag{36}$$

where v_f is the reference speed on the road without traffic, $k(t)$ represents the density of vehicles on the road at time t, and k_{jam} is the saturated density for a traffic jam. Note that we can replace $k(t)/k_{jam}$ in (36) with $f(t)/f_{jam}$, where $f(t)$ is the vehicle flow on the road at time t and f_{jam} represents the maximum capacity of the road. The parameters p and q are empirically identified for actual traffic flows. Given a network topology (i.e., a road map), the distances d_{ij} between nodes and the capacity of links, f_{jam}^{ij}, are known. Let us assume that EVs enter the network at a rate of R veh./min. We then evenly divide the EV inflow into N subflows, and the total flow entering link (i, j) becomes $f_{ij} = \sum_k x_{ij}^k \dfrac{R}{N}$. Then, the time a vehicle spends on link (i, j) becomes

$$\tau_{ij}(\mathbf{x_{ij}}) = \frac{d_{ij}}{v_f\left(1 - \left(\frac{f_{ij}}{f_{jam}^{ij}}\right)^p\right)^q} \tag{37}$$

Note that in order to prevent the inflow entering each link from exceeding its capacity, we add the following inequality constraint to the problem formulations (26) and (35)

$$\sum_k x_{ij}^k \frac{R}{N} \leqslant f_{jam}^{ij} \tag{38}$$

As for $e_{ij}^k(\mathbf{x_{ij}})$, we assume that the energy consumption rates of subflows on link (i, j) are all identical, proportional to the distance between nodes i and j, giving $e_{ij}^k(\mathbf{x_{ij}}) = ed_{ij}R/N$.

Therefore, we aim to solve the multi-vehicle routing problem using (26) which in this case becomes

$$\begin{gathered}
\min_{\substack{x_{ij}^k \\ i,j \in \mathcal{N}}} \quad \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{N}\left(\frac{d_{ij}x_{ij}^k\frac{R}{N}}{v_f\left(1-\left(\frac{R/N\sum_k x_{ij}^k}{f_{jam}^{ij}}\right)^p\right)^q} + egd_{ij}\frac{R}{N}x_{ij}^k\right) \\
s.t. \text{ for each } k \in \{1, \ldots, N\}: \\
\sum_{j\in O(i)} x_{ij}^k - \sum_{j \in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N} \\
b_1 = 1,\ b_n = -1,\ b_i = 0, \text{ for } i \neq 1, n \\
\sum_k x_{ij}^k \frac{R}{N} \leqslant f_{jam}^{ij},\ \forall (i,j) \in \mathcal{A} \\
x_{ij}^k \in \{0, 1\}
\end{gathered} \tag{39}$$

3.4. Numerical examples

For simplicity, we let $v_f = 1$ mile/min, $R = 1$ veh./min, $p = 2$, $q = 2$, $eg = 1$, and $f_{jam}^{ij} = 1$ veh./min $\forall (i,j) \in \mathcal{A}$. The network topology used is that of Figure 1, where the distance of each link is shown in Table I. To solve the nonlinear binary programming problem (39), we use the optimization solver *Opti* (MATLAB toolbox for optimization). The results are shown in Table II for different values of $N = 1, \ldots, 30$. It can be observed that vehicles are mainly distributed through three routes, and the traffic congestion effect makes the flow distribution differ from following the shortest path. The number of decision variables (hence, the solution search space) rapidly increases with the number of subflows. However, looking at Figure 2, which gives the performance in terms of our objective function in (39) as a function of the number of subflows, one can observe that the optimal objective value quickly converges with no significant fluctuations beyond $N = 10$. Thus, even though the best solution is found when $N = 25$, a near-optimal solution can be determined under a small number of subflows. This suggests that one can rapidly approximate the asymptotic solution of the multi-vehicle problem (dealing with individual vehicles routed so as to optimize a system-wide objective) based on a relatively small value of N.

Table I. d_{ij} values for network of Figure 1 (miles).

d_{12}	d_{14}	d_{15}	d_{23}	d_{24}	d_{46}	d_{56}	d_{37}	d_{47}	d_{67}
5	6.2	7	3.5	5	3.6	4.3	6	6	4

Table II. Numerical results for sample problem.

N	1	2
Obj	1.22e9	37.077
Routes	$1 \to 4 \to 7$	$1 \to 4 \to 7$ $1 \to 2 \to 3 \to 7$
N	3	4
Obj	31.7148	32.8662
Routes	$(1 \to 4 \to 7)$ $1 \to 2 \to 3 \to 7$ $1 \to 5 \to 6 \to 7$	$(1 \to 4 \to 7) \times 2$ $1 \to 2 \to 3 \to 7$ $1 \to 5 \to 6 \to 7$
N	5	6
Obj	32.1921	31.7148
Routes	$(1 \to 4 \to 7) \times 2$ $(1 \to 2 \to 3 \to 7) \times 2$ $1 \to 5 \to 6 \to 7$	$(1 \to 4 \to 7) \times 2$ $(1 \to 2 \to 3 \to 7) \times 2$ $(1 \to 5 \to 6 \to 7) \times 2$
N	10	15
Obj	31.5279	31.4851
Routes	$(1 \to 4 \to 7) \times 4$ $(1 \to 2 \to 3 \to 7) \times 3$ $(1 \to 5 \to 6 \to 7) \times 3$	$(1 \to 4 \to 7) \times 5$ $(1 \to 2 \to 3 \to 7) \times 5$ $(1 \to 5 \to 6 \to 7) \times 4$ $(1 \to 4 \to 6 \to 7) \times 1$
N	25	30
Obj	31.4513	31.4768
Routes	$(1 \to 4 \to 7) \times 9$ $(1 \to 2 \to 3 \to 7) \times 8$ $(1 \to 5 \to 6 \to 7) \times 7$ $(1 \to 4 \to 6 \to 7) \times 1$	$(1 \to 4 \to 7) \times 11$ $(1 \to 2 \to 3 \to 7) \times 10$ $(1 \to 5 \to 6 \to 7) \times 8$ $(1 \to 4 \to 6 \to 7) \times 1$

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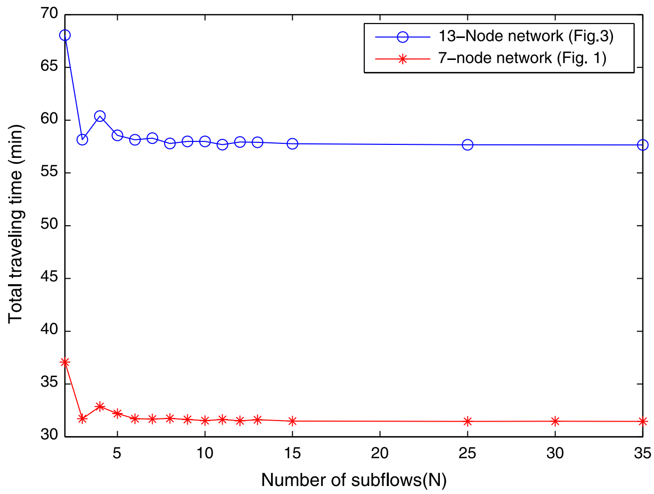

Figure 2. Performance as a function of N (no. of subflows).

Next, we obtain a solution to the same problem (39) using the alternative NLP formulation (35), where $0 \leqslant x_{ij}^k \leqslant 1$. Because in this example all subflows are identical, solving the NLP relaxed problem results in the same routing probabilities for all subflows, that is, $x_{ij}^1 =, \ldots, = x_{ij}^N$. Therefore, we can further combine all x_{ij}^k over each link (i, j) and formulate the following N-subflow relaxed problem, referred to as N-NLP, giving the total normalized flow on each link, $x_{ij}, \forall (i, j) \in \mathcal{A}$, which is independent of N

$$
\begin{aligned}
\min_{x_{ij},\, i,j \in \mathcal{N}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{d_{ij} x_{ij} R}{v_f (1 - (R x_{ij} / f_{jam}^{ij})^p)^q} + e g d_{ij} R x_{ij} \right) \\
s.t. \quad & \sum_{j \in O(i)} x_{ij} - \sum_{j \in I(i)} x_{ji} = b_i, \quad \text{for each } i \in \mathcal{N} \\
& b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1, n \\
& x_{ij} R \leqslant f_{jam}^{ij} \\
& 0 \leqslant x_{ij} \leqslant 1
\end{aligned}
\tag{40}
$$

This is a relatively easy to solve NLP problem. It can be readily shown that the objective function is convex. In particular, $\dfrac{d_{ij} x_{ij} R}{v_f (1 - (x_{ij})^p)^q)}$ is convex over $0 \leqslant x_{ij} \leqslant 1$, and $e g d_{ij} R x_{ij}$ is a linear function; therefore, their positive weighted sum is a convex function, and (40) is a convex optimization problem whose solution is a global optimum. Using the same parameter settings as before, we obtain the objective value of 31.45 min, and the optimal routes are as follows:

$$
\begin{aligned}
35.88\% \text{ of vehicle flow: } & (1 \to 4 \to 7) \\
31.74\% \text{ of vehicle flow: } & (1 \to 2 \to 3 \to 7) \\
27.98\% \text{ of vehicle flow: } & (1 \to 5 \to 6 \to 7) \\
4.44\% \text{ of vehicle flow: } & (1 \to 4 \to 6 \to 7)
\end{aligned}
$$

Compared with the best solution ($N = 25$) in Table II and Figure 2, the difference in objective values between the integer and flow-based solutions is less than 0.1%. This supports the effectiveness of a solution based on a limited number of subflows in the MINLP problem.

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3.4.1. Performance improvement over uncontrolled traffic systems. Next, we address the extent to which this optimization approach offers improvements over an uncontrolled traffic network. We simulate the vehicle routing problem on the discrete event simulator SimEvents, where the vehicle arrivals to the source are randomly generated with a random initial energy. As a simple example, we model the routing for each vehicle at each node to be round-robin, while the recharging amount of the vehicle is just adequate to reach the next node. The total traveling time under this uncontrolled routing for the network shown in Figure 1 is 38.52 min compared with our optimal policy with the objective value of 31.45 min, showing an improvement of 18.36%.

3.4.2. Larger networks. We have also considered a more topologically complex network with 13 nodes and 20 links as shown in Figure 3. The number on each link indicates the distance between adjacent nodes. We assume all other numerical values to be similar to the previous example. Figure 2 shows the performance in terms of the objective function in (39) versus the number of subflows for this network. We can see that the optimal objective value converges around $N = 10$.

Now, let us solve the N-subflow relaxed problem (40) for this network with the same parameter settings as those in Section 3.4 to check for its accuracy. We obtain the optimal objective function value as 57.63 min, which is almost equal to the optimal traveling time of 57.65 min obtained for $N = 35$ in the MINLP formulation. The optimal routing probabilities are as follows:

$$34.77\% \text{ of vehicle flow: } (1 \to 2 \to 3 \to 4 \to 5 \to 13)$$
$$27.52\% \text{ of vehicle flow: } (1 \to 9 \to 10 \to 11 \to 12 \to 13)$$
$$24.89\% \text{ of vehicle flow: } (1 \to 6 \to 10 \to 7 \to 8 \to 13)$$
$$10.81\% \text{ of vehicle flow: } (1 \to 6 \to 3 \to 8 \to 13)$$
$$1.71\% \text{ of vehicle flow: } (1 \to 9 \to 10 \to 7 \to 8 \to 13)$$
$$0.31\% \text{ of vehicle flow: } (1 \to 6 \to 3 \to 4 \to 5 \to 13)$$

3.4.3. CPU time comparison. Based on our simulation results, we conclude that the flow control formulation is a good approximation of the original MINLP problem. Table III compares the

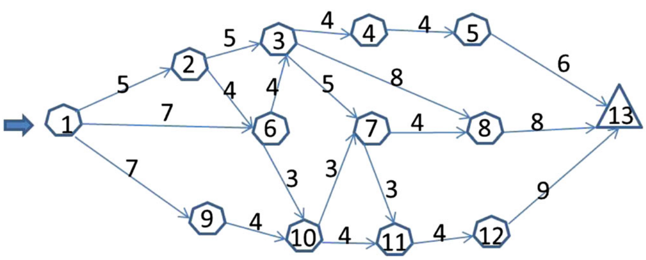

Figure 3. A 13-node network example for routing with recharging nodes.

Table III. CPU time for sample problems.

Figure 1 net.	MINLP	MINLP	NLP approx.
N	2	10 (near opt)	—
Obj	37.08	31.53	31.45
CPU time (s)	312	9705	0.07
Figure 3 net.	**MINLP**	**MINLP**	**NLP approx.**
N	2	15 (near opt)	—
Obj	68.05	57.76	57.63
CPU time (s)	820	10037	0.2

MINLP, Mixed-Integer Nonlinear Programming; NLP, nonlinear programming problem.

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computational effort in terms of CPU time for both formulations to find optimal routes for the two sample networks we have considered. Our results show that the flow control formulation results in a reduction of about five orders of magnitude in CPU time with virtually identical objective function values (the difference between objective values of NLP and MINLP with near-optimal N is less than 1%).

3.4.4. Effect of recharging speed on optimal routes. Once we determine the optimal routes, we can also ascertain the total time spent traveling and recharging, respectively, that is, the first and second terms in (40). Obviously, the value of eg, which captures the recharging speed, determines the proportion of traveling and recharging amount as well as the route selection. As shown in Table IV, the larger the product eg is, the slower the recharging speed, therefore, the more weighted the recharging time in the objective function becomes. In this case, flows tend to select the shortest paths in terms of energy consumption. Conversely, if the recharging speed is fast, the routes are selected to prioritize the traveling time on paths.

3.4.5. Price of anarchy. In order to compare system performance under a user-optimal (single-vehicle routing problem) policy and a system-optimal (multiple-vehicle routing problem) policy, we investigate the price of anarchy for this problem. To make this comparison, we consider two different scenarios.

1. A single driver acts selfishly. We control all vehicles to follow system-optimal paths and assume that a single driver acts selfishly. We then investigate this driver's total traveling time and the possible gain resulting from this deviation.

   Let us consider the numerical example in Section 3.4 for the network shown in Figure 1. The system-optimal flows are obtained by solving the NLP problem (40). Under these flows, let us calculate the traveling time, τ^c_{ij}, of all links $(i,j) \in \mathcal{A}$ with positive flows using (37) as shown in Table V. Assuming that the energy consumption on each link is equal to the distance of that link, the total traveling time experienced by an individual EV, T_{EV}, depends on the system-optimal path assigned to its subflow, P

$$T_{EV} = \sum_{(i,j)\in P} (\tau^c_{ij} + e_{ij} g)$$

Total traveling time for a single EV in flow $(1 \to 4 \to 7)$: 28.94 min
Total traveling time for a single EV in flow $(1 \to 2 \to 3 \to 7)$: 32.43 min
Total traveling time for a single EV in flow $(1 \to 5 \to 6 \to 7)$: 33.59 min
Total traveling time for a single EV in flow $(1 \to 4 \to 6 \to 7)$: 31.24 min

Table IV. Numerical results for different values of eg for network of Figure 1.

eg	0.1	1	10
Total time	18.94	31.45	154.48
Time on paths	17.55	17.58	19.45
Time at stations	1.39	13.87	135.03
Optimal routes	31.53% : $(1 \to 2 \to 3 \to 7)$ 32.97% : $(1 \to 4 \to 7)$ 28.58% : $(1 \to 5 \to 6 \to 7)$ 5.78% : $(1 \to 4 \to 6 \to 7)$ 1.14% : $(1 \to 2 \to 4 \to 7)$	31.74% : $(1 \to 2 \to 3 \to 7)$ 35.88% : $(1 \to 4 \to 7)$ 27.98% : $(1 \to 5 \to 6 \to 7)$ 4.40% : $(1 \to 4 \to 6 \to 7)$	32.35% : $(1 \to 2 \to 3 \to 7)$ 49.63% : $(1 \to 4 \to 7)$ 18.02% : $(1 \to 5 \to 6 \to 7)$

Table V. Traveling time on each link for the network shown in Figure 1 under system-optimal flows.

τ^c_{12}	τ^c_{14}	τ^c_{15}	τ^c_{23}	τ^c_{24}	τ^c_{46}	τ^c_{56}	τ^c_{37}	τ^c_{47}	τ^c_{67}
6.18	8.83	8.24	4.33	5	3.61	5.06	7.42	7.90	4.99

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Table VI. Normalized Nash-equilibrium flows.

x_{12}^e	x_{14}^e	x_{15}^e	x_{23}^e	x_{24}^e	x_{46}^e	x_{56}^e	x_{37}^e	x_{47}^e	x_{67}^e
30.7	47.9	21.4	30.7	0	1.5	21.4	30.7	46.4	22.9

Now, if we solve the single-vehicle routing problem for a lone EV in the network, the user-optimal path is $(1 \to 4 \to 7)$ with a traveling time of 28.94 min. Thus, in the system-optimal problem, vehicles assigned to the subflows following this path experience the same traveling time as if they act selfishly and follow the user-optimal path. However, vehicles assigned to other subflows will experience longer traveling times in order to reduce the total elapsed time for the whole inflow. For instance, a single EV can gain 13.86% in its traveling time by acting selfishly and deviating from the subflow assigned to path $(1 \to 5 \to 6 \to 7)$ and joining path $(1 \to 4 \to 7)$.

2. All drivers act selfishly. In this case, the flow will be in a Nash equilibrium where no single user can incur a gain by changing its own strategy [22]. Based on Wardrop's principle, the equilibrium occurs at flows that minimize the potential function [23]

$$\phi(f) = \sum_{(i,j)\in\mathcal{A}} \int_0^{f_{ij}} \eta_{ij}(x)dx \tag{41}$$

where $\eta_{ij}(x)$ is the travel time incurred by traffic that traverses link (i, j) as a function of the link congestion (Rx_{ij} in (40)). The price of anarchy is defined as the ratio of the total system cost (the total elapsed time) under Nash equilibrium to the total cost under the social-optimal flows. Table VI shows the normalized Nash-equilibrium flow, x_{ij}^e, on each link (i, j) of the network shown in Figure 1 resulting in the following selfish routing:

$$\begin{aligned}
46.4\% \text{ of vehicle flow: } & (1 \to 4 \to 7)\\
30.7\% \text{ of vehicle flow: } & (1 \to 2 \to 3 \to 7)\\
21.4\% \text{ of vehicle flow: } & (1 \to 5 \to 6 \to 7)\\
1.5\% \text{ of vehicle flow: } & (1 \to 4 \to 6 \to 7)
\end{aligned}$$

Applying Nash-equilibrium flows into the system-wide objective function (40), the total traveling time is 32.27 min, which is higher than the optimal cost of 31.45 obtained under the social-optimal policy, and the price of anarchy is $PoA = 1.038$.

4. SELECTION OF THE NUMBER OF SUBFLOWS

We begin with the observation that the objective function as well as the constraints of the flow control formulation (NLP) (35) are the same as those of the MINLP formulation (26), except for the relaxed binary constraints, that is, $0 \leqslant x_{ij}^k \leqslant 1$. Thus, in general, the optimal objective value of the NLP problem will be equal or lower than that of the MINLP problem. We seek the best value of N to render the problem computationally manageable.

Similar to the numerical examples in Section 3.4, we focus on the case where we divide the total vehicle inflow, R, into N sublows each with a rate of R/N. In this case, solving the NLP problem results in the same routing probabilities for all subflows, that is, $x_{ij}^1 =, \dots, = x_{ij}^N$. Therefore, we can combine them and reformulate the problem as an N-subflow relaxed problem, referred to as 'N-NLP', giving the total normalized flow on each link, x_{ij}, $(i, j) \in \mathcal{A}$ (40).

In Section 3.4, the numerical results show that the optimal objective value quickly converges for a small value of N. Thus, even though the best solution may be found for a larger N, a near-optimal solution can be determined under a small number of subflows. This suggests that we can approximate the asymptotic solution of the multi-vehicle problem based on a relatively small value of N. Our goal is to find a lower bound, N^*, for the number of subflows, such that by selecting any

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$N \geqslant N^*$, we can guarantee that the N-NLP solution will be in a given neighborhood of the MINLP solution. To do so, first, we proceed as follows.

Let π be the number of different paths from the origin node to the destination node in a given graph, and let x_p denote the normalized amount of flow through path $p,\ p = 1, \ldots, \pi$ determined using the solution of the N-NLP problem (35). Based on that, we define *active paths* to be those with non-zero flow, that is, paths with $x_p > 0$. Let us assume that there are $q \leqslant \pi$ active paths; then, let $\tilde{x}_p$ denote the normalized flow on active path p, $p = 1, \ldots, q$.

Next, we define $N_p(N)$ to be the number of subflows assigned to active path p obtained by the MINLP solution with N subflows. We write $N_p(N)$ to emphasize that the MINLP solution depends on the choice of N. Then, $\dfrac{N_p(N)}{N}$ is the normalized flow on path p obtained by solving the MINLP problem with N subflows. Noting that N is an integer, the best N is the one that minimizes the deviation from the normalized flows obtained by solving the NLP problem (lower bounds to the MINLP solutions), that is,

$$\min_{N \geqslant 1} \left| \frac{N_p(N)}{N} - x_p \right|, \ p = 1, \ldots, \pi$$

Because the computational complexity of the MINLP problem increases with N, the selection of N is a trade-off between a near-optimal solution and the computational effort required to solve the problem. To address this trade-off, let us define the *average deviation* between the optimal routing probabilities of the active paths obtained by solving the N-NLP problem, $\tilde{x}_p,\ p = 1, \ldots, q$, and the normalized flows obtained by solving the MINLP problem, $N_p(N)/N$, as a near-optimality metric, that is, $\dfrac{1}{q} \sum_{p=1}^{q} |\dfrac{N_p(N)}{N} - \tilde{x}_p|$. Then, we define a 'desired accuracy', δ, as the upper bound for this metric and seek to determine values of N that satisfy

$$\frac{1}{q} \sum_{p=1}^{q} \left| \frac{N_p(N)}{N} - \tilde{x}_p \right| \leqslant \delta \tag{42}$$

Based on (42), we seek the critical N^* such that by selecting $N \geqslant N^*$, the average deviation between the N-NLP and MINLP solutions does not exceed δ, that is,

$$N \geqslant \frac{\sum_{p=1}^{q} |N_p(N) - N\tilde{x}_p|}{q\delta} \tag{43}$$

We define

$$N^* = \left\lceil \frac{\max_{N \geqslant 1} \left(\sum_{p=1}^{q} |N_p(N) - N\tilde{x}_p| \right)}{q\delta} \right\rceil \tag{44}$$

Because the numerator of N^* is an upper bound for the numerator in the right-hand side of (43) and noting that δ and q are constants, choosing $N \geqslant N^*$ guarantees that the average deviation between the NLP and MINLP solutions never exceeds our desired accuracy, δ. However, because $N_p(N)$ is a function of N, finding a closed-form expression for $\max_{N \geqslant 1} \left(\sum_{p=1}^{q} |N_p(N) - N\tilde{x}_p| \right)$ in the numerator of (44) is not easy. To address this issue, we propose a method that efficiently and accurately estimates the MINLP solution. Then, using these estimates, to be referred as $\hat{N}_p(N),\ p = 1, \ldots, \pi$, for a large range of the number of subflows, N, we can find $\max_{N \geqslant 1} \left(\sum_{p=1}^{q} | \hat{N}_p(N) - N\tilde{x}_p| \right.$ and select the proper N^* using (44).

As described in Algorithm 1, first, we solve the N-NLP problem and find the optimal objective value and optimal normalized flow on each link (i, j), x_{ij}. Next, we determine the active paths and their corresponding optimal normalized flow, $\tilde{x}_p,\ p = 1, \ldots, q$. Because the objective functions of

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Algorithm 1 MINLP solution estimation algorithm

Input: N

Output: estimation for MINLP solution, $\hat{N}_p,\ p = 1, .., \pi$

Initialization: Set $\hat{N}_p = 0,\ p = 1, .., \pi$.

1: Solve N-NLP problem and identify active paths and corresponding $\tilde{x}_p,\ p = 1, .., q$

2: Form the set of all possible combinations for assigning N subflows to q active paths, $\mathcal{S}_N$, for each $i \in \mathcal{S}_N$, N_p^i is the number of subflows allocated to path p for the ith such assignment.

3: Find the best assignment, i^*, so that $i^* = \arg\min_{i \in \mathcal{S}_N} \left(\sum_{p=1}^{q} |N_p^i/N - \tilde{x}_p| \right)/q$

4: Set $\hat{N}_p = N_p^{i^*}$ for active paths.

End

the MINLP and NLP problems are the same, the MINLP solution for each N assigns the subflows to the active paths such that the deviation between the corresponding normalized flows, $N_p(N)/N$, and the N-NLP solution, $\tilde{x}_p,\ p = 1, \ldots, q$, is minimized. Therefore, to estimate the MINLP solution, $\hat{N}_p$, for each value of N, we consider all possible combinations of assigning N subflows to the q active paths and form a set $\mathcal{S}_N$. There are $\begin{pmatrix} N+q-1 \\ N \end{pmatrix}$ different such assignments (cardinality of set $\mathcal{S}_N$). Let N_p^i denote the number of subflows allocated to path $p,\ p = 1, \ldots, q$ in the ith such assignment, $i \in \mathcal{S}_N$. For each assignment i, the equivalent normalized flows on active paths become $[N_1^i/N, \ldots, N_q^i/N]$, where $\sum_{p=1}^{q} N_p^i = N$. For each value of N, i^* is the 'best assignment' if it results in the minimum average deviation from the N-NLP solution among all $i \in \mathcal{S}_N$, that is,

$$i^* = \arg\min_{i \in \mathcal{S}_N} \frac{1}{q} \sum_{p=1}^{q} \left| \frac{N_p^i}{N} - \tilde{x}_p \right| \tag{45}$$

For each value of N, we set the i^* assignment as the estimate of the optimal routing of subflows (MINLP solution), $\hat{N}_p = N_p^{i^*},\ p = 1, \ldots, q$, and $\hat{N}_p = 0$ for non-active paths. Finally, for a given graph, one can create a lookup table of the estimates of the MINLP solution for a range of N and find $\max_{N \geqslant 1} \left(\sum_{p=1}^{q} |\hat{N}_p(N) - N\tilde{x}_p| \right)$ to calculate N^* for a desired δ using (44).

Remark 1

A simple way of intuitively determining the critical N is as follows: Defining $\hat{x} = \min_p \tilde{x}_p$, $\hat{x}$ is the least fraction of inflow obtained by the N-NLP problem to flow through an individual path. If we choose $N = \left\lceil \frac{1}{\hat{x}} \right\rceil$, the MINLP solution will have a chance to send at least one subflow through the same path, and the normalized flows obtained by the MINLP will be close to the NLP solution. Obviously, it may not be the best N and there is no guarantee for such N to satisfy the bound defined in (42). However, our simulation results show that it is a *good* 'rule of thumb' for selecting N.

4.1. Numerical example

Consider the seven-node graph shown in Figure 1 with the same parameter values as in Section 3.4. The N-NLP solution is shown in Table VII.

Table VII. Optimal normalized flow on each link (x_{ij}) obtained by solving N-NLP problem.

x_{12}	x_{14}	x_{15}	x_{23}	x_{24}
31.73%	40.28%	27.98%	31.73 %	0
x_{46}	x_{56}	x_{37}	x_{47}	x_{67}
4.40%	27.98%	31.73%	35.88%	32.39%

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Using the optimal x_{ij}s, we determine each active path and corresponding $\tilde{x}_p$

$$\begin{aligned}
\tilde{x}_1 &= 31.73\% \quad \text{Path}_1 : (1 \to 2 \to 3 \to 7)\\
\tilde{x}_2 &= 4.40\% \quad \text{Path}_2 : (1 \to 4 \to 6 \to 7)\\
\tilde{x}_3 &= 35.88\% \quad \text{Path}_3 : (1 \to 4 \to 7)\\
\tilde{x}_4 &= 27.98\% \quad \text{Path}_4 : (1 \to 5 \to 6 \to 7)
\end{aligned}$$

In this example, there are four active paths. Figure 4 shows the average deviation between the NLP solution and the normalized flows obtained by the estimated MINLP solutions for $N = 1, \ldots, 72$. Table VIII shows the best assignments (estimates of the MINLP solutions) in the form $[\hat{N}_1\ \hat{N}_2\ \hat{N}_3\ \hat{N}_4]$ (corresponds to [Path_1 Path_2 Path_3 Path_4]) for different values of N. In Figure 4, we observe that the minimum average deviation occurs for $N = 25$ with the closest objective value to the NLP problem and the following normalized flows:

$$\begin{aligned}
N_1/N &= 8/25 = 32\% \quad \text{Path}_1 : (1 \to 2 \to 3 \to 7)\\
N_2/N &= 1/25 = 4\% \quad \text{Path}_2 : (1 \to 4 \to 6 \to 7)\\
N_3/N &= 9/25 = 36\% \quad \text{Path}_3 : (1 \to 4 \to 7)\\
N_4/N &= 7/25 = 28\% \quad \text{Path}_4 : (1 \to 5 \to 6 \to 7)
\end{aligned}$$

which are almost identical to the NLP solution, $\tilde{x}_p$, $p = 1, \ldots, 4$. In this particular example, $\hat{x} = \min_{p=1,\ldots,4} \tilde{x}_p = 0.044$, which suggests the same number of subflows based on the simple 'rule of thumb' in Remark 1.

Finally, we investigate the correctness of the bound defined in (43) for different values of δ. It can be seen in Figure 4 that by selecting $N \geqslant N^*$, calculated for different values of δ in Table IX, the average deviation never exceeds our desired accuracy, δ, which shows the validity of the proposed criterion in (44). Increasing the upper bound of the average deviation allows us to select smaller N, and consequently, the problem size and associated computational complexity decrease. This demonstrates the trade-off between proximity to optimality and computational effort required to solve the problem.

Our numerical results show that the optimal routing obtained by solving the MINLP problem is exactly the same as our estimate, that is, the best assignment corresponding to the minimum average deviation with the NLP solution for each N. This can be verified by comparing the results in Tables II and VIII for different values of N.

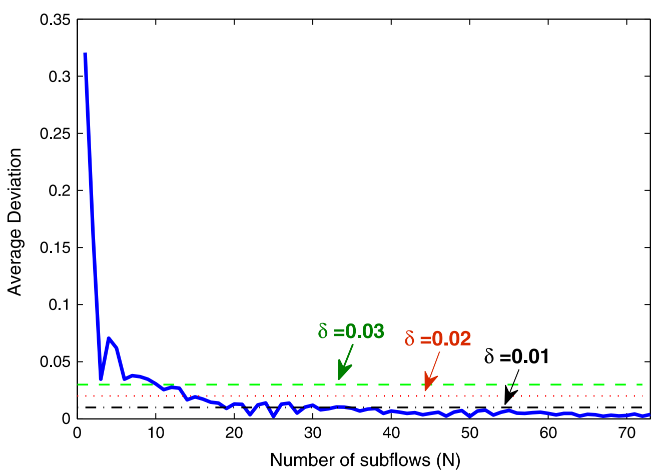

Figure 4. Average deviation between the solution of the NLP and estimated solution of the MINLP problem for different values of N.

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Table VIII. Estimates for the MINLP solution for different values of N.

N	Estimated $\hat{N}_1, \ldots, \hat{N}_4$	N	Estimated $\hat{N}_1, \ldots, \hat{N}_4$	N	Estimated $\hat{N}_1, \ldots, \hat{N}_4$
1	0 0 1 0	25	8 1 9 7	49	15 2 18 14
2	1 0 1 0	26	8 1 10 7	50	16 2 18 14
3	1 0 1 1	27	9 1 10 7	51	16 2 19 14
4	1 0 2 1	28	9 1 10 8	52	16 2 19 15
5	2 0 2 1	29	9 1 11 8	53	17 2 19 15
6	2 0 2 2	30	10 1 11 8	54	17 3 19 15
7	2 0 3 2	31	10 1 11 9	55	18 2 20 15
8	3 0 3 2	32	10 1 12 9	56	18 2 20 16
9	3 0 3 3	33	11 1 12 9	57	18 3 20 16
10	3 0 4 3	34	11 1 12 10	58	18 3 21 16
11	4 0 4 3	35	11 1 13 10	59	19 3 21 16
12	4 1 4 3	36	11 2 13 10	60	19 3 21 17
13	4 0 5 4	37	12 2 13 10	61	19 3 22 17
14	4 1 5 4	38	12 2 13 11	62	20 3 22 17
15	5 1 5 4	39	12 2 14 11	63	20 3 22 18
16	5 1 6 4	40	13 2 14 11	64	20 3 23 18
17	5 1 6 5	41	13 2 15 11	65	21 3 23 18
18	6 1 6 5	42	13 2 15 12	66	21 3 24 18
19	6 1 7 5	43	14 2 15 12	67	21 3 24 19
20	6 1 7 6	44	14 2 16 12	68	22 3 24 19
21	7 1 7 6	45	14 2 16 13	69	22 3 25 19
22	7 1 8 6	46	15 2 16 13	70	22 3 25 20
23	7 1 8 7	47	15 2 17 13	71	23 3 25 20
24	8 1 8 7	48	15 2 17 14	72	23 3 26 20

MINLP, Mixed-Integer Nonlinear Programming.

Table IX. Critical number of subflows, N^*, for different values of δ.

$\delta = 0.01$	$N^* = 33$
$\delta = 0.02$	$N^* = 17$
$\delta = 0.03$	$N^* = 11$

5. MULTIPLE-VEHICLE ROUTING PROBLEM IN THE PRESENCE OF NON-ELECTRIC VEHICLE FLOWS

In this section, we extend our approach by involving both EV and NEV flows. Let all vehicles enter the network at node 1, and let R denote the rate of vehicles arriving at this node. Assuming a fraction P of NEVs in the inflow, NEVs and EVs enter the network with flow rates given by RP and $R(1-P)$, respectively. We propose two different ways to incorporate the effect of NEV flows. In the first method, we assume that the flow of NEVs on each link (i, j), f_{ij}^{NEV}, is known (e.g., Nash equilibrium flows or socially optimal flows are determined), and we can calculate the *residual* capacity for each link accordingly, that is, $f_{jam}^{ij} - f_{ij}^{NEV}$. Thus, the problem is reduced to the multi-vehicle routing problem with all arriving vehicles as EVs with the residual capacity for links, which have already been discussed.

Our second method is to reformulate an optimization problem in order to control both EV and NEV flows. Similar to our approach in Section 3, we group EVs as well as NEVs into subflows. In particular, we divide the inflow of NEVs into a fixed number of subflows, M, (e.g., the number of distinct paths from the origin to the destination node) and the inflow of EVs into N subflows. Our objective is to determine optimal routes for NEV subflows and optimal routes, as well as energy recharging amounts, for each EV subflow so as to minimize the total elapsed time of these subflows from the origin to the destination. Note that for NEVs, we do not consider the refueling process as

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part of this optimization problem. The decision variables consist of (i) $x_{ij}^k \in \{0,1\},\ k = 1,\dots,M$ and $y_{ij}^l \in \{0,1\},\ l = 1,\dots,N$, corresponding to the selection of link (i,j) by NEV and EV subflows, respectively; and (ii) charging amounts r_i^l for EV subflows for all nodes $i = 1,\dots,n-1$ and subflows $l = 1,\dots,N$. Given traffic congestion effects, the time and energy consumption on each link depends on the values of x_{ij}^k, y_{ij}^l and the fraction of the total flow rate R associated with the kth NEV subflow or the lth EV subflow. As in Section 3, the simplest such flow allocation is to assign each subflow the same rate, that is, every NEV subflow $k = 1,\dots,M$ is assigned a rate RP/M and every EV subflow $l = 1,\dots,N$ is assigned a rate $R(1-P)/N$. Let $\mathbf{x_{ij}} = (x_{ij}^1,\dots,x_{ij}^M,y_{ij}^1,\dots,y_{ij}^N)^T$ and $\mathbf{r_i} = (r_i^1,\dots,r_i^N)^T$, where r_i^l is the amount of charge selected by the lth EV subflow at node i. Similar to Section 3, we denote the traveling time (delay) a vehicle will experience through link (i,j) by some nonlinear function $\tau_{ij}(\mathbf{x_{ij}})$. The corresponding energy consumption for the lth subflow of EVs through (i,j) is a nonlinear function denoted by $e_{ij}^l(\mathbf{x_{ij}})$. Finally, E_i^l represents the residual energy of subflow l of EVs at node i given by the aggregated residual energy of all EVs in the subflow. The optimization problem is formulated as follows:

$$\min_{\mathbf{x_{ij}},\mathbf{r_i},\, i,j\in\mathcal{N}} \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{M}\tau_{ij}(\mathbf{x_{ij}})x_{ij}^k\frac{RP}{M} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{N}\left(\tau_{ij}(\mathbf{x_{ij}})y_{ij}^l\frac{R(1-P)}{N} + r_i^l g y_{ij}^l\right)\right] \tag{46}$$

$$s.t.\ \text{for each } k \in \{1,\dots,M\}:$$

$$\sum_{j\in O(i)} x_{ij}^k - \sum_{j\in I(i)} x_{ji}^k = b_i, \quad \text{for each } i \in \mathcal{N} \tag{47}$$

$$b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1,n \qquad \text{for each } l \in \{1,\dots,N\}: \tag{48}$$

$$\sum_{j\in O(i)} y_{ij}^l - \sum_{j\in I(i)} y_{ji}^l = b_i, \quad \text{for each } i \in \mathcal{N} \tag{49}$$

$$b_1 = 1,\ b_n = -1,\ b_i = 0,\ \text{for } i \neq 1,n \tag{50}$$

$$E_j^l = \sum_{i\in I(j)} \left(E_i^l + r_i^l - e_{ij}^l(\mathbf{x_{ij}})\right) y_{ij}^l, \quad j = 2,\dots,n \tag{51}$$

$$E_1^l \text{ is given}, \quad E_i^l \geqslant 0, \quad \text{for each } i \in \mathcal{N} \tag{52}$$

$$x_{ij}^k \in \{0,1\}, \quad y_{ij}^l \in \{0,1\}, \quad r_i^l \geqslant 0 \tag{53}$$

In the aforementioned formulation, (46) is the objective function which for NEVs is the first sum representing the overall traveling time from the origin to the destination by adding the link traveling times $\tau_{ij}(\mathbf{x_{ij}})$ when $x_{ij}^k = 1$. For EVs, the second sum includes the charging times $r_i^l g$ when $y_{ij}^l = 1$, and an EV subflow selects node l for charging. The constraints (47)–(48) and (49)–(50) represent flow conservation for NEV and EV subflows, respectively, while (51)–(52) show the energy dynamics for each EV subflow. This is an MINLP with $(M+N)m + 2(n-1)N$ variables. Similar to our discussion in Section 3.1, one can exploit some properties of the optimal solution and energy

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dynamics in order to decompose this problem into route selection and recharging amount determination and reduce the problem dimensionality. We omit numerical results, which lead to observations similar to those presented in Section 3.4, including a behavior of performance as a function of the number of subflows similar to that of Figure 2.

6. CONCLUSIONS AND FUTURE WORK

We have introduced energy constraints into vehicle routing in traffic networks and studied the problem of minimizing the total elapsed time for vehicles to reach their destinations by determining routes, as well as recharging amounts, when there is no adequate energy for the entire journey. For a single vehicle, we have shown how to decompose this problem into two simpler problems. For a multi-vehicle problem, we solved the problem by aggregating vehicles into subflows and seeking optimal routing decisions for each such subflow. One critical factor in this problem is the selection of the number of subflows. Our numerical results showed that a small number of subflows are adequate to obtain convergence to near-optimal solutions. Thus, we defined a critical number of subflows, which guarantee near optimality. In particular, we show that by selecting the number of subflows to be equal to or larger than a critical number N^*, that is, $N \geqslant N^*$, the average deviation never exceeds the predefined upper bound. Therefore, by selecting a desired accuracy, one can choose between proximity to optimality and computational complexity needed to solve the problem. We also reformulated the multi-vehicle routing problem in order to incorporate the effect of NEVs on traffic congestion.

Our ongoing work focuses on the case of *inhomogeneous* charging nodes with different charging rate characteristics and possibly queuing capacities. In this case, we can show that a similar decomposition still holds, although we can no longer obtain an LP problem. Thus, we must now seek alternative methods to reduce the computational complexity of obtaining solutions, possibly at the expense of optimality guarantees. We also believe that extensions to multiple vehicle origins and destinations are straightforward, as is the case where only a subset of nodes has recharging resources, or not all vehicles in the network are BPVs. Finally, we are exploring extensions into stochastic vehicle flows, which can incorporate various random effects.

In the multi-vehicle case, an important issue is that of implementing an optimal routing and recharging policy. This is a challenging problem for two reasons. First, individual drivers need to be provided explicit guidance by the central controller who determines this policy. Second, a driver needs to have the proper incentives to follow this policy. While the first difficulty may be handled through communication capabilities that are increasingly being made available to vehicles, the second one is more fundamental, because it concerns the behavior of drivers who are generally ‘selfish’ and concerned with their own individual optimal policy (i.e., the one we determined in our single-vehicle analysis). This is a topic of an ongoing research, which has recently been fostered by the emerging trend towards connected vehicles: If vehicles can exchange state information in real time, this creates a new environment promoting cooperation among drivers and, ultimately, a traffic system consisting entirely of autonomous vehicles, which can automatically implement a centrally derived system-wide optimal policy.

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