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The Price of Anarchy in Transportation Networks by Estimating User Cost Functions from Actual Traffic Data*

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***Abstract*— We consider a large-scale road network in Eastern Massachusetts. Using real traffic data in the form of spatial average speeds and the flow capacity for each road segment of the network, we convert the speed data to flow data and estimate the origin-destination flow demand matrices for the network. Assuming that the observed traffic data correspond to user (Wardrop) equilibria for different times-of-the-day and days-of-the-week, we formulate appropriate inverse problems to recover the per-road cost (congestion) functions determining user route selection for each month and time-of-day period. Then, we formulate a system-optimum problem in order to find socially optimal flows for the network. We investigate the network performance, in terms of the total latency, under a user-optimal policy versus a system-optimal policy. The ratio of these two quantities is defined as the Price of Anarchy (POA) and quantifies the efficiency loss of selfish actions compared to socially optimal ones. Our findings contribute to efforts for a smarter and more efficient city.**

***Index Terms*— Transportation networks, variational inequalities, price of anarchy, smart cities, optimization.**

I. INTRODUCTION

A transportation (traffic) network is a system with non-cooperative agents (drivers) in which each driver seeks to minimize her own cost by choosing the best route (resources) to reach her destination without taking into account the overall system performance. In these systems, the cost for each agent depends on the resources it chooses as well as the number of agents choosing the same resources [1]. In such a non-cooperative setting, one often observes convergence to a Nash equilibrium, a point where no agent can benefit by altering its actions assuming that the actions of all the other agents remain fixed [2]. However, it is known that the Nash equilibrium is not always the best strategy from the system's point of view and results in a suboptimal behavior compared to the *socially optimal* policy. In a transportation network with selfish drivers, each agent (driver) follows the path (we will use "path" and "route" interchangeably) derived from a *user optimal* policy. In order to quantify the social suboptimality of selfish driving, we use the Price of Anarchy (POA) as a measure to compare system performance under a user-optimal policy vs. a system-optimal policy.

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The equilibrium flow in traffic networks, known as "Wardrop equilibrium," is the solution of the *Traffic Assignment Problem (TAP)* [3], [4]. In the transportation science literature, the TAP, which will be termed "forward problem" in what follows, has been extensively explored; see, e.g., [4] and the references therein. To solve the TAP, we need to know *a priori* the specific cost function, as well as the Origin-Destination (O-D) demand matrix.

Recent developments in data-driven inverse optimization techniques [5] enable the estimation of the cost (usually, the travel time) functions given the observations of the equilibrium flows from a large-scale transportation network. This facilitates a better understanding of the underlying dynamics of the transportation system itself. In addition, with cost function estimates at our disposal, we can address the issue of improving a traffic network's performance by controlling traffic flows, hence, contributing to the design of better transportation systems that serve smart cities.

In this paper, we leverage actual traffic data provided to us by the Boston Region Metropolitan Planning Organization (MPO). Applying a traffic flow model, we first infer equilibrium flows on each segment from the spatial average speed data. Then, by adopting the estimated traffic flows we obtain O-D demand matrices which pave the way to the derivation of cost function estimators by solving a set of inverse variational inequality problems. Finally, we formulate a system-centric problem in which agents, here drivers, cooperate to optimize the overall system performance. This allows us to estimate the POA for a sub-network so as to determine the difference in network performance between selfish routing (non-cooperative) and system-optimal routing (cooperative).

The rest of the paper is organized as follows. In Sec. II we present the models and methods we apply to the traffic data. In Sec. III we provide descriptions of the datasets we use. We elaborate on data processing tasks in Sec. IV and show our findings for the cost function estimators in Sec. V. We quantify the POA for the transportation network in Sec. VI, where numerical results for a subnetwork are included to illustrate our approaches. We conclude in Sec. VII.

Notational conventions: All vectors are column vectors. For economy of space, we write $\mathbf{x} = (x_1, \ldots, x_{\dim(\mathbf{x})})$ to denote the column vector $\mathbf{x}$, where $\dim(\mathbf{x})$ is its dimensionality. We use "prime" to denote the transpose of a matrix or vector. Unless otherwise specified, $\|\cdot\|$ denotes the ℓ_2 norm. $|\mathscr{D}|$ denotes the cardinality of a set $\mathscr{D}$. $A \overset{\text{def}}{=} B$ indicates A is defined using B.

II. Models and Methods

We first describe the model for single-class transportation networks, which we adopt throughout the paper. We then provide two equivalent formulations for the forward problem. Finally we provide a formulation for the inverse problem which will play a key role in estimating the cost functions.

A. Model for single-class transportation network

Consider a directed graph, denoted by $(\mathscr{V}, \mathscr{A})$, where $\mathscr{V}$ denotes the set of nodes and $\mathscr{A}$ the set of links. Assume it is strongly connected. Let $\mathbf{N} \in \{0, 1, -1\}^{|\mathscr{V}| \times |\mathscr{A}|}$ be the node-link incidence matrix, and $\mathbf{e}_a$ the vector with an entry equal to 1 corresponding to link a and all the other entries equal to 0.

Let $\mathbf{w} = (w_s, w_t)$ denote an origin-destination (O-D) pair and $\mathscr{W} = \{\mathbf{w}_i : \mathbf{w}_i = (w_{si}, w_{ti}), i = 1, \ldots, |\mathscr{W}|\}$ the set of all O-D pairs. Denote by $d^{\mathbf{w}} \geq 0$ the amount of the flow demand from w_s to w_t. Let $\mathbf{d}^{\mathbf{w}} \in \mathbb{R}^{|\mathscr{V}|}$ be the vector which is all zeros, except for a $-d^{\mathbf{w}}$ in the coordinate corresponding to node w_s and a $d^{\mathbf{w}}$ in the coordinate corresponding to node w_t.

Let $\mathscr{R}_i$ be the index set of simple routes (a *simple route* is a route without cycles) connecting O-D pair i. For all $a \in \mathscr{A}$, $r \in \mathscr{R}_i$, and $i \in \{1, \ldots, |\mathscr{W}|\}$, define the link-route incidence by

$$\delta_{ra}^{i} = \begin{cases} 1, & \text{if route } r \in \mathscr{R}_i \text{ uses link } a, \\ 0, & \text{otherwise.} \end{cases}$$

Let x_a be the total link flow on link $a \in \mathscr{A}$ and $\mathbf{x}$ the vector of these flows. Let $t_a(\mathbf{x}) : \mathbb{R}_+^{|\mathscr{A}|} \to \mathbb{R}_+$ be the cost function for link $a \in \mathscr{A}$; in particular, when $t_a(\mathbf{x})$ only depends on x_a, we also write $t_a(\mathbf{x})$ as $t_a(x_a)$. In addition, denote by $\mathbf{t}(\mathbf{x})$ the vector-valued function whose ath component is $t_a(\mathbf{x})$.

Throughout the paper, we assume that the cost functions have the following form [5], [6]:

$$t_a(x_a) = t_a^0 g\left(\frac{x_a}{m_a}\right), \tag{1}$$

where t_a^0 is the free-flow travel time of $a \in \mathscr{A}$, $g(\cdot)$ is strictly increasing and continuously differentiable on $\mathbb{R}_+$, and m_a is the effective flow capacity of $a \in \mathscr{A}$.

Let $\mathscr{F}$ be the set of feasible flow vectors defined by

$$\left\{\mathbf{x} : \exists \mathbf{x}^{\mathbf{w}} \in \mathbb{R}_+^{|\mathscr{A}|} \text{ s.t. } \mathbf{x} = \sum_{\mathbf{w} \in \mathscr{W}} \mathbf{x}^{\mathbf{w}},\ \mathbf{N}\mathbf{x}^{\mathbf{w}} = \mathbf{d}^{\mathbf{w}},\ \forall \mathbf{w} \in \mathscr{W}\right\},$$

where $\mathbf{x}^{\mathbf{w}}$ is the flow vector attributed to O-D pair $\mathbf{w}$. As a stepping stone for formulating the problems of interest, we present next the definition of Wardrop equilibrium.

Definition 1 *([5])*. A feasible flow $\mathbf{x}^* \in \mathscr{F}$ is a *Wardrop equilibrium* if for every O-D pair $\mathbf{w} \in \mathscr{W}$, and any route connecting (w_s, w_t) with positive flow in $\mathbf{x}^*$, the cost of traveling along that route is less than or equal to the cost of traveling along any other route that connects (w_s, w_t). Here, the cost of traveling along a route is the sum of the costs of each of its constituent links.

B. The forward problem

In this work, the forward problem (i.e., the TAP) refers to finding the Wardrop equilibrium for a given single-class transportation network with a given O-D demand matrix. We first formulate the TAP as a Variational Inequalities (VI) problem (see Def. 2 below), and then present an optimization problem as an alternative.

Definition 2 *([5])*. The VI problem, denoted $\mathrm{VI}(\mathbf{t}, \mathscr{F})$, is to find an $\mathbf{x}^* \in \mathscr{F}$ s.t.

$$\mathbf{t}(\mathbf{x}^*)'(\mathbf{x} - \mathbf{x}^*) \geq 0, \quad \forall \mathbf{x} \in \mathscr{F}. \tag{2}$$

We need the following assumption for the theorem that follows.

Assumption 1 *([4], [5])*. $\mathbf{t}(\cdot)$ is strongly monotone (see [4] or [5] for the definition of strong monotonicity) and continuously differentiable on $\mathbb{R}_+^{|\mathscr{A}|}$. $\mathscr{F}$ is nonempty and contains an interior point (Slater's condition).

Theorem II.1 *([4], [5]). Suppose Assump. 1 holds. Then, there exists a Wardrop equilibrium of the single-class transportation network, which is the unique solution to VI($\mathbf{t}, \mathscr{F}$).*

Proof: This can be established by applying [4, Thms. 3.14, 3.17, and 3.19]. Recall that, in the single-class transportation network model, we assume the network to be strongly connected and the demand for each O-D pair to be fixed (given) and nonnegative. In addition, having assumed the additivity for the route costs in the definition of the Wardrop equilibrium (Def. 1), we see that, under Assump. 1, [4, Assump. 3.A], which yields [4, Thm. 3.14], [4, Assump. 3.C], which in turn yields [4, Thm. 3.17], and [4, Assump. 3.D], which implies [4, Thm. 3.19], are all satisfied. ■

Now we present an equivalent formulation of the forward problem. It is a well-known fact that the TAP can also be formulated as the following optimization problem [3], [4]:

$$\min_{\mathbf{x} \in \mathscr{F}} \sum_{a \in \mathscr{A}} \int_0^{x_a} t_a(s)\, ds. \tag{3}$$

Note that here the objective function is different from the one we will use in the formulation for finding the social optimum (see Sec. VI); for a detailed explanation, see [3].

C. Inverse VI problem formulation

Now, given $\varepsilon > 0$, we present the definition of an ε-*approximate solution* to $\mathrm{VI}(\mathbf{t}, \mathscr{F})$ by replacing the right-hand side of (2) with $-\varepsilon$:

Definition 3 *([5])*. Let $\varepsilon > 0$. Then, $\hat{\mathbf{x}} \in \mathscr{F}$ is said to be an ε-*approximate solution* to $\mathrm{VI}(\mathbf{t}, \mathscr{F})$ if

$$\mathbf{t}(\hat{\mathbf{x}})'(\mathbf{x} - \hat{\mathbf{x}}) \geq -\varepsilon, \quad \forall \mathbf{x} \in \mathscr{F}. \tag{4}$$

Assume we are given observations $(\mathbf{x}_k, \mathscr{F}_k)$, $k = 1, \ldots, K$, with $\mathbf{x}_k \in \mathscr{F}_k$ and each $\mathscr{F}_k$ being a set of feasible flow vectors satisfying Slater's condition accordingly. The inverse VI problem amounts to seeking a function $\mathbf{t}$ such that $\mathbf{x}_k$ is an ε_k-approximate solution to $\mathrm{VI}(\mathbf{t}, \mathscr{F}_k)$ for each k. Therefore, we can formulate the inverse VI problem as [5]

$$\begin{aligned} \min_{\mathbf{t}, \boldsymbol{\varepsilon}} \quad & \|\boldsymbol{\varepsilon}\| \\ \text{s.t.} \quad & \mathbf{t}(\mathbf{x}_k)'(\mathbf{x} - \mathbf{x}_k) \geq -\varepsilon_k, \quad \forall \mathbf{x} \in \mathscr{F}_k,\ \forall k. \end{aligned} \tag{5}$$

We now apply the nonparametric estimation approach of [5] which expresses the congestion function in a Reproducing Kernel Hilbert Space (RKHS) [7]. In particular, we use the polynomial kernel, i.e., $\phi(x,y) \overset{\text{def}}{=} (c+xy)^n$ for some choice of $c \geq 0$ and $n \in \mathbb{N}$. Take the costs as in (1). Assume we are given networks $(\mathscr{V}_k, \mathscr{A}_k, \mathscr{W}_k)$, $k = 1,\ldots,K$, and the normalized link flow data $\{\mathbf{x}^k = (x_a^k / m_a^k; a \in \mathscr{A}_k); k = 1,\ldots,K\}$ are available, where k is the network index and x_a^k (resp., m_a^k) is the flow (resp., capacity) for link $a \in \mathscr{A}_k$ accordingly. Let $M = \sum_{k=1}^K |\mathscr{A}_k|$ and $\mathbf{z} = (z_1,\ldots,z_M) \overset{\text{def}}{=} ((\mathbf{x}^1)',\ldots,(\mathbf{x}^K)')$. Define the kernel matrix as $\mathbf{\Phi} = [\phi(z_i,z_j)]_{i,j=1}^M$. Then, using conic duality, by [5, Thm. 2], we reformulate the inverse VI problem (5) as the following Quadratic Programming (QP) problem [5]:

$$
\begin{aligned}
\min_{\boldsymbol{\alpha},\mathbf{y},\boldsymbol{\varepsilon}} \quad & \boldsymbol{\alpha}'\mathbf{\Phi}\boldsymbol{\alpha} + \gamma\|\boldsymbol{\varepsilon}\| \qquad (6)\\
\text{s.t.} \quad & \mathbf{e}_a'\mathbf{N}_k'\mathbf{y}^{\mathbf{w}} \leq t_a^0 \boldsymbol{\alpha}'\mathbf{\Phi}\mathbf{e}_a, \ \forall \mathbf{w} \in \mathscr{W}_k, \ a \in \mathscr{A}_k, \ k = 1,\ldots,K,\\
& \boldsymbol{\alpha}'\mathbf{\Phi}\mathbf{e}_a \leq \boldsymbol{\alpha}'\mathbf{\Phi}\mathbf{e}_{\tilde{a}}, \ \forall a, \tilde{a} \in \mathscr{A}_0 \ \text{ s.t. } \frac{x_a}{m_a} \leq \frac{x_{\tilde{a}}}{m_{\tilde{a}}},\\
& \sum_{a \in \mathscr{A}_k} t_a^0 x_a \boldsymbol{\alpha}'\mathbf{\Phi}\mathbf{e}_a - \sum_{\mathbf{w} \in \mathscr{W}_k} (\mathbf{d}^{\mathbf{w}})'\mathbf{y}^{\mathbf{w}} \leq \varepsilon_k, \ \forall k = 1,\ldots,K,\\
& \boldsymbol{\alpha}'\mathbf{\Phi}\mathbf{e}_{a_0} = 1,
\end{aligned}
$$

where $\boldsymbol{\alpha}$, $\mathbf{y} = (\mathbf{y}^{\mathbf{w}}; \mathbf{w} \in \mathscr{W}_k, k = 1,\ldots,K)$, and $\boldsymbol{\varepsilon} = (\varepsilon_k; k = 1,\ldots,K)$ are decision vectors, γ is a regularization parameter, $\mathscr{A}_0 \subset \bigcup_{k=1}^K \mathscr{A}_k$, and a_0 is some (arbitrary) link chosen for normalization purposes. The second constraint in (6) forces the function $g(\cdot)$ to be non-decreasing on the links in $\mathscr{A}_0$.

We can derive an estimator $\hat{g}(\cdot)$ of the cost function $g(\cdot)$ by solving the QP (6), thereby, obtaining an optimal $\boldsymbol{\alpha}^*$. In particular, if writing $\boldsymbol{\alpha}^* = (\alpha_1^*,\ldots,\alpha_M^*)$, then, by [5, Thm. 4], we obtain

$$
\hat{g}(\cdot) = \sum_{m=1}^M \alpha_m^* \boldsymbol{\phi}(z_m,\cdot). \qquad (7)
$$

III. Data Set Description

A. Speed dataset description

The actual traffic data provided by the MPO is a dataset of 51.2 GB consisting of 861 CSV files, each with more than 1 million lines of data. The dataset includes the spatial average speeds for major roadways and arterial streets in Eastern Massachusetts for the year 2012. The average speed within a given unit of spatial reference is calculated by aggregating observed speeds from billions of data points. Specifically, it is derived by combining data from physical traffic sensors (e.g., induction loop sensors, toll tag readers, etc), as well as all available data from probe vehicles (equipped with on-board GPS devices returning speed and location back to a central system) that fall within a specific segment of a road for a particular time window.

The dataset includes traffic data for more than 13,000 road segments (with the average distance of 0.7 miles; see Fig. 1) of Eastern Massachusetts, covering the average speed for every minute of the year 2012.

For each road segment, identified with a unique *tmc (traffic message channel)* code, the dataset provides information such as speed data (instantaneous, average and free-flow speed) in mph, date and time, and traveling time (minute) through that segment. Note that a road typically consists of many segments.

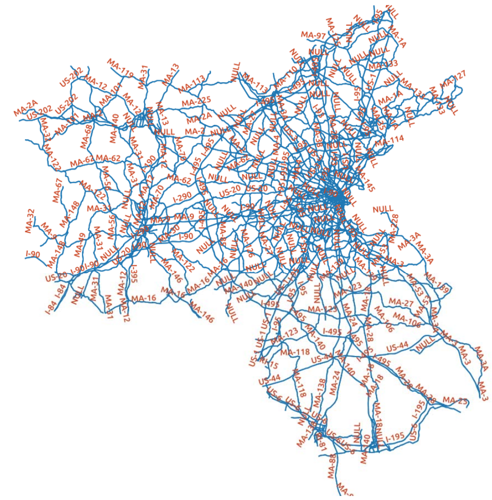
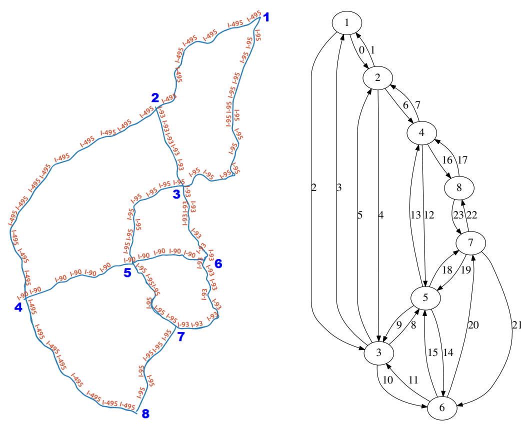


Fig. 1. All available road segments in the road map of Eastern Mass.



(a) (b)

Fig. 2. (a) An interstate highway sub-network of Eastern Massachusetts (a sub-map of Fig. 1; the blue numbers indicate node indices); (b) The topology of the sub-network (the numbers beside arrows are link indices, and the numbers inside ellipses are node indices).

B. Capacity dataset description

The flow capacity (vehicles/hour) dataset, provided by the MPO, includes capacity data – vehicle counts for each road segment – for more than 100,000 road segments (average distance of 0.13 miles) in Eastern Massachusetts. In particular, the capacity data is given for four different time periods (AM: 6 am – 9 am, MD: 9 am – 3 pm, PM: 3 pm – 6 pm, and NT: 6 pm – 6 am) in a day. For each time period, the total roadway capacity for all available lanes for that time period is given. These values are calculated based on the share of daily traffic counts in each hour of that time period. For each time period there exists a period capacity factor applied to represent peak hour conditions within that period. These factors are as follows: 2.5 for AM, 4.75 for MD, 2.5 for PM, and 7 for NT. Then, the total roadway capacity for a time period is the product of the capacity/lane/hour, the number of lanes, and the capacity factor. In our experiments, we need flow capacity on each segment in vehicle counts per hour. Thus, for each time period we scale the given vehicle counts by the inverse of the corresponding capacity factor.

C. Matching capacity data with speed data

Note that, in the capacity dataset, the ID for a road segment is named *road inventory ID*, and the segments are not absolutely identical with those in the speed dataset. Based on the geographic longitude and latitude, we have built up a dictionary mapping segments with *tmc* code to capacity dataset *road inventory ID*, through which we can read the capacity data for each road segment in the speed dataset.

IV. Data Processing

A. Preprocessing

1) Calculating average speed and free-flow speed: First, we select the time instances set $\mathscr{T}$ consisting of each minute of AM (7 am – 9 am), MD (11 am – 1 pm), PM (5 pm – 7 pm), and NT (9 pm – 11 pm) for each day of January, April, July, and October, all in 2012. Note that the selected AM (resp., MD, PM, NT) period is a subinterval of the AM (resp., MD, PM, NT) period in the capacity dataset. Then, we calculate the average speed for each segment separately for the four time periods, each of which lasts 120 minutes. Finally, for each segment, we compute a reliable proxy of the free-flow speed by using the 85th-percentile point of the observed speeds on that segment for all the time instances belonging to $\mathscr{T}$.

2) Selecting a sub-network: To reduce the computational burden while capturing the key elements of the Eastern Massachusetts road network, we only consider a representative interstate highway sub-network as shown in Fig. 2(a), where there are 701 road segments, composing a road network with 8 nodes and 24 links. We depict the topology of this sub-network in Fig. 2(b).

3) Aggregating flows of the segments on each link: Let $\{v_a^j, t_a^j, v_a^{0j}, t_a^{0j}, m_a^j;\, j = 1,\ldots,J_a\}$ denote the available observations (v_a^j, t_a^j), and parameters (v_a^{0j}, t_a^{0j}, m_a^j) of the segments composing link $a \in \mathscr{A}$, where, for each segment j, v_a^j (resp., v_a^{0j}) is the speed (resp., free-flow speed; miles/hour), t_a^j (resp., t_a^{0j}) is the travel time (resp., free-flow travel time; hour), and m_a^j is the flow capacity (vehicles/hour). Then, using Greenshield's model [8], we calculate the flow on segment j by

$$x_a^j = \frac{4m_a^j}{v_a^{0j}} v_a^j - \frac{4m_a^j}{(v_a^{0j})^2}(v_a^j)^2. \tag{8}$$

In our analysis, we enforce $v_a^j \le v_a^{0j}$ to make sure that the flow given by (8) is nonnegative. In particular, if for some time instance $v_a^j > v_a^{0j}$ (this rarely happens), we set $v_a^j = v_a^{0j}$ in (8), thus leading to a zero flow estimation for this time instance.

Aggregating over all segments composing link a we compute:

$$x_a = \frac{\sum_{j=1}^{J_a} x_a^j t_a^j}{\sum_{j=1}^{J_a} t_a^j},\ \ t_a^0 = \sum\nolimits_{j=1}^{J_a} t_a^{0j},\ \ m_a = \frac{\sum_{j=1}^{J_a} m_a^j t_a^{0j}}{\sum_{j=1}^{J_a} t_a^{0j}},$$

where x_a^j is given by (8) and $t_a^{0j} = v_a^j t_a^j / v_a^{0j}$, $j = 1,\ldots,J_a$.

4) Processing flow data such that the flow conservation law is satisfied: For $a \in \mathscr{A}$, let $\hat{x}_a$ denote the original estimate of the flow on link a, and x_a its adjustment. We then solve the following *Least Squares Problem (LSP)*:

$$\begin{aligned} \min_{\mathbf{x}} \quad & \sum_{a \in \mathscr{A}} (x_a - \hat{x}_a)^2 \\ \text{s.t.} \quad & \sum_{a \in I(i)} x_a = \sum_{a \in O(i)} x_a, \qquad \forall i \in \mathscr{V}, \\ & x_a \ge 0, \qquad \forall a \in \mathscr{A}, \end{aligned} \tag{9}$$

where the first constraint enforces flow conservation for each node $i \in \mathscr{V}$, where $I(i)$ (resp., $O(i)$) denotes the set of links entering (resp., outgoing) to (resp., from) node i.

B. Estimating O-D demand matrices

Note that we need to know the O-D demand information (compiled into a matrix) in both the forward problem formulation (3) and the inverse problem formulation (6). Based on the parameters and flows of the road network, we borrow the General Least Squares (GLS) method [9] to estimate the desired O-D demand matrix, using the following steps:

1) Obtaining link-route incidence matrix: We assume that each node could be an origin and a destination; for the subnetwork shown in Fig. 2(a), there are $8 \times (8-1) = 56$ O-D pairs in total. We then identify feasible routes for each O-D pair, thereby obtaining a 24×314 link-route incidence matrix.

2) Implementing the GLS method: Vectorize the O-D demand matrix as $\boldsymbol{\lambda}$. Let $\mathbf{A}$ be the link-route incidence matrix obtained in Sec. IV-B.1 and $\mathbf{P} = [p_{ir}]$ the route choice probability matrix, where p_{ir} is the probability that a traveler between O-D pair i uses route r. Let $\{\mathbf{x}^{(k)}; k = 1,\ldots,K\}$ denote K observations of the flow vector and $\bar{\mathbf{x}}$ the average. Then, the O-D demand matrix estimation problem is equivalent to the following generalized least squares problem [9]:

$$\begin{aligned} \text{(P0)} \quad \max_{\mathbf{P}\succeq\mathbf{0},\,\boldsymbol{\lambda}\succeq\mathbf{0}} \quad & -\frac{1}{2}\sum_{k=1}^{K}\left(\mathbf{x}^{(k)} - \mathbf{A}\mathbf{P}'\boldsymbol{\lambda}\right)'\mathbf{S}^{-1}\left(\mathbf{x}^{(k)} - \mathbf{A}\mathbf{P}'\boldsymbol{\lambda}\right) \\ \text{s.t.} \quad & p_{ir} = 0 \quad \forall (i,r) \in \{(i,r) : r \notin \mathscr{R}_i\}, \\ & \mathbf{P}\mathbf{1} = \mathbf{1}, \end{aligned}$$

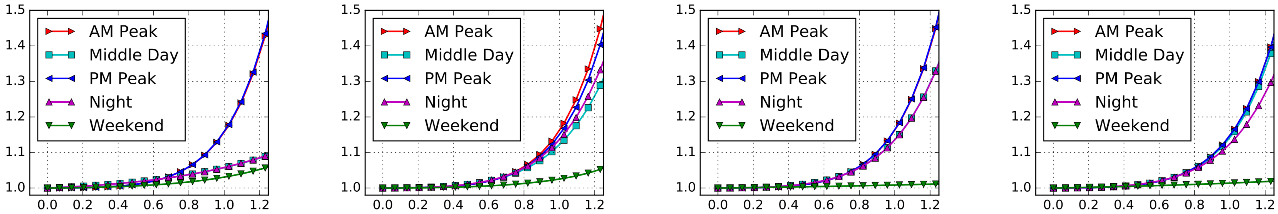
where $\mathbf{S} = (1/(K-1))\sum_{k=1}^{K}\left(\mathbf{x}^{(k)} - \bar{\mathbf{x}}\right)\left(\mathbf{x}^{(k)} - \bar{\mathbf{x}}\right)'$ is the sample covariance matrix. Letting $\boldsymbol{\xi} = \mathbf{P}'\boldsymbol{\lambda}$ and expanding the objective function in (P0), we see that the problem (P0) reduces to the following two problems:

$$\text{(P1)} \quad \min_{\boldsymbol{\xi}\succeq\mathbf{0}} \quad \frac{K}{2}\boldsymbol{\xi}'\mathbf{Q}\boldsymbol{\xi} - \mathbf{b}'\boldsymbol{\xi}, \tag{10}$$

where $\mathbf{Q} = \mathbf{A}'\mathbf{S}^{-1}\mathbf{A}$ and $\mathbf{b} = \sum_{k=1}^{K}\mathbf{A}'\mathbf{S}^{-1}\mathbf{x}^{(k)}$, and

$$\begin{aligned} \text{(P2)} \quad \min_{\mathbf{P}\succeq\mathbf{0},\,\boldsymbol{\lambda}\succeq\mathbf{0}} \quad & h(\mathbf{P},\boldsymbol{\lambda}) \\ \text{s.t.} \quad & p_{ir} = 0 \quad \forall (i,r) \in \{(i,r) : r \notin \mathscr{R}_i\}, \\ & \mathbf{P}'\boldsymbol{\lambda} = \boldsymbol{\xi}^*, \\ & \mathbf{P}\mathbf{1} = \mathbf{1}, \end{aligned} \tag{11}$$

where $h(\mathbf{P},\boldsymbol{\lambda})$ can be taken as any smooth scalar-valued function, and $\boldsymbol{\xi}^*$ is the optimal solution to (P1). Note that (P1) is a typical Quadratic Programing (QP) problem, and (P2) is a Quadratically Constrained Programming (QCP) problem. Letting $(\mathbf{P}^*,\boldsymbol{\lambda}^*)$ be an optimal solution to (P2),



(a) Jan. (b) Apr. (c) Jul. (d) Oct.

Fig. 3. Comparison of the estimated cost functions corresponding to different time periods.

then $\boldsymbol{\lambda}^*$ is our final estimation of the vectorized O-D demand matrix.

We solve (P1) and (P2) using data corresponding to five different time periods (AM, MD, PM, NT, and weekend) of four months (Jan., Apr., Jul., and Oct.) in 2012, thus obtaining 20 different O-D demand matrices for these scenarios; for details, see [10].

It is worth pointing out that the GLS method assumes the traffic network to be uncongested; therefore, the estimated O-D demand matrices for non-peak periods (MD/NT/weekend) are relatively more accurate than those for peak periods (AM/PM), and we only take the latter as a rough approximation of the corresponding true O-D demand matrices.

C. Estimating cost functions

Using the estimated flow data and the O-D demand matrices, we then estimate the costs for 20 different scenarios by solving the QP (6) accordingly. We use polynomial kernels $\mathbf{\Phi}$ and obtain the estimated cost functions $g(\cdot)$ as polynomial functions. To make the estimated costs reliable, for each scenario, we perform a 3-fold cross-validation; for details, see [10].

V. Numerical Results for Cost Functions

We list the comparison results of the cost functions in Fig. 3, where in each sub-figure, we plot the curves of the estimated $g(\cdot)$ corresponding to five different time periods.

We observe from Figs. 3(a) – 3(d) that the costs for peak periods (AM/PM) are more sensitive to traffic flows than for non-peak periods (MD/NT/weekend). This can be explained as follows: during rush hour, it is very common for vehicles to pass through a congested road network while during non-rush hour, drivers mostly enjoy an uncongested road network.

In addition, it is seen that, for different months, the cost curves for non-peak periods differ more greatly than for peak periods. Aside from the observation and modeling errors, this can also be explained by seasonal traveling patterns.

VI. Price of Anarchy

In this section we quantify the POA in a traffic network as the ratio between the total latency, i.e., the total travel time over all drivers in different O-D pairs, obtained under Wardrop flows and that obtained under social-optimal flows. Assuming a network with multiple O-D pairs, the total latency of the network is defined as follows:

$$L(\mathbf{x}) = \sum_{a \in \mathscr{A}} x_a t_a(x_a). \tag{12}$$

Let now $\mathbf{x}^*$ and $\mathbf{x}^{ne}$ denote the socially optimum and the Wardrop link flow vectors respectively. Then, the POA is defined as

$$\text{POA} = \frac{L(\mathbf{x}^{ne})}{L(\mathbf{x}^*)}. \tag{13}$$

In order to find socially optimum flows we formulate the following optimization problem [4] [11]:

$$\min_{\mathbf{x} \in \mathscr{F}} \sum_{a \in \mathscr{A}} x_a t_a(x_a). \tag{14}$$

The problem above is a Non-Linear Programming (NLP) problem in which the non-linearity comes from the cost function $t_a(x_a)$.

We calculate x_a^*'s via (14) using the corresponding O-D demand matrix and cost function, $t_a(\mathbf{x})$, derived from the speed and capacity datasets. Once we have socially optimum flows, we can calculate the POA by considering the link flow data derived from the speed and capacity datasets as the user-optimum flows, x_a^{ne}, $\forall a \in \mathscr{A}$. In particular, for a specific date and time period, we calculate the POA as follows:

$$\text{POA} = \frac{\sum_{a \in \mathscr{A}} x_a^{ne} t_a(x_a^{ne})}{\sum_{a \in \mathscr{A}} x_a^* t_a(x_a^*)}.$$

A. Numerical example

In this subsection, we investigate the POA for the network shown in Fig. 2(a). First, we calculate the POA for a specific time period in a day and then for a whole month during the year 2012. As an example we calculate the POA for the PM (5 pm – 7 pm) period of Wed., Apr. 11, 2012. To do so, we calculate socially optimal flows by solving problem (14). Tab. I shows the flow values on each link obtained under

TABLE I

Social-Optimum vs Wardrop-Equilibrium Flows

edge No.	0	1	2	3	4	5
x_a^*	1.79e3	0.7e3	1.08e3	1.67e3	3.43e3	1.94e3
x_a^{ne}	2.05e3	1.57e3	1.95e3	2.42e3	3.28e3	1.69e3
edge No.	6	7	8	9	10	11
x_a^*	0.85e3	0.35e3	5.21e3	4.2e3	3e3	1.47e3
x_a^{ne}	1.12e3	2.23e3	5.08e3	5.23e3	4.94e3	3.67e3
edge No.	12	13	14	15	16	17
x_a^*	1.19e3	2.28e3	1.14e3	1.4e3	0.13e3	0.38e3
x_a^{ne}	1.41e3	3.12e3	3e3	2.82e3	1.69e3	1.09e3
edge No.	18	19	20	21	22	23
x_a^*	3.35e3	4.34e3	1.48e3	2.26e3	1.44e3	2.24e3
x_a^{ne}	2.5e3	4.56e3	5.39e3	3.93e3	1.26e3	1.86e3

the socially optimal policy, x_a^*, as well as the average flow during the PM period extracted from data, x_a^{ne}, for Apr. 11. These flows result in the total latency of $1.6871e4$ for the selfish driving vs. 1.0869e4 for the socially optimal routing which yields POA $= 1.5522$.

We then look at the POA for a specific time period in a whole month. Fig. 4 shows the POA for the PM period during April 2012. It is observed that POA > 1 for all days in April during the PM period. In the worst case, on April 12 and April 22, POA $\simeq 2$, which means that the system is considerably inefficient under selfish driving. On the other hand, POA $= 1.23$ in the best case showing that we can reduce the total latency in the network by at least 23% if drivers follow socially optimal paths. Fig. 5 shows the socially optimal vs. average user-optimal flows on each link for all days of April 2012 during the PM period. We can observe that for some links (e.g. links 7, 11, 19 and 23), there exist significant differences between selfish behavior and system-centric behavior suggesting several potential opportunities to improve the system performance.

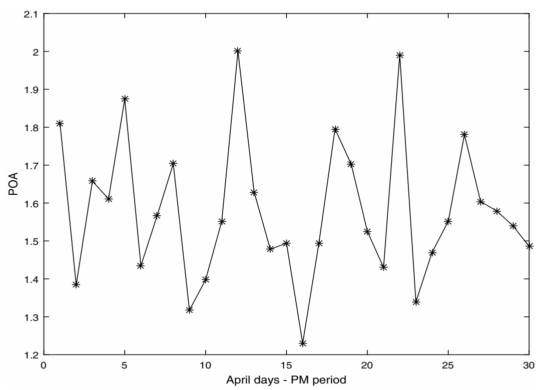
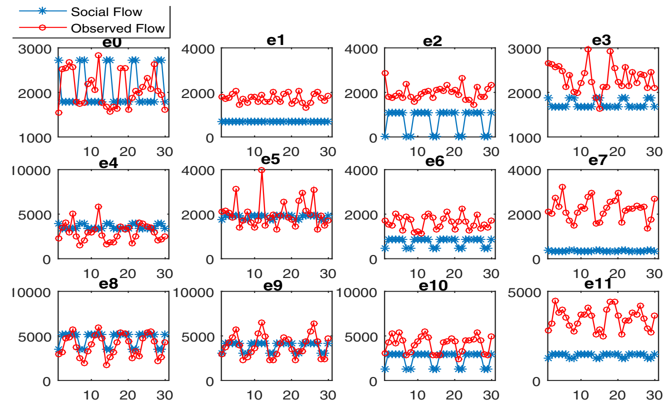
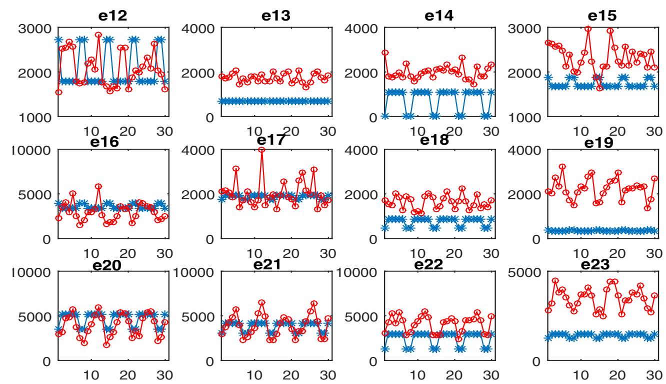


Fig. 4. POA for PM period in Apr. based on avg. flow on each link.



(a)



(b)

Fig. 5. (a) Social-opt. (blue) and user-opt. (red) flows on links 0 to 11; (b) Social-opt. (green) and user-opt. (red) flows on links 12 to 23.

VII. Conclusions

In this paper, we study a large-scale transportation network (Eastern Massachusetts) using vehicle probe data obtained from the Boston Region MPO for the year 2012, and obtain estimates for the cost functions determining users' route choices. We then quantify the Price of Anarchy (POA). Our findings help elucidate the underlying operation of a large transportation system and provide useful insights contributing to efforts for building a smarter city.

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