Perturbation Analysis of a 2-Class Stochastic Fluid Model with Finite Buffer Capacity

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Abstract

We use a Stochastic Fluid Model (SFM) for a communication network node with finite buffer capacity processing two classes of traffic: one is uncontrolled and the other is subject to threshold-based buffer control. We derive gradient estimators for class-dependent packet loss and workload related performance metrics with respect to the threshold parameter. These estimators are shown to be unbiased and directly observable from a sample path without any knowledge of underlying stochastic characteristics of the traffic processes. This renders them computable in on-line environments and easily implementable for network management and control.

1 Introduction

In this paper, we use the framework of Stochastic Fluid Models (SFM), together with Perturbation Analysis (PA) techniques, in order to develop means for on-line control and performance optimization of communication networks with multiple traffic classes. In [1], we considered a node (switch) with infinite buffer capacity which processes two traffic streams: one traffic stream is uncontrolled and the other is subject to threshold-based buffer control. We used Infinitesimal Perturbation Analysis (IPA) to obtain sensitivity estimators for packet loss and workload metrics with respect to a threshold parameter. In this paper, we analyze a similar model, except that the node has finite buffer capacity, which introduces several complications.

SFMs provide an alternative to queueing models when the latter become impractical due to huge traffic volumes and complex stochastic phenomena that cannot be handled by tractable analytical derivations. It foregoes the identity and dynamics of individual packets and focuses instead on the aggregate flow rate. SFMs have recently been shown to be especially useful for analyzing various kinds of high-speed networks (see [2], [1] and references). As argued in [2] and [1], such models may not always be accurate for the purpose of performance analysis, but they capture the salient features of the underlying "real" system, which is often sufficient to solve control and optimization problems. In this case, estimating the gradient of a given cost function with respect to key parameters becomes an essential task, for which Perturbation Analysis (PA) methods [3], [4] are therefore suitable, if appropriately adapted to a SFM viewed as a discrete-event system [5], [6], [2], [1]. In a single-class node with threshold-based buffer control [2], Infinitesimal Perturbation Analysis (IPA) has been shown to yield simple nonparametric sensitivity estimators for packet loss and workload metrics with respect to threshold or buffer size parameters. In the multiclass case studied in [1], the estimators generally depend on traffic rate information, but not on the stochastic characteristics of the arrival and service processes involved. In addition, the estimators obtained are unbiased under very weak structural assumptions on the defining traffic processes. As a result, they can be evaluated based on data observed on a sample path of the actual (discrete-event) system and combined with gradient-based optimization schemes as shown in [2] and [1].

In this paper, we consider a node with a finite buffer and two traffic streams (see Fig. 1) one controlled and the other subject to threshold-based control. Thus, we model a typical network node where the controlled stream represents a source of new traffic into the network at that node and the uncontrolled stream represents "interfering traffic", i.e., traffic originating at other nodes on its way to various destinations. This is an essential step towards the study of a complete network, which is the ultimate objective of this line of re-

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search. Interestingly, this model also captures the operation of the Differentiated Services (DS) protocol that has been proposed for supporting QoS requirements ([7] and its references). In contrast to [1], the uncontrolled traffic class may now experience loss when the buffer exceeds its capacity. Thus, new events have to be incorporated in the model and, as a result, a different way of analyzing a sample path is introduced. Moreover, there are now two loss metrics to consider, one for each traffic class. The contribution of this paper is the derivation of IPA gradient estimators for two classdependent loss metrics and a workload metric with respect to the threshold parameter in the model (from workload metrics it is possible to obtain delay metrics using appropriate forms of Little's law; see [8]). In addition, these estimators are also shown to be unbiased.

2 The Stochastic Fluid Model (SFM)

The SFM studied in this paper is shown in Fig. 1. There are two "classes" of traffic: controlled (class 1) and uncontrolled (class 2). Uncontrolled traffic has a time-varying arrival rate $\alpha_2(t)$. A threshold θ is associated with class 1 traffic, which has a time-varying arrival rate $\alpha_1(t)$. A control policy is exercised so that when the total buffer content reaches a threshold θ , class 1 traffic is rejected, while class 2 traffic is not affected. The two traffic streams share a common FIFO buffer which has a finite capacity $b > \theta$. The service rate is also time-varying and denoted by $\beta(t)$. In addition, let $\gamma_1(\theta;t)$ be the loss rate of class 1 when the buffer content exceeds the designated threshold level θ , and let $\gamma_2(\theta;t)$ be the loss rate of class 2 when the buffer content exceeds the buffer size b. Finally, $x(\theta;t)$ denotes the buffer content at time t. The notational dependence on θ indicates that we will analyze performance metrics as functions of the given θ . In the infinite buffer system in [1], only class 1 traffic may experience loss, whereas in the current finite buffer case, class 2 traffic will overflow when the buffer content reaches the capacity b. We are interested in studying sam-

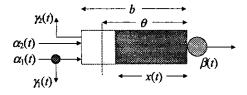


Figure 1: Stochastic Fluid Model (SFM) with two Traffic Classes

ple paths of the SFM over a time interval [0,T] for a given fixed $0 < T < \infty$. We assume that the processes $\{\alpha_1(t)\}, \{\alpha_2(t)\},$ and $\{\beta(t)\}$ are independent of θ (thus, we consider network settings operating with protocols

such as ATM and UDP, but not TCP) and they are right-continuous piecewise continuously differentiable w.p.1. Viewed as a discrete-event system, an event in a sample path of the above SFM may be either exogenous or endogenous. An exogenous event is any event that causes the difference function $[\alpha_1(t) + \alpha_2(t) - \beta(t)]$ or $[\alpha_2(t)-\beta(t)]$ to change sign. For our purposes, we identify the following exogenous events of interest: (e_1) an event where the buffer ceases to be empty, (e_2) an event where the buffer content leaves the value $x(\theta;t) = \theta$ after it has maintained it for some finite length of time, and (e_3) an event where the buffer content leaves the buffer limit b. An endogenous event is defined to occur whenever: (e_4) the buffer becomes empty, (e_5) the buffer content reaches the value $x(\theta;t) = \theta$ and then maintains it for some finite length of time, (e_6) the buffer content crosses the value $x(\theta;t) = \theta$ from either below or above, and (e_7) the buffer content reaches b. Observe that any exogenous event time is locally independent of θ , whereas any endogenous event time is generally a function of θ .

We will assume that the real-valued parameter θ is confined to a closed and bounded (compact) interval Θ ; to avoid unnecessary technical complications, we assume that $0 < \theta < b$ for all $\theta \in \Theta$. Let $\mathcal{L}(\theta) : \Theta \to \mathbb{R}$ be a random function defined over the underlying probability space (Ω, \mathcal{F}, P) . Strictly speaking, we write $\mathcal{L}(\theta, \omega)$ to indicate that this sample function depends on the sample point $\omega \in \Omega$, but will suppress ω unless it is necessary to stress this fact. In what follows, we will consider three performance metrics, the Loss Volume of Class 1, $L_{1T}(\theta)$, the Loss Volume of Class 2, $L_{2T}(\theta)$, and the Cumulative Workload (or just Work) $Q_T(\theta)$, all defined on the interval [0,T] as follows:

$$L_{iT}(\theta) = \int_0^T \gamma_i(\theta;t)dt, \quad i = 1, 2$$
 (1)

$$Q_T(\theta) = \int_0^T x(\theta; t) dt, \qquad (2)$$

where, for simplicity, we assume that $x(\theta;0) = 0$. We may then formulate optimization problems such as the determination of θ^* that minimizes a cost function of the form

$$J_{T}(\theta) = \frac{1}{T} E[Q_{T}(\theta)] + \frac{R}{T} \{ E[L_{1T}(\theta)] + wE[L_{2T}(\theta)] \}$$

$$\equiv \frac{1}{T} J_{Q}(\theta) + \frac{R}{T} \{ J_{1L}(\theta) + wJ_{2L}(\theta) \}$$
(3)

where R represents a rejection cost due to loss, and w is a weight factor to reflect the relative importance of class 1 and class 2 losses. In order to solve such problems, we rely on estimates of $dJ_{1L}(\theta)/d\theta$, $dJ_{2L}(\theta)/d\theta$ and $dJ_{Q}(\theta)/d\theta$ for use in stochastic gradient-based schemes. Henceforth we shall use the "prime" notation to denote derivatives with respect to θ .

3 Infinitesimal Perturbation Analysis (IPA) with respect to Threshold

Our objective here is to estimate the derivatives $J_{iL}'(\theta)$, $J_{Q}'(\theta)$ through the sample derivatives $L_{iT}'(\theta)$ and $Q_{T}'(\theta)$ which are commonly referred to as the Infinitesimal Perturbation Analysis (IPA) estimators; comprehensive discussions of IPA and its applications can be found in [3], [4]. The IPA derivative-estimation technique computes the derivative of a sample function $\mathcal{L}(\theta)$ along an observed sample path ω .

Contrary to [1], where the sample path was partitioned into alternating empty periods and busy periods, here, we shall proceed by partitioning a sample path into alternating Boundary Periods (BP) and Non-Boundary Periods (NBP). A NBP refers to the time interval such that $0 < x(\theta;t) < b$. A BP is either an Empty Period (EP) in which $x(\theta;t) = 0$ or an Overflow Period (OP) in which $x(\theta;t) = b$. Suppose that a sample path includes K NBPs denoted by \mathcal{N}_k , k = 1, ..., K. The starting point of any \mathcal{N}_k is one where the buffer ceases to be either empty or full, which corresponds to the exogenous events e_1 and e_3 respectively, and it is, therefore, locally independent of θ . The ending point of \mathcal{N}_k generally depends on θ . Denoting the starting point of \mathcal{N}_k by ξ_k and its ending point by $\eta_k(\theta)$ respectively, we express \mathcal{N}_k as $\mathcal{N}_k = [\xi_k, \eta_k(\theta)), k = 1, \dots, K$, for some random integer K which is also locally independent of θ . Note that if there were no limit b to the buffer content, then a NBP would be identical to a busy period as used in [1].

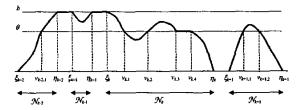


Figure 2: Typical Sample Path Segment

Let us now focus on a typical NBP $\mathcal{N}_k = [\xi_k, \eta_k(\theta))$. Let $v_{k,i}$, $i = 0, ..., S_k$, be the occurrence times of all exogenous and endogenous events $(e_1, ..., e_7)$ as previously defined in the NBP. Note that $v_{k,0} = \xi_k$ and $v_{k,S_k} = \eta_k(\theta)$. Depending on buffer content level, we can divide \mathcal{N}_k into periods (intervals) $p_{k,i}(\theta)$, which is defined as

$$p_{k,i}(\theta) := [v_{k,i-1}(\theta), v_{k,i}(\theta)), i = 1, ..., S_k,$$
 (4)

so that each one belongs to one of the following three sets, defined as in earlier work [1]:

1. Partial Loss Period Set $U_k(\theta)$. During such periods, the buffer content is $x(\theta;t) = \theta$ and class 1 traffic experiences partial loss. In particular,

$$\frac{dx(t)}{dt^{+}} = 0 (5)$$

$$\alpha_1(t) + \alpha_2(t) - \beta(t) > 0 \tag{6}$$

$$\alpha_2(t) - \beta(t) < 0 \tag{7}$$

Therefore, the loss rate of class 1 is

$$\gamma_1(\theta;t) = \alpha_1(t) + \alpha_2(t) - \beta(t) \tag{8}$$

Formally, we define $U_k(\theta)$ as follows:

$$U_k(\theta) := \{ p_{k,i}(\theta) : x(t) = \theta, \ t \in p_{k,i}(\theta) \}$$
 (9)

where the end point $v_{k,i}$ of each period is locally independent of θ . In Fig. 2, $[v_{k,3}, v_{k,4})$ is an example of a partial loss period within \mathcal{N}_k .

2. Full Loss Period Set $V_k(\theta)$. In a full loss period, the buffer content is $x(t;\theta) > \theta$ (excluding the starting point $v_{k,i-1}(\theta)$) and all class 1 traffic is lost:

$$V_k(\theta) := \{ p_{k,i}(\theta) : x(t) > \theta, \ t \in (v_{k,i-1}(\theta), v_{k,i}(\theta)) \}$$
(10)

and we have

$$\frac{dx(t)}{dt^{+}} = \alpha_2(t) - \beta(t) \tag{11}$$

$$\gamma_1(\theta;t) = \alpha_1(t) \tag{12}$$

In Fig. 2, $[\xi_k, v_{k,1}), [v_{k,2}, v_{k,3})$ are examples of full loss periods within \mathcal{N}_k .

3. No Loss Period Set $W_k(\theta)$. During such periods the buffer content is $x(t;\theta) < \theta$ (excluding the starting point $v_{k,i-1}(\theta)$) and no loss occurs:

$$W_k(\theta) := \{ p_{k,i}(\theta) : x(t) < \theta, \ t \in (v_{k,i-1}(\theta), v_{k,i}(\theta)) \}$$
(13)

and we have

$$\frac{dx(t)}{dt^{+}} = \alpha_1(t) + \alpha_2(t) - \beta(t) \tag{14}$$

$$\gamma_1(\theta;t) = 0 \tag{15}$$

In Fig. 2, $[v_{k,1}, v_{k,2}), [v_{k,4}, \eta_k)$ are examples of no loss periods within \mathcal{N}_k .

Before proceeding, let us identify conditions under which the sample derivatives exists.

Assumption 1.

- a. $\alpha_1(t) < \infty, \ \alpha_2(t) < \infty \ \text{and} \ \beta(t) < \infty \ \text{for all} \ t \in [0,T]$
- b. For every $\theta \in \Theta$, w.p. 1, no two events may occur at the same time

c. W.p. 1, there exists no interval $(v_{k,i}(\theta), v_{k,i}(\theta) + \tau)$, $\tau > 0$, such that $x(t) = \theta$ for all $t \in (v_{k,i}(\theta), v_{k,i}(\theta) + \tau)$, and either $\alpha_2(t) - \beta(t) = 0$ or $\alpha_1(t) + \alpha_2(t) - \beta(t) = 0$.

All three parts of Assumption 1 are mild technical conditions. Regarding part c, one-side derivatives may still be used if a sample path happens to contain a partial loss period in which $\alpha_2(t) - \beta(t) = 0$ or $\alpha_1(t) + \alpha_2(t) - \beta(t) = 0$.

We further define, for $i = 1, ..., S_k - 1$:

$$A_{k,i} \equiv \alpha_1(v_{k,i}(\theta)) + \alpha_2(v_{k,i}(\theta)) - \beta(v_{k,i}(\theta))$$
 (16)

$$B_{k,i} \equiv \alpha_2(v_{k,i}(\theta)) - \beta(v_{k,i}(\theta)) \tag{17}$$

Note that if $S_k=1$ for a NBP $[\xi_k,\eta_k(\theta))$, this means that the buffer content is either always above or always below the threshold θ and no $v_{k,i}(\theta)$ of interest exists. No such NBP contributes to the derivatives, therefore, in the following, we consider only the case $S_k>1$, which means the buffer content reaches θ at least once during the NBP.

The following lemma shows that all event time derivatives of interest, $v'_{k,i}(\theta)$, are expressed in terms of $A_{k,i}$ and $B_{k,i}$ above; by convention, we shall set $A_{k,0} \equiv 1$ and $B_{k,0} \equiv 1$. Moreover, we establish the fact that after a Partial Loss period occurs, all ensuing event time derivatives are $v'_{k,i}(\theta) = 0$.

Lemma 3.1 Suppose that $p_{k,m+1}(\theta)$, $1 \le m < S_k - 1$, is the first Partial Loss period in a NBP $[\xi_k, \eta_k(\theta))$ with $S_k > 1$. Then:

1. For $v_{k,i} < v_{k,m}$, if $x(\theta; \xi_k) = 0$:

$$v'_{k,1}(\theta) = \frac{A_{k,0}}{A_{k,1}} \tag{18}$$

$$v'_{k,2n}(\theta) = \prod_{i=1}^{n} \frac{B_{k,2i-1}}{B_{k,2i}} \cdot \frac{A_{k,2i-2}}{A_{k,2i-1}}$$
(19)

where $1 \le n \le \frac{m}{2}$ if m is even, and $1 \le n \le \frac{m-1}{2}$ if m is odd, and m > 1; and

$$v'_{k,2n+1}(\theta) = \frac{A_{k,2n}}{A_{k,2n+1}} \prod_{i=1}^{n} \frac{B_{k,2i-1}}{B_{k,2i}} \cdot \frac{A_{k,2i-2}}{A_{k,2i-1}}$$
(20)

where $1 \le n \le \frac{m-2}{2}$ if m is even, and $1 \le n \le \frac{m-1}{2}$ if m is odd, and m > 2.

If $x(\theta; \xi_k) = b$:

$$v_{k,1}'(\theta) = \frac{B_{k,0}}{B_{k,1}} \tag{21}$$

$$v'_{k,2n}(\theta) = \prod_{i=1}^{n} \frac{A_{k,2i-1}}{A_{k,2i}} \cdot \frac{B_{k,2i-2}}{B_{k,2i-1}}$$
(22)

where $1 \le n \le \frac{m}{2}$ if m is even, and $1 \le n \le \frac{m-1}{2}$ if m is odd, and m > 1; and

$$v'_{k,2n+1}(\theta) = \frac{B_{k,2n}}{B_{k,2n+1}} \prod_{i=1}^{n} \frac{A_{k,2i-1}}{A_{k,2i}} \cdot \frac{B_{k,2i-2}}{B_{k,2i-1}}$$
(23)

where $1 \le n \le \frac{m-2}{2}$ if m is even, and $1 \le n \le \frac{m-1}{2}$ if m is odd, and m > 2.

2. For all $v_{k,i} \geq v_{k,m+1}$:

$$v'_{k,i}(\theta) = 0$$

Due to space limitation, we omit all the proofs in this paper.

Remark. It should be obvious that if there is no Partial Loss Period in a NBP, then (18)-(23) apply for all $v'_{k,i}(\theta)$, $i = 1, ..., S_k - 1$.

3.1 Class 1 Loss Derivatives

We begin by considering the class 1 loss metric $L_{1T}(\theta)$. In view of our sample path partitioning into NBPs $[\xi_k, \eta_k(\theta)), k = 1, \ldots, K$, and BPs that precede and follow them, we may write (1) as follows:

$$L_{1T}(\theta) = \sum_{k=1}^K \int_{\xi_k}^{\eta_k(\theta)} \gamma_1(\theta;t) dt + \sum_{k=1}^K \int_{\eta_k(\theta)}^{\xi_{k+1}} \gamma_1(\theta;t) dt$$
(24)

If the sample path ends with an incomplete NBP, we simply set $\eta_K = \xi_{K+1} = T$; similarly for a possibly incomplete last BP. By differentiating with respect to θ we obtain

$$L'_{1T}(\theta) = \sum_{k=1}^{K} \frac{d}{d\theta} \int_{\xi_k}^{\eta_k(\theta)} \gamma_1(\theta; t) dt + \sum_{k=1}^{K} \frac{d}{d\theta} \int_{\eta_k(\theta)}^{\xi_{k+1}} \gamma_1(\theta; t) dt.$$
 (25)

Let us now define

$$\lambda_{1,k}(\theta) = \int_{\xi_k}^{\eta_k(\theta)} \gamma_1(\theta;t)dt + \int_{\eta_k(\theta)}^{\xi_{k+1}} \gamma_1(\theta;t)dt,$$

so that, from (25), we can write

$$L'_{1T}(\theta) = \sum_{k=1}^{K} \lambda'_{1,k}(\theta)$$
 (26)

and our objective is to evaluate $\lambda'_{1,k}(\theta)$ for any $k=1,\ldots,K$.

The next lemma provides an expression for the derivative $\lambda'_{1,k}(\theta)$ required in (26). Let $\bar{\Phi}(\theta)$ denote the set

of NBPs containing no Partial Loss periods and set

$$\begin{split} \bar{\Phi}_{00}(\theta) &= \{k : k \in \bar{\Phi}(\theta), \ x(\theta; \xi_k) = 0, \ x(\theta; \eta_k(\theta)) = 0\} \\ \bar{\Phi}_{0b}(\theta) &= \{k : k \in \bar{\Phi}(\theta), \ x(\theta; \xi_k) = 0, \ x(\theta; \eta_k(\theta)) = b\} \\ \bar{\Phi}_{b0}(\theta) &= \{k : k \in \bar{\Phi}(\theta), \ x(\theta; \xi_k) = b, \ x(\theta; \eta_k(\theta)) = 0\} \\ \bar{\Phi}_{bb}(\theta) &= \{k : k \in \bar{\Phi}(\theta), \ x(\theta; \xi_k) = b, \ x(\theta; \eta_k(\theta)) = b\} \end{split}$$

and we note that all the above sets are locally independent of θ . Further we define for any $k=1,\ldots,K,$ $n=2,4,\ldots$

$$R_{k,n} = \prod_{i=1}^{n/2} \frac{A_{k,2i}}{B_{k,2i}} \cdot \frac{B_{k,2i-1}}{A_{k,2i-1}},\tag{27}$$

$$\bar{R}_{k,n} = \prod_{i=1}^{n/2} \frac{B_{k,2i}}{A_{k,2i}} \cdot \frac{A_{k,2i-1}}{B_{k,2i-1}},$$
(28)

and define $R_{k,0} = 1$, $\bar{R}_{k,0} = 1$.

Lemma 3.2 For any $k = 1, \ldots, K$,

$$\lambda_{1,k}^{'}(\theta) = -1 + \begin{cases} R_{k,S_{k}-1} & \text{if } k \in \bar{\Phi}_{00}(\theta) \\ \frac{B_{k,S_{k}-1}}{A_{k,S_{k}-1}} R_{k,S_{k}-2} & \text{if } k \in \bar{\Phi}_{0b}(\theta) \\ \frac{A_{k,S_{k}-1}}{B_{k,S_{k}-1}} \bar{R}_{k,S_{k}-2} & \text{if } k \in \bar{\Phi}_{b0}(\theta) \\ \bar{R}_{k,S_{k}-1} & \text{if } k \in \bar{\Phi}_{bb}(\theta) \\ 0 & \text{otherwise} \end{cases}$$

Moreover,

$$-1 < \lambda_{1,k}'(\theta) \le 0. \tag{30}$$

Theorem 3.1 The sample derivative $L_{1T}^{'}(\theta)$ is given by

$$L'_{1T}(\theta) = -K + \sum_{k \in \bar{\Phi}_{00}} R_{k,S_{k}-1} + \sum_{k \in \bar{\Phi}_{0b}} \frac{B_{k,S_{k}-1}}{A_{k,S_{k}-1}} R_{k,S_{k}-2} + \sum_{k \in \bar{\Phi}_{bb}} \frac{A_{k,S_{k}-1}}{B_{k,S_{k}-1}} \bar{R}_{k,S_{k}-2} + \sum_{k \in \bar{\Phi}_{bb}} \bar{R}_{k,S_{k}-1}$$
(31)

where K is the (random) number of NBPs contained in [0,T], including a possibly incomplete last one.

The expression in (31) provides the IPA estimator for the loss metric defined in (1). We show the unbiasedness of this estimator in Section 4. Note that $L'_{1T}(\theta)$ above does not depend on any distributional information regarding the traffic arrival and service processes and involves only flow rates at event times $v_{k,i}(\theta)$ which may be estimated on line. If NBPs include at least one Partial Loss period, then the only implementation requirement is that such a period be detected and the contribution of this entire NBP is simply -1.

3.2 Class 2 Loss Derivatives

Class 2 traffic will experience loss only during an OP where the buffer level is b and the loss rate is $\gamma_2(\theta;t) = \alpha_2(t) - \beta(t)$. Let $[\eta_k(\theta), \xi_{k+1})$ denote the kth BP in [0,T] and suppose it is an OP. Note that ξ_{k+1} is independent of θ since it is the occurrence time of an exogenous event e_3 . Define the loss of class 2 during that period to be $\lambda_{2,k}(\theta)$:

$$\lambda_{2,k}(\theta) = \int_{\eta_k(\theta)}^{\xi_{k+1}} \gamma_2(\theta;t) dt = \int_{\eta_k(\theta)}^{\xi_{k+1}} [\alpha_2(t) - \beta(t)] dt,$$
(32)

so, upon taking derivatives, we have

$$\lambda_{2,k}^{'}(\theta) = \frac{d}{d\theta} \int_{\eta_k(\theta)}^{\xi_{k+1}} [\alpha_2(t) - \beta(t)] dt$$
$$= -[\alpha_2(\eta_k(\theta)) - \beta(\eta_k(\theta))] \eta_k^{'}(\theta). \tag{33}$$

Define

$$\Psi(\theta) = \{k \in \{1, \dots, K\} : x(\theta; \eta_k(\theta)) = x(\theta; \xi_{k+1}) = b\}$$

to be the set of OPs in the sample path (locally independent of θ), so that

$$L'_{2T}(\theta) = \sum_{k \in \Psi(\theta)} \lambda'_{2,k}(\theta)$$
 (34)

Lemma 3.3 Let $[\eta_k(\theta), \xi_{k+1})$ denote the kth BP, k = 1, ..., K, and assume it is an OP. Then,

$$\lambda_{2,k}^{'}(\theta) = -\lambda_{1,k}^{'}(\theta) \tag{35}$$

where $\lambda_{1,k}(\theta)$ is the class 1 loss incurred over the preceding NBP $[\xi_k, \eta_k(\theta))$.

Lemma 3.3 greatly simplifies the task of evaluating the sample derivatives of both traffic classes, since $L'_{2T}(\theta)$ in (34) is obtained as a byproduct of the evaluation of $L'_{1T}(\theta)$ in Theorem 3.1.

3.3 Work Derivative

Similar to L_{1T} in (24), we can rewrite (2) as

$$Q_T(\theta) = \sum_{k=1}^K \int_{\xi_k}^{\eta_k(\theta)} x(\theta;t)dt + \sum_{k=1}^K \int_{\eta_k(\theta)}^{\xi_{k+1}} x(\theta;t)dt$$

Define

$$q_k(\theta) = \int_{\xi_k}^{\eta_k(\theta)} x(\theta;t)dt + \int_{\eta_k(\theta)}^{\xi_{k+1}} x(\theta;t)dt,$$

so that

$$q_{k}^{'}(\theta) = \frac{d}{d\theta} \int_{\xi_{k}}^{\eta_{k}(\theta)} x(\theta; t) dt + \frac{d}{d\theta} \int_{\eta_{k}(\theta)}^{\xi_{k+1}} x(\theta; t) dt$$

Lemma 3.4 For any NBP $[\xi_k, \eta_k(\theta))$, suppose that $S_k > 1$ and $p_{k,m+1}(\theta), 1 \le m < S_k - 1$ is the first Partial Loss period in the NBP. Then,

$$q_{k}'(\theta) = \sum_{i=1}^{m-1} (v_{k,i+1} - v_{k,i}) \phi_{k,i} + (v_{k,S_{k}} - v_{k,m})$$
 (36)

where

$$\phi_{k,i} = \begin{cases} 1 - A_{k,i} v'_{k,i}, & \text{if } p_{k,i+1} \text{ is a No Loss period} \\ 1 - B_{k,i} v'_{k,i}, & \text{if } p_{k,i+1} \text{ is a Full Loss period} \end{cases}$$

$$(37)$$

Moreover, if $S_k = 1$, then

$$q_{k}^{'}(\theta) = 0.$$

Theorem 3.2 The sample derivative $Q_T'(\theta)$ is given by

$$Q_{T}^{'}(\theta) = \sum_{k=1}^{K} q_{k}^{'}(\theta)$$
 (38)

where K is the (random) number of NBP contained in [0,T], including a possibly incomplete last NBP.

The expression in (38) provides the IPA estimator for the work metric defined in (2). Its implementation requires the same information as that for the loss metric with the addition of timers to measure the duration of periods $p_{k,i+1}$ within each NBP observed in [0,T], as well as $(v_{k,S_k} - v_{k,m_k})$ if one or more Partial Loss periods are included.

4 IPA Estimator Unbiasedness

In general, the unbiasedness of an IPA derivative $\mathcal{L}'(\theta)$ is ensured by the following two conditions (see [9], Lemma A2, p.70):

Condition 1. For every $\theta \in \Theta$, the sample derivative $\mathcal{L}'(\theta)$ exists w.p.1.

Condition 2. W.p.1, the random function $\mathcal{L}(\theta)$ is Lipschitz continuous throughout Θ , and the (generally random) Lipschitz constant has a finite first moment.

Consequently, establishing the unbiasedness of $L_{1T}^{'}(\theta)$, $L_{2T}^{'}(\theta)$ and $Q_{T}^{'}(\theta)$ as estimators of $dE[L_{1T}(\theta)]/d\theta$, $dE[L_{2T}(\theta)]/d\theta$ and $dE[Q_{T}(\theta)]/d\theta$, respectively, reduces to verifying the Lipschitz continuity of $L_{1T}(\theta)$, $L_{2T}(\theta)$ and $Q_{T}(\theta)$ with appropriate Lipschitz constants.

Theorem 4.1 The IPA estimates $L_{1T}^{'}(\theta)$, $L_{2T}^{'}(\theta)$ and $Q_{T}^{'}(\theta)$ are unbiased estimates of $dE[L_{1T}(\theta)]/d\theta$,

 $dE[L_{2T}(\theta)]/d\theta$ and $dE[Q_T(\theta)]/d\theta$ respectively. In other words,

$$egin{align} E\left[L_{iT}^{'}(heta)
ight] &= rac{dE[L_{iT}(heta)]}{d heta}, \; i=1,2 \quad and \ E\left[Q_{T}^{'}(heta)
ight] &= rac{dE[Q_{T}(heta)]}{d heta} \end{split}$$

References

- [1] C. Cassandras, G. Sun, C. Panayiotou, and Y. Wardi, "Perturbation analysis and control of multiclass stochastic fluid models for communication networks," *IEEE Transactions on Automatic Control*, 2002. Submitted; a preprint can be found at http://vita.bu.edu/cgc/fullpubs/TwoClassSFM.pdf.
- [2] C. G. Cassandras, Y. Wardi, B. Melamed, G. Sun, and C. G. Panayiotou, "Perturbation analysis for on-line control and optimization of stochastic fluid models," *IEEE Transactions on Automatic Control*, vol. AC-47, no. 8, pp. 1234–1248, 2002.
- [3] Y. Ho and X. Cao, Perturbation Analysis of Discrete Event Dynamic Systems. Boston, Massachusetts: Kluwer Academic Publishers, 1991.
- [4] C. G. Cassandras and S. Lafortune, *Introduction to Discrete Event Systems*. Kluwer Academic Publishers, 1999.
- [5] Y. Wardi, B. Melamed, C. Cassandras, and C. Panayiotou, "IPA gradient estimators in single-node stochastic fluid models," *Journal of Optimization Theory and Applications*, 2001. To appear.
- [6] Y. Liu and W. Gong, "Perturbation analysis for stochastic fluid queueing systems," in *Proc. 38th IEEE Conf. Dec. and Ctrl*, pp. 4440–4445, 1999.
- [7] C. Panayiotou and C. Cassandras, "On-line predictive techniques for "differentiated services" networks," in *Proceedings IEEE Conference on Decision and Control*, pp. 4529–4534, Dec 2001.
- [8] T. Konstantopoulos, M. Zazanis, and G. De Veciana, "Conservation laws and reflection mappings with an application to multiclass mean value analysis for stochastic fluid queues," *Stochastic Processes and their Applications*, pp. 139–146, 1996.
- [9] R. Y. Rubinstein and A. Shapiro, Discrete Event Systems: Sensitivity Analysis and Stochastic Optimization by the Score Function Method. New York, New York: John Wiley and Sons, 1993.