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Brief paper

Optimal control of batteries with fully and partially available rechargeability[☆]

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ABSTRACT

Motivated by the increasing dependence of many systems on battery energy, we study the problem of optimally controlling how to discharge and recharge a non-ideal battery so as to maximize the work it can perform over a given time period and still maintain a desired final energy level. Modeling a battery as a dynamic system, we adopt a Kinetic Battery Model (KBM) and formulate a finite-horizon optimal control problem when recharging is always feasible under the constraint that discharging and recharging cannot occur at the same time. The solution is shown to be of bang–bang type with the property that the battery is always in recharging mode during the last part of the interval. When the length of the time horizon exceeds a critical value, we also show that the optimal policy includes chattering. Numerical results are included to illustrate our analysis. We then extend the problem to settings where recharging is only occasionally feasible and show that it can be reduced to a nonlinear optimization problem which can be solved at least numerically.

1. Introduction

With the increasing importance of energy management in wireless environments, batteries are playing a critical role in fields such as consumer electronics, public transportation, and military applications. The same is true for energy-aware systems encountered in robotics, mobile sensor networks and embedded computer systems. Systems of thiskind have been studied through techniques such as Dynamic Voltage Scheduling (DVS) (Mao, Cassandras, & Zhao, 2007; Rao, Singhal, & Kumar, 2004; Yao, Demers, & Shenker, 1995) where a battery is often modeled as a queueing system with the possibility of recharging (Kar, Krishnamurthy, & Jaggi, 2006). In these studies, it is normally assumed that the battery is ideal, i.e., it maintains a constant voltage throughout the discharge process and a constant capacity for all discharge profiles, which is not generally true. To date, most batteries are electrochemical with complex dynamics characterizing nonlinear discharge behaviors (Chiasserini & Rao, 2001; Rao, Vrudhula, & Rakhmatov, 2003). In fact, the energy amount delivered by the battery heavily depends on the discharge profile and it is generally not possible to extract all the capacity stored in the battery (Panigrahi et al., 2001). This is due to the *rate capacity effect* (Doyle & Newman, 1997) that leads to the loss of capacity with increasing load current and to the *recovery effect* (Martin, 1999) which would make the battery appear to regain portions of its capacity after some resting time. Therefore, in order to optimize the use of battery power, it is necessary to take these factors into account which incurs significant computational complexity for the purpose of real-time power control. Thus, to use an efficient battery model in energy-aware systems requires a combination of accuracy and speed in expressing battery discharge behaviors under various profiles.

In early models, the electrochemical processes in a battery were described by partial differential equations (PDE) (Fuller, Doyle, & Newman, 1993; Panigrahi et al., 2001). Other efforts include an approximate Single Particle Model (Chaturvedi, Klein, Christensen, Ahmed, & Kojic, 2010), Ning and Popov (2004), Santhanagopalan, Guo, Ramadass, and White (2006) and a diffusion-based model (Vrudhula & Rakhmatov, 2003) transformed into an equivalent linear state space model (Zhang & Shi, 2009), which facilitates energy optimization but is still computation-intensive (Barbarisi, Vasca, & Glielmo, 2006; Rao, Singhal, Kumar, & Navet, 2005). More recent work Chandy, Low, Topcu, and Xu (2010) takes advantage of renewable energy in optimal power flow problems, but a battery is still modeled as a simple linear system. A Kinetic Battery Model (KBM) was originally proposed to provide a fast and comprehensive battery model for embedded systems (Manwell & McGowan, 1994). It takes into account not only the recovery effect, but also the rate capacity effect. The modification of the KBM in Rao et al. (2005) enhances

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model accuracy while still preserving computational speed. More recently, the KBM was introduced in a lifetime maximization problem for wireless sensor networks (Ning & Cassandras, 2009), revealing its applicability to large-scale systems, and to introduce (Wang & Cassandras, 2010) the optimal control problem we will analyze in what follows.

In this paper, we use the KBM to formulate a state-constrained optimal control problem with the added feature of a battery recharging capability. We seek to maximize the work performed by the battery over a given time interval $[0, T]$ with the requirement that its energy is at a desired level at the end of this interval. Our motivation comes from several application areas, including (i) mobile battery-based robotic systems which must periodically interrupt operation for recharging purposes; (ii) wireless sensor nodes, which must also be periodically recharged, sometimes through standard power sources or possibly energy harvesting from sources such as solar, wind or vibrations (Beeby et al., 2007); and (iii) electric vehicles, where the emerging "smart grid" provides considerable flexibility for controlling the timing of recharging intervals in between usage of the vehicle (Foster & Caramanis, 2010). In many such applications, the desired performance is directly controlled by the discharge rate of the battery through DVS techniques mentioned earlier. By controlling the discharge and recharge rates, we derive an optimal policy shown to be of bang–bang type with the property that the battery is always in recharging mode during the last part of the interval and there is an optimal time to switch from discharging to recharging, within the constraints of the problem. We extend the analysis to settings where recharging is only occasionally feasible in some given time intervals contained within $[0, T]$ (e.g., during sunlight intervals for solar recharging). We show that this can be reduced to a nonlinear parametric optimization problem, which can be efficiently solved at least numerically.

In Section 2, we briefly review the basics of battery dynamics and propose a modified KBM to include a recharging capability. In Section 3, a battery output maximization problem is formulated, structural properties of the optimal solution are derived, and a full solution is provided using a standard optimal control approach. Numerical examples are included to illustrate the properties of the optimal solution. In Section 4 we study an extension with a three-interval optimal control problem where recharging is possible in only one of them and derive a solution. Finally, conclusions and further research directions are described in Section 5.

2. Battery model

An electrochemical battery cell consists of an anode, a cathode and the electrolyte that separates the two electrodes. The electric current derives from the electrochemical reactions occurring at the electrode-electrolyte interface. The two important effects (Rao et al., 2005) that make battery performance nonlinear (unlike an ideal linear battery model) and sensitive to the discharge profile are: (i) the Rate Capacity effect, and (ii) the Recovery effect. The battery lifetime relies on the availability and reachability of active reaction sites in the cathode. When the load current goes high, the deviation of the concentration of active reaction sites from the average increases, thus resulting in a lower state of charge as well as less cell voltage, compared with the battery under a low load current. This phenomenon is called the *Rate Capacity Effect* (Doyle & Newman, 1997). On the other hand, the diffusion process could compensate for the depletion of the active materials taking place during the current drain, which results in voltage recovery after resting. This nonlinearity in the battery is termed the *Recovery Effect* (Martin, 1999).

The KBM, which was originally proposed in Manwell and McGowan (1994), models the battery as two wells of charge, as shown in Fig. 1 (except for the input $h(t)$). The available-charge well (R-well) directly supplies electrons to the load while the bound-charge well (B-well) only supplies electrons to the R-well. The energy levels in the two wells are denoted by $r(t)$ and $b(t)$ respectively. The rate of energy flow from the B-well to the R-well is $k(b(t) - r(t))$, where k depends on the battery characteristics. The output $u(t)$ is the workload of the battery at time t. The battery is said to be depleted when $r(t)$ becomes 0.

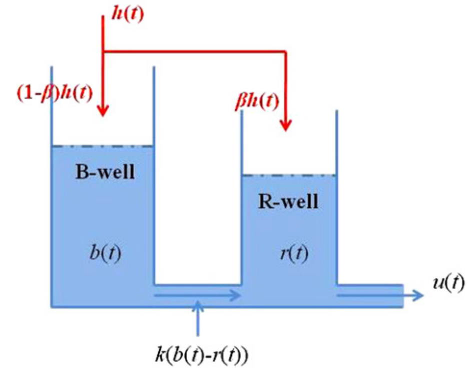

Fig. 1. Kinetic battery model modified to include recharging.

Since we are interested in a battery with rechargeability capabilities, we modify the KBM by adding a controllable input flow $h(t)$. For the sake of generality, we distribute the inflow $h(t)$ to both wells by adding a constant coefficient β ($0 \le \beta \le 1$), as seen in Fig. 1. The resulting model is as:

$$\dot{r}(t) = -c_1 u(t) + c_2 \beta h(t) + k(b(t) - r(t))$$
$$\dot{b}(t) = c_2(1 - \beta)h(t) - k(b(t) - r(t))$$

where c_1 and c_2 are battery-specific influencing factors for the discharge outflow $u(t)$ and the recharge inflow $h(t)$ respectively; since, in general, a battery discharges faster than it can recharge, we assume $c_1 > c_2 \ge 0$ where the special case $c_2 = 0$ simply means the battery is not rechargeable. Empirical evidence for the accuracy of the KBM is provided in Rao et al. (2005) in terms of capturing the recovery and rate capacity effects, which are the basic phenomena affecting the solution of the optimal control problem we will consider next. However, finding alternative simple and accurate models for non-ideal batteries remains a crucial research topic.

3. Output maximization problem

We will start with the assumption that the option to recharge the battery is always available over $[0, T]$; this will be relaxed in Section 4 where this option will be available only occasionally. Thus, we seek to control the discharging and recharging processes so as to maximize the battery output over a finite time interval $[0, T]$. Note that the continuous operation of a battery is broken down into periods of length T rather than considering the infinite future. This provides the ability to periodically return the total battery charge to a desired level and to control battery performance over individual cycles. Then, the objective of our problem is

$$\min_{(u(t),h(t))\in\mathfrak{U}} -q_T = -\int_0^T u(t)dt \tag{1}$$

where q_T is the total output over $[0, T]$ and $\mathfrak{U}$ is a feasible control set defined as

$$\mathfrak{U} = \{(u, h) \in \mathbb{R}^2 : 0 \le u(t) \le 1,\ 0 \le h(t) \le 1,\ u(t)h(t) = 0\}. \tag{2}$$

The constraint $u(t)h(t) = 0$ restricts the discharge and recharge processes so that they cannot occur simultaneously. This requirement is application-dependent and may be relaxed as shown later, but we consider the problem in the presence of this constraint for the sake of generality. The optimization problem in (1)–(2) is subject to the dynamics of the state variables $r(t)$ and $b(t)$ in the KBM described in the previous section with appropriate constraints and boundary conditions as follows:

$$\dot{r}(t) = -c_1 u(t) + c_2 \beta h(t) + k(b(t) - r(t)) \tag{3}$$

$$\dot{b}(t) = c_2(1-\beta)h(t) - k(b(t) - r(t)) \tag{4}$$

$$r(T) = r(0) \tag{5}$$

$$0 \le r(t) \le B, \quad t \in [0, T] \tag{6}$$

$$0 \le b(t) \le B, \quad t \in [0, T]. \tag{7}$$

The boundary condition (5) reflects the requirement to end a battery operating cycle of length T with the same energy level as the initial one, so as to exercise periodic control allowing the battery to be used over a potentially infinite horizon, as mentioned above. Alternatively, we may set $r(T) = r_f$ for any given $r_f \ge 0$, adding flexibility without affecting the analysis that follows. The constraints (6) and (7) capture the physical limitations of the battery which must maintain a non-negative available-charge well level throughout the interval $[0, T]$ while not exceeding an upper bound in each well level.

Before proceeding with a solution to the problem above, the following lemma establishes simple properties of the KBM state dynamics (proofs of all lemmas can be found in Wang and Cassandras (2011)).

Lemma 1. *Regarding the state bounds:*

(1) *$b(\tau) = 0$ for any $\tau \in (0, T]$ if and only if $b(0) = r(0) = 0$ and $u(t) = h(t) = 0$ for all $t \in [0, \tau]$; otherwise $b(\tau) > 0$.*
(2) *If $\beta = 0$ ($\beta = 1$), then $r(\tau) = B$ ($b(\tau) = B$) for any $\tau \in (0, T]$ if and only if $b(0) = r(0) = B$ and $u(t) = h(t) = 0$ for all $t \in [0, \tau]$; otherwise $r(\tau) < B$ ($b(\tau) < B$).*
(3) *$r(\tau) = b(\tau) = B$ for any $\tau \in (0, T]$ if and only if $b(0) = r(0) = B$ and $u(t) = h(t) = 0$ for all $t \in [0, \tau]$; otherwise either $r(\tau) < B$ when $b(\tau) = B$ or $b(\tau) < B$ when $r(\tau) = B$.*

Remark 1. Given (1), it is obvious that $u(t) = 0$ is not an optimal solution. Therefore an optimal trajectory is always characterized by: (1) $b(t) > 0$; (2) $r(t) < B$ ($b(t) < B$) if $\beta = 0$ ($\beta = 1$); and (3) $r(t) < B$ ($b(t) < B$) when $b(t) = B$ ($r(t) = B$) for $t \in (0, T]$.

We begin by analyzing the unconstrained case in which (6) and (7) are relaxed. We will then extend the analysis to incorporate these constraints.

Unconstrained case. In this case, the optimal state trajectory consists of an interior arc over the entire interval $[0, T]$. Let $\mathbf{x}(t) = (r(t), b(t))^T$ and $\boldsymbol{\lambda}(t) = (\lambda_1(t), \lambda_2(t))^T$ denote the state and costate vector respectively. The Hamiltonian for this problem is

$$\begin{aligned} H(\mathbf{x}, \boldsymbol{\lambda}, u, h) &= -u(t) + \lambda_1 \dot{r}(t) + \lambda_2 \dot{b}(t) \\ &= [-c_1\lambda_1(t) - 1]u(t) \\ &\quad + c_2[\beta\lambda_1(t) + (1-\beta)\lambda_2(t)]h(t) \\ &\quad + k[\lambda_1(t) - \lambda_2(t)][b(t) - r(t)]. \end{aligned} \tag{8}$$

Note that we do not incorporate the constraint $u(t)h(t) = 0$ into the Hamiltonian for the time being. Omitting function arguments for simplicity, the corresponding Lagrangian is:

$$L = H + \mu_e uh + \mu_1(u-1) - \mu_2 u + \mu_3(h-1) - \mu_4 h$$

where $\mu_i,\ i = 1, \dots, 4$ and μ_e are the multipliers corresponding to the constraints $0 \le u(t) \le 1$, $0 \le h(t) \le 1$ and $u(t)h(t) = 0$ respectively. The multipliers μ_i satisfy $\mu_i \ge 0, i = 1, \dots, 4$, and the complementary slackness conditions: $\mu_1(u-1) = 0$, $\mu_2 u = 0$, $\mu_3(h-1) = 0$, $\mu_4 h = 0$. The costate (Euler–Lagrange) equations $\dot{\lambda} = -\frac{\partial L}{\partial \mathbf{x}}$ give

$$\begin{cases} \dot{\lambda}_1(t) = k(\lambda_1(t) - \lambda_2(t)) \\ \dot{\lambda}_2(t) = -k(\lambda_1(t) - \lambda_2(t)) \end{cases} \tag{9}$$

and, due to (5), we must satisfy $\lambda(T) = \frac{\partial \Phi(\mathbf{x}(T))}{\partial \mathbf{x}}$ where $\Phi(\mathbf{x}(T)) = \nu[r(T) - r(0)]$ and ν is an unknown multiplier, so that

$$\lambda_1(T) = \nu, \qquad \lambda_2(T) = 0. \tag{10}$$

Solving (9) with the boundary conditions (10), we get

$$\lambda_1(t) = \frac{\nu}{2}[1 + e^{2k(t-T)}], \qquad \lambda_2(t) = \frac{\nu}{2}[1 - e^{2k(t-T)}]. \tag{11}$$

Looking at (8), let us define the switching functions $s_1(t)$ and $s_2(t)$ corresponding to $u(t)$ and $h(t)$:

$$\begin{aligned} &s_1(t) = -c_1\lambda_1(t) - 1, \\ &s_2(t) = c_2[\beta\lambda_1(t) + (1-\beta)\lambda_2(t)] \end{aligned} \tag{12}$$

and apply the Pontryagin minimum principle:

$$H(\mathbf{x}^*, \lambda^*, u^*, h^*) = \min_{(u,h)\in\mathfrak{U}} H(\mathbf{x}, \lambda, u, h) \tag{13}$$

where $u^*(t), h^*(t), t \in [0, T)$, denote the optimal controls. We can then see that

$$u^*(t) = \begin{cases} 1 & s_1(t) < 0 \\ 0 & s_1(t) > 0, \end{cases} \qquad h^*(t) = \begin{cases} 1 & s_2(t) < 0 \\ 0 & s_2(t) > 0 \end{cases} \tag{14}$$

where the singular case with $s_1(t) = s_2(t) = 0$ is excluded, since, by (11)–(12), the monotonicity of $s_1(t)$ and $s_2(t)$ makes it impossible to have $s_1(t) = s_2(t) = 0$ for any interval of finite length. In addition, the case $u^*(t) = h^*(t) = 0$ can be immediately excluded based on the following observation. If $s_2(t) > 0$ in (14), then $h^*(t) = 0$; in addition, by (11)–(12), we must have $\nu > 0$, which in turn implies $\lambda_1(t) > 0$. It follows that $s_1(t) = -c_1\lambda_1(t) - 1 < 0$, implying that $u^*(t) = 0$ cannot be optimal, therefore $u^*(t) = h^*(t) = 0$ is not an optimal control pair. If, on the other hand, $s_2(t) < 0$, then $h^*(t) = 0$ cannot be optimal and $u^*(t) = h^*(t) = 0$ cannot be an optimal control pair.

Recall that the constraint $u(t)h(t) = 0$ was not included in (8). Since we have established that $u^*(t) = h^*(t) = 0$ may be excluded, it follows that $h^*(t) = 1 - u^*(t)$ and we can rewrite $H(x, \lambda, u, h)$ with $h(t) = 1 - u(t)$ without affecting the optimality conditions:

$$\begin{aligned} H(\mathbf{x}, \lambda, u, h) = {} & (-c_1\lambda_1 - 1 - c_2[\beta\lambda_1 + (1-\beta)\lambda_2])u \\ & + c_2[\beta\lambda_1 + (1-\beta)\lambda_2] + k(\lambda_1 - \lambda_2)(b - r). \end{aligned}$$

We now define a new switching function

$$\sigma(t) = -c_1\lambda_1(t) - 1 - c_2[\beta\lambda_1(t) + (1-\beta)\lambda_2(t)] \tag{15}$$

which, using (11), becomes

$$\sigma(t) = -\frac{\nu}{2}[(c_1 + c_2) + (c_1 - (1-2\beta)c_2)e^{2k(t-T)}] - 1. \tag{16}$$

Note that since $c_1 > c_2$ and $|1 - 2\beta| \le 1$, the bracketed term in (16) is positive. Then, the optimal control on an interior arc is

$$\begin{cases} u^*(t) = 0, & h^*(t) = 1 \quad \text{if } \sigma(t) > 0 \\ u^*(t) = 1, & h^*(t) = 0 \quad \text{if } \sigma(t) < 0 \end{cases} \tag{17}$$

where the singular case $\sigma(t) = 0$ can be excluded for the same reason that the case $s_1(t) = s_2(t) = 0$ was excluded. The optimal solution in (17) is a simple bang–bang control with a switch occurring when (and if) $\sigma(t)$ changes sign for some $t \in [0, T)$. In view of (16), let us consider two cases regarding the sign of the

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unknown constant ν. First, suppose $\nu \geq 0$, in which case $\sigma(t) < 0$ for all $t \in [0, T]$. It follows from (17) that $u^*(t) = 1, h^*(t) = 0$ for all $t \in [0, T]$. Using these values in (3)–(4), we can solve the two state equations with the added condition $r(T) = r(0)$ in (5) and obtain the following equation which must be satisfied by $r(0)$, $b(0)$ and T:

$$-\frac{c_1}{2}T + \frac{1}{2}\left[b(0) - r(0) - \frac{c_1}{2k}\right](1 - e^{-2kT}) = 0. \tag{18}$$

Clearly, (18) cannot be satisfied for arbitrary $r(0)$, $b(0)$, T and constitutes a special case of little or no interest. Thus, in the sequel we concentrate on the remaining case where $\nu < 0$. We can then have either $\sigma(t) > 0$ or $\sigma(t) < 0$ for $t \in [0, T]$ and there are two possible cases to consider:

Case 1. $\sigma(t) < 0$ for all $t \in [0, T]$. This case results in the exact same analysis as $\nu \geq 0$ above and can only hold if (18) is satisfied, which is only true for specific $r(0)$, $b(0)$, T and, hence, of little or no interest.

Case 2. $\sigma(t) < 0$ for $t \in [0, t_s)$ and $\sigma(t) > 0$ for $t \in (t_s, T]$, where t_s is a *switching time*. Since $\nu < 0$ and recalling that $c_1 > c_2$, observe in (16) that $\sigma(t)$ is monotonically increasing, so if $\sigma(t) > 0$ this must happen in an interval that ends at T, thus justifying the assertion that $\sigma(t) > 0$ for $t \in (t_s, T]$. It then follows from (17) that

$$u^*(t) = \begin{cases} 1 & t \leq t_s \\ 0 & t > t_s, \end{cases} \qquad h^*(t) = \begin{cases} 0 & t \leq t_s \\ 1 & t > t_s \end{cases} \tag{19}$$

where t_s can be obtained by solving the state equations:

$$\begin{cases} \dot{r}(t) = -c_1 + k(b(t) - r(t)) \\ \dot{b}(t) = -k(b(t) - r(t)) \end{cases} \quad \text{for } t \in [0, t_s]$$

$$\begin{cases} \dot{r}(t) = c_2\beta + k(b(t) - r(t)) \\ \dot{b}(t) = c_2(1-\beta) - k(b(t) - r(t)) \end{cases} \quad \text{for } t \in (t_s, T]$$

with $r(T) = r(0)$, $r(t_s^-) = r(t_s^+)$, and given $r(0)$, $b(0)$. Following some straightforward calculations, the switching time t_s is obtained as the root of

$$\frac{c_1 + c_2}{2}t_s + \frac{b(0) - r(0) - \frac{c_1}{2k}}{2}(e^{-2kT} - 1) + \frac{c_1 - c_2(1 - 2\beta)}{4k}(e^{-2k(T - t_s)} - 1) - \frac{c_2}{2}T = 0. \tag{20}$$

Remark 2. If $c_2 = 0$ (i.e., no recharging is possible), then $u(t)$ is the only control variable. Our analysis still fully applies with $c_2 = 0$ in (20). In this case, the battery is discharged at full rate over $[0, t_s]$ and remains idle over $(t_s, T]$ while the R-well in the battery is replenished through the B-well because of the battery's recovery effect (see Fig. 1).

Further, assuming that $b(0) < B$, we can easily see that (7) is not relevant over the interval $[0, t_s)$ and we can proceed to analyze whether the constraint $r(t) \geq 0$ in (6) may become active depending on the value of T, given the values of $r(0)$, $b(0)$, c_1, c_2. Let t_s^* be the switching time of the critical case $r(t_s^*) = 0$. Since we can solve the state Eq. (3), setting $r(t_s^*) = 0$ allows us to obtain t_s^* as the root of the equation:

$$r(0) - \frac{c_1}{2}t_s^* - \frac{1}{2}\left[b(0) - r(0) - \frac{c_1}{2k}\right](e^{-2kt_s^*} - 1) = 0. \tag{21}$$

Moreover, we can determine the associated critical value T^* by solving (3) over $(t_s^*, T]$ with $r(t_s^*) = 0$ and $r(T^*) = r(0)$, leading to the equation:

$$\frac{1}{2}\left[b(0) - r(0) - c_1 t_s^* - \frac{c_2(1 - 2\beta)}{2k}\right][1 - e^{-2k(T^* - t_s^*)}] + \frac{c_2}{2}(T^* - t_s^*) - r(0) = 0. \tag{22}$$

If $T \leq T^*$ (obtained through (22)), then the optimal solution is the one derived from the unconstrained problem we have analyzed up to this point. If $T > T^*$, then it is necessary to consider the constraint (6) and we expect that part of the optimal state trajectory must include one or more boundary arcs. In addition, the constraint (7) may also become active in this constrained optimal trajectory. Next, we consider the constrained case with $r(t) \geq 0$.

Constrained case. $r(t) \geq 0$. Given that we can characterize whether this constraint is active or not by checking the condition $T \leq T^*$ with T^* obtained through (22), we will now assume that the given interval satisfies $T > T^*$, i.e., $r^*(t) = 0$ for some $t \in (0, T)$. If this happens, given that $r(T) = r(0) > 0$, then the final part of the optimal state trajectory $r^*(t)$ must be such that $r^*(t) > 0$, $t \in (t_l, T]$, over an interior arc that starts at some t_l such that $t_l = \sup\{t : t \in (0, T),\ r^*(t) = 0\}$. We now employ the indirect adjoining approach (Hartl, Sethi, & Vickson, 1995) to deal with the constrained optimal control problem under $r(t) \geq 0$. The Hamiltonian is still the same as (8), but the Lagrangian is modified to incorporate the state constraint as follows:

$$L = H + \eta_1(t)\dot{S}_1(\mathbf{x}(t)) + \mu_e u(t)h(t) + \mu_1(u(t) - 1) - \mu_2 u(t) + \mu_3(h(t) - 1) - \mu_4 h(t) \tag{23}$$

where $S_1 : \mathbb{R}^2 \to \mathbb{R}$ is a function assumed to be analytic, which in our case is $S_1(\mathbf{x}(t)) = -r(t) \leq 0$, and $\eta_1(t)$ is an associated multiplier satisfying $\eta_1(t) \geq 0$, $\eta_1(t)S_1(\mathbf{x}^*(t)) = 0$, $\dot{\eta}_1(t) \leq 0$. Then, using the state dynamics (3), we get

$$\dot{S}_1(\mathbf{x}(t)) = c_1 u(t) - c_2\beta h(t) - k(b(t) - r(t)).$$

Given the introduction of $\eta_1(t)$, the costate equations $\dot{\lambda} = -\frac{\partial L}{\partial \mathbf{x}}$ become

$$\begin{cases} \dot{\lambda}_1(t) = k[\lambda_1(t) - \lambda_2(t) - \eta_1(t)] \\ \dot{\lambda}_2(t) = -k[\lambda_1(t) - \lambda_2(t) - \eta_1(t)] \end{cases}$$

with the same boundary conditions as (10). Thus, when the state belongs to an interior arc, $\eta_1(t) = 0$ and it does not affect the costate equations. However, when the state is on a boundary arc, i.e., $r(t) = 0$, then $\eta_1(t)$ becomes some function dependent on $\lambda(t)$. The determination of $\eta_1(t)$ follows from the solution of $\frac{\partial L}{\partial u} = 0$ and $\frac{\partial L}{\partial h} = 0$. Proceeding backwards in time from T, as already argued, the optimal solution consists of an interior arc $(t_l, T]$. The first question we need to address is whether this interval includes a switching time t_s such that $\sigma(t_s) = 0$, $t_s \in (t_l, T]$. The following lemma provides an important property that will allow us to show this is not possible.

Lemma 2. *Let $t_1 \in (0, T)$ be such that $u(t_1) = 1$ and $\dot{r}(t_1) > 0$ regardless of $h(t)$. Then, under $(u(t), h(t)) \in \mathfrak{U}$, $\dot{r}(t) > 0$ for all $t \in [0, t_1]$.*

Now, let us assume there exists a switching time $t_s \in (t_l, T]$. Since $\sigma(t) > 0$ for $t \in (t_s, T]$, it follows from (17) that $u^*(t) = 0$, $h^*(t) = 1$ for $t \in (t_s, T]$, therefore $u^*(t) = 1$, $h^*(t) = 0$ for $t \in (t_l, t_s]$. Since $r(t_l) = 0$ and $r(t_s) > 0$, there exists some $t_1 \in (t_l, t_s]$ such that $\dot{r}(t_1) > 0$. Thus, Lemma 2 applies and we conclude that $\dot{r}(t) > 0$ for all $t \in [0, t_1]$. This implies that $\dot{r}(t) > 0$ for some $t < t_l$ and since $r(t_l) = 0$ we must have $r(t) < 0$, hence violating the constraint $r(t) \geq 0$. Consequently, there can be no sign switch in $\sigma(t)$ for $t \in (t_l, T]$ and the optimal control over the entire ending interior arc $(t_l, T]$ is

$$u^*(t) = 0, \qquad h^*(t) = 1, \quad t \in (t_l, T]. \tag{24}$$

Next, we consider the interval preceding the ending interior arc in order to determine the optimal trajectory for $t \leq t_l$. There are two possible cases: (i) there is a finite-length boundary arc ending at t_l, and (ii) the preceding arc is also an interior arc and t_l is a contact point. Here, we define a "contact point" $t_c \in (0, T)$ to be such that $r(t_c) = 0$ and $r(t) > 0$ for $t \neq t_c$ in a neighborhood of t_c (Maurer, 1997). We study each of these cases next with the aid of one more lemma regarding any finite-length boundary arc $[t_{en}, t_{ex}]$.

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Lemma 3. *If there exists a finite-length boundary arc* $[t_{en}, t_{ex}]$, $t_{en} < t_{ex}$, $r(t) = 0$ *for all* $t \in [t_{en}, t_{ex}]$, *then*

$$0 < \frac{kb(t)}{c_1} < 1, \quad t \in (t_{en}, t_{ex}].$$

Case 1. There exists a finite-length boundary arc $[t_b, t_l]$ ending at t_l with $r(t_l) = 0$. In this case, $\dot{r}(t) = 0$, $t \in (t_b, t_l)$. It follows from (3) and the constraint $u(t)h(t) = 0$ that the control on this boundary arc is $u_b(t) = \frac{kb(t)}{c_1}$ and $h_b(t) = 0$ for $t \in (t_b, t_l)$. Moreover, to satisfy the Pontryagin principle, the Hamiltonian $H(\mathbf{x}, \lambda, u, h)$ must satisfy:

$$H(\mathbf{x}^*, \lambda^*, u_b, 0) \le \min_{0\le h\le 1} H(\mathbf{x}^*, \lambda^*, 0, h) \tag{25}$$

$$H(\mathbf{x}^*, \lambda^*, u_b, 0) \le \min_{0\le u\le \frac{kb(t)}{c_1}} H(\mathbf{x}^*, \lambda^*, u, 0). \tag{26}$$

In addition, to account for a possible costate discontinuity at $t = t_l$, the following condition must be satisfied (Bryson & Ho, 1975):

$$\lambda(t_l^-) = \lambda(t_l^+) + \pi \cdot \frac{\partial S_1}{\partial x}(t_l)$$

where $\pi = (\pi_1(t), \pi_2(t))^T \ge 0$ is a multiplier. In view of $S_1(\mathbf{x}(t)) = -r(t)$ in our case,

$$\lambda_1(t_l^-) = \lambda_1(t_l^+) - \pi_1, \qquad \lambda_2(t_l^-) = \lambda_2(t_l^+). \tag{27}$$

Since $\nu < 0$ and (11) applies at t_l^+, we have $\lambda_1(t_l^+) < 0$, $\lambda_2(t_l^+) < 0$, therefore, $\lambda_1(t_l^-) < 0$, $\lambda_2(t_l^-) < 0$. Using this fact along with (8) in (25) gives

$$\begin{aligned}(-c_1\lambda_1(t_l^-) - 1) \cdot \frac{kb(t_l^-)}{c_1} &\le c_2[\beta\lambda_1(t_l^-) + (1-\beta)\lambda_2(t_l^-)] \\ &\le 0.\end{aligned}$$

The rightmost equality can only hold when $c_2 = 0$. Since, by Lemma 3, $0 < \frac{kb(t)}{c_1} < 1$ for all t on the boundary arc, it follows that

$$-c_1\lambda_1(t_l^-) - 1 \le c_2[\beta\lambda_1(t_l^-) + (1-\beta)\lambda_2(t_l^-)]. \tag{28}$$

Furthermore, recall that $\sigma(t) \ge 0$ applies to the ending interior arc, i.e., $\sigma(t_l^+) \ge 0$, so that

$$-c_1\lambda_1(t_l^+) - 1 \ge c_2[\beta\lambda_1(t_l^+) + (1-\beta)\lambda_2(t_l^+)] \tag{29}$$

which, combined with (27) and the fact that $\pi_1 \ge 0$, gives

$$-c_1\lambda_1(t_l^-) - 1 \ge c_2[\beta\lambda_1(t_l^-) + (1-\beta)\lambda_2(t_l^-)] \tag{30}$$

which contradicts (28) unless $c_2 = 0$. Consequently, this case is only feasible for the special case $c_2 = 0$ (implying that there is no recharging capability.).

Case 2. t_l is a contact point. In this case, the preceding arc is also an interior arc, say (t_k, t_l). Therefore, recalling the constraint $u(t)h(t) = 0$, there are three possible cases regarding $u^*(t_l^-), h^*(t_l^-)$: $u^*(t_l^-) = 1, h^*(t_l^-) = 0$ or $u^*(t_l^-) = 0, h^*(t_l^-) = 1$ or $u^*(t_l^-) = h^*(t_l^-) = 0$. However, given that t_l is a contact point and $r(t)$ is continuous, we have $\dot{r}(t_l^-) \le 0$, $r(t_l^-) = 0$, and since $b(t_l^-) > 0$ by Lemma 1, it follows from (3) that $u(t_l^-) > 0$, which excludes $u^*(t_l^-) = 0$. Therefore, the only feasible case is $u^*(t_l^-) = 1, h^*(t_l^-) = 0$ and, in view of (24), t_l is a switching point. Next, there are two cases regarding the existence of any additional control switch in (t_k, t_l).

First, consider the case that there is no switch in (t_k, t_l). Then, $u^*(t) = 1$, $h^*(t) = 0$ throughout $t \in (t_k, t_l)$. If $t_k > 0$ and since $r(t_k) = 0$, there must exist some $t_1 \in [t_k, t_l)$ such that $\dot{r}(t_1) > 0$. Now Lemma 2 applies and rules out this case by the same argument used above to exclude $u^*(t) = 1, h^*(t) = 0$ on the ending interior arc. Consequently, $t_k = 0$, which would make the optimal control:

$$\begin{cases} u^*(t) = 0, & h^*(t) = 1 \quad t \in (t_l, T] \\ u^*(t) = 1, & h^*(t) = 0 \quad t \in [0, t_l]. \end{cases}$$

This is identical to the critical case we identified in the unconstrained case requiring that $T = T^*$ in (22) and that $t_l = t_s^*$ in (21). Thus, it cannot be satisfied in general, since we assume that $T > T^*$, and this possibility is excluded.

This leaves only the second possible case, i.e., that there exists a control switch in (t_k, t_l), in addition to the one at t_l. Recalling that an interior arc starts at t_l, note that (29) applies and allowing for a possible discontinuity of the costates at the contact point, (27) still holds leading to (30). On the other hand, since (t_k, t_l) is also an interior arc, the Hamiltonian is given by (8) and, recalling that $u^*(t_l^-) = 1$, this implies that $-c_1\lambda_1(t_l^-) - 1 \le c_2[\beta\lambda_1(t_l^-) + (1-\beta)\lambda_2(t_l^-)]$. Comparing this with (30), we conclude that

$$-c_1\lambda_1(t_l^-) - 1 = c_2[\beta\lambda_1(t_l^-) + (1-\beta)\lambda_2(t_l^-)]. \tag{31}$$

It follows from (27) and $\pi_1 \ge 0$ that (29) can only be satisfied as an equality with $\pi_1 = 0$, i.e., the costates $\lambda_1(t)$, $\lambda_2(t)$ are both continuous at $t = t_l$ and, since (t_k, t_l) is an interior arc, (11) continues to apply, hence (16) also applies. Recalling that $\nu < 0$ and $c_1 > c_2$, one can easily see in (16) that $\sigma(t)$ is monotonically increasing regardless of β. In view of (31), this implies that $\sigma(t) < 0$ for $t < t_l$ and there can be no further switch in (t_k, t_l). Consequently, this possibility is excluded as well, leading to the conclusion that t_l cannot be a contact point in addition to the fact that it can also not be the end of a boundary arc with $r(t) = 0$ over some interval with $t \le t_l$.

We can now conclude that an optimal trajectory always includes a terminal interior arc over $(t_l, T]$ with $u^*(t) = 0, h^*(t) = 1$ for all $t \in (t_l, T]$ and $r(t_l) = 0$. However, there can be no boundary arc ending at t_l nor can t_l be a contact point. We can further show that there exists no boundary arc satisfying $r(t) = 0$ over any segment of an optimal trajectory nor can there exist a contact point *anywhere* in $[0, T]$. This is established in Theorem 1 with the aid of the following lemma and leads to the conclusion that the optimal trajectory when $T > T^*$ includes an interval over which the control variables *chatter* until the terminal arc over $(t_l, T]$ takes place. Lemma 4 is used to extend the argument made under Case 1 above to exclude finite-length boundary arcs, while the argument for excluding contact points is the same as the one in Case 2 above.

Lemma 4. *Let* $[\tau, T)$, $\tau \ge 0$, *be an interval over which an optimal trajectory contains no finite-length boundary arc. Then,*

$$\lambda_1(t) < \lambda_2(t) < 0 \quad \textit{for all } t \in [\tau, T).$$

Moreover, $\lambda_1(t)$ *is monotonically decreasing and* $\lambda_2(t)$ *is monotonically increasing for all* $t \in [\tau, T]$.

Theorem 1. *Suppose* $T > T^*$, *i.e., the constraint* $r(t) \ge 0$ *is active on the optimal trajectory. There exists no finite-length boundary arc nor any contact point on the optimal trajectory when* $c_2 > 0$.

Proof. We first prove there can be *no finite-length boundary arc*. Assuming there exists at least one finite-length boundary arc in the optimal trajectory, we consider the last one, i.e., $r(t) = 0$ for $t \in [t_{en}, t_{ex}]$, $t_{en} < t_{ex}$, and there exists no finite-length boundary arc in $(t_{ex}, T]$. Under these conditions, Lemma 4 applies over (t_{ex}, T), i.e.,

$$\lambda_1(t) < \lambda_2(t) < 0, \quad t \in (t_{ex}, T). \tag{32}$$

Now if the boundary arc $[t_{en}, t_{ex}]$ is part of the optimal trajectory, then the Hamiltonian $H(x, \lambda, u, h)$ must satisfy (25) and (26).

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Thus, (25) at t_{ex}^- implies that

$$\min_{0\le h\le 1} c_2[\beta\lambda_1(t_{ex}^-)+(1-\beta)\lambda_2(t_{ex}^-)]h(t_{ex}^-) \ge (-c_1\lambda_1(t_{ex}^-)-1)\cdot\frac{kb(t_{ex}^-)}{c_1}. \quad (33)$$

Accounting for possible discontinuities in $\lambda_1(t)$, $\lambda_2(t)$ at $t = t_{ex}$, it follows from (27) and (32) that $\lambda_2(t_{ex}^-) = \lambda_2(t_{ex}^+) < 0$ and (33) becomes:

$$(-c_1\lambda_1(t_{ex}^-)-1)\cdot\frac{kb(t_{ex}^-)}{c_1} \le c_2[\beta\lambda_1(t_{ex}^-)+(1-\beta)\lambda_2(t_{ex}^-)] < 0.$$

Moreover, since, by Lemma 3, on the boundary arc we have $0 < \frac{kb(t_{ex})}{c_1} < 1$, we get

$$-c_1\lambda_1(t_{ex}^-)-1 < c_2[\beta\lambda_1(t_{ex}^-)+(1-\beta)\lambda_2(t_{ex}^-)]. \quad (34)$$

In the interior arc $(t_{ex}, T]$, in view of (8) and (32), $h^*(t) = 1$ and the possibility of $u^*(t) = h^*(t) = 0$ can be excluded. Further, there is no finite-length boundary arc in $(t_{ex}, T]$. We can thus set $h(t) = 1 - u(t)$ for $t \in (t_{ex}, T]$ and use the switching function $\sigma(t) = -c_1\lambda_1(t) - 1 - c_2\lambda_2(t)$. This implies that the optimal control for $t \in (t_{ex}, T]$ is either $u^*(t) = 1, h^*(t) = 0$ or $u^*(t) = 0, h^*(t) = 1$. Since $r(t_{ex}) = 0$ and the last boundary arc ends at t_{ex}, then $\dot r(t_{ex}^+) \ge 0$, which, from (3), requires $u^*(t_{ex}^+) \le \frac{kb(t_{ex})}{c_1} < 1$. Therefore, $u^*(t_{ex}^+) = 0$, $h^*(t_{ex}^+) = 1$, which results in

$$\sigma(t_{ex}^+) = -c_1\lambda_1(t_{ex}^+) - 1 - c_2[\beta\lambda_1(t_{ex}^+)+(1-\beta)\lambda_2(t_{ex}^+)] \ge 0.$$

Using (27) and the inequality above, we get:

$$\begin{aligned} &-c_1\lambda_1(t_{ex}^-)-1-c_2[\beta\lambda_1(t_{ex}^-)+(1-\beta)\lambda_2(t_{ex}^-)] \\ &\quad = -c_1\lambda_1(t_{ex}^+)+c_1\pi_1-1-c_2[\beta\lambda_1(t_{ex}^+)+(1-\beta)\lambda_2(t_{ex}^+)] \\ &\quad\quad + c_2\beta\pi_1 \ge 0 \end{aligned} \quad (35)$$

which implies that

$$-c_1\lambda_1(t_{ex}^-)-1 \ge c_2[\beta\lambda_1(t_{ex}^-)+(1-\beta)\lambda_2(t_{ex}^-)]$$

contradicting (34). Consequently, there is no finite-length boundary arc in the optimal trajectory for the constrained optimal control problem. We now show that there can be *no contact point*. We have already shown that there exists a terminal interior arc that starts at t_l. Since there can be no finite-length boundary interval, t_l could only be a contact point. However, this possibility was also excluded in our analysis of the constrained case $r(t) \ge 0$ considered earlier. The same argument can be used for any other possible contact point t_c, since it must be preceded and followed by an interior arc and since Lemma 4 applies over $[0, T)$ precluding any control switch other than the one at t_l. □

In practice, chattering is clearly undesirable and it prevents us from keeping track of the optimal state trajectory $b^*(t)$ and, therefore, the evaluation of at least the start time of the final interior arc during which the battery is charging. The obvious way to avoid chattering is to select an interval length T such that $T \le T^*$, specified through (22), if the problem setting allows it, e.g., if one wishes to control the long-term behavior of the battery over periods whose length is T. To summarize, the optimal control solution for the general case with $c_2 > 0$ is described by

$$\begin{cases} u^*(t)=1, \quad h^*(t)=0, & t\in[0,t_j) \\ u^*(t)=u_{ch}, h^*(t)=h_{ch} & t\in[t_j,t_l] \\ u^*(t)=0, \quad h^*(t)=1, & t\in(t_l,T] \end{cases} \quad (36)$$

where u_{ch} and h_{ch} are unspecified values corresponding to the chattering interval $[t_j, t_l]$. The final step is to determine the two critical times t_j and t_l. The former is analytically obtained by simply solving the state Eq. (3)–(4) with given initial conditions $r(0)$, $b(0)$ and $u(t) = 1$, $h(t) = 0$ for $t \in [0, t_j)$ with $r(t_j) = 0$. This results in the equation

$$r(0) - \frac{c_1}{2}t_j - \frac{1}{2}\left[b(0) - r(0) - \frac{c_1}{2k}\right](e^{-2kt_j} - 1) = 0 \quad (37)$$

which can be solved for t_j. On the other hand, determining t_l by a similar approach is not possible. This is because chattering prevents us from keeping track of $b(t)$ over $[t_j, t_l]$, hence $b(t_l)$ is unknown. Thus, we cannot fully solve (3)–(4) to determine t_l such that $r(t_l) = 0$ knowing only that $r(T) = r(0)$, but not $b(T)$. What we can make use of, however, is the fact that t_l is such that $\dot r(t_l) > 0$ while $\dot r(t_l^-) \le 0$. The optimal solution can still be obtained by numerical techniques.

Remark 3. In the special case $c_2 = 0$, we have $u^*(t) = \frac{kb(t)}{c_1}$ for the optimal solution on the boundary, while $h(t)$ is irrelevant since it only affects the system through $c_2h(t)$ in (4).

Constrained case. $0 \le r(t) \le B$, $0 \le b(t) \le B$. We begin by observing that in the special case $c_2 = 0$, (3)–(4) imply $r(t) < B$, $b(t) < B$, $t \in [0, T]$ regardless of the control. Thus, we consider only $c_2 > 0$. Moreover, according to Remark 1, the constraints $b(t) \le B$ and $r(t) \le B$ cannot be active simultaneously in the optimal solution. We now allow all constraints $0 \le r(t) \le B$, $0 \le b(t) \le B$ to become active on an optimal trajectory. The analysis is similar to the unconstrained case and the constrained case with only $r(t) \ge 0$ active, and the only difference arises if in the terminal interior arc (where $r(t)$ and $b(t)$ are increasing) the constraint $b(t) - B \le 0$ ($r(t) - B \le 0$) becomes active. If this happens, then the Hamiltonian is minimized by $h^*(t) = \frac{k(B-r(t))}{c_2(1-\beta)}$ ($h^*(t) = \frac{k(B-b(t))}{c_2\beta}$) rather than $h^*(t) = 1$. Thus, omitting derivation details, we can determine the full solution to the case with all the state constraints active as:

$$\begin{cases} u^*(t)=1, \quad h^*(t)=0, & t\in[0,t_j) \\ u^*(t)=u_{ch}, h^*(t)=h_{ch} & t\in[t_j,t_k] \\ u^*(t)=0, \quad h^*(t)=1, & t\in(t_k,t_l) \\ u^*(t)=0, \quad h^*(t)=\dfrac{k(B-r(t))}{c_2(1-\beta)} \left(\dfrac{k(B-b(t))}{c_2\beta}\right), & t\in[t_l,T]. \end{cases} \quad (38)$$

An example where all constraints become active at some points over $[0, T]$ is shown in Fig. 2. The optimal objective is $q_T^* = 8.8243$. The solution was obtained using the generic numerical solver *Tomlab/PROPT* (Tomlab Optimization Inc.) and it can be seen to be consistent with (38). Finally, we need to point out that once chattering occurs, we lose track of the value of $b(t)$, which prevents us from analytically obtaining t_k, t_l. As already mentioned, the obvious way to avoid chattering is to select an interval length T such that $T \le T^*$, specified through (22).

Remark 4. From an implementation standpoint, the optimal control switching structure in (19) or (38) is very simple. However, determining the exact value of switching times such as t_s in (20) requires knowledge of the battery characteristics expressed through c_1, c_2, β and k. Measuring these values may not be an easy task and involves a model identification process which represents a research effort parallel to the one of optimal discharging and recharging control. Alternatively, knowing the optimal control structure one may empirically obtain the optimal value of t_s (or other switching times) by using a known type of battery so that this value applies to all batteries of this type characterized by parameters c_1, c_2, β and k; this can be done without any explicit knowledge of the parameter values.

Solution with $u(t)h(t) = 0$ relaxed. When the constraint $u(t)h(t) = 0$ is relaxed, the Hamiltonian in (8) is unaffected, but the

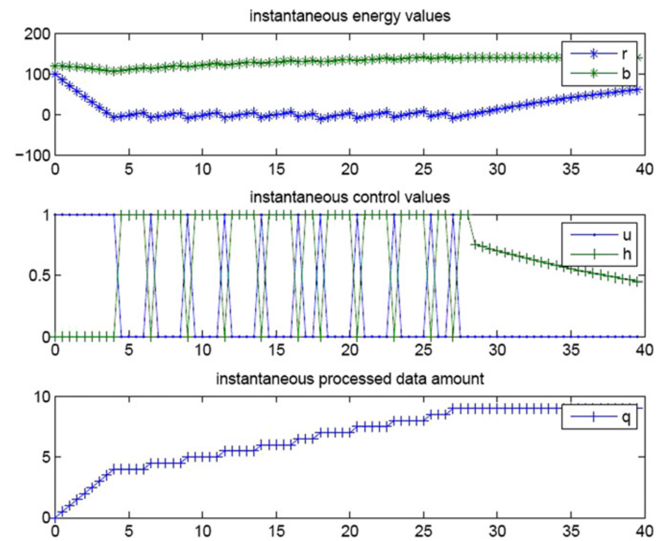

Fig. 2. Optimal solution under $r(0) = 100$, $b(0) = 120$, $c_1 = 30$, $c_2 = 10$, $k = 0.05$, $\beta = 0$, $T = 40$, and $B = 150$ with the constraints $r(t) \geq 0$ and $b(t) \leq B$.

Lagrangian no longer includes the term $\mu_e u(t)h(t)$. In addition, the admissible control set becomes $\mathfrak{U}' = \{(u, h) \in \mathbb{R}^2 : 0 \leq u(t) \leq 1, 0 \leq h(t) \leq 1\}$. The approach is exactly the same as before and the solution can be categorized for each of the four cases: (i) unconstrained, (ii) constrained case with $r(t) \geq 0$, (iii) constrained case with $b(t) \leq B$ ($r(t) \leq B$), and (iv) constrained case with $0 \leq r(t) \leq B$, $0 \leq b(t) \leq B$. In the unconstrained case, the analysis is exactly the same. In the constrained cases, the only difference arises when the state constraints are active, where it is no longer required that one of $u(t)$ and $h(t)$ be zero. The analysis is again the same and we omit details to give directly the optimal solution as follows:

$$\begin{cases} u^*(t) = 1, \quad h^*(t) = 1, \quad t \in [0, t_j) \\ u^*(t) = \dfrac{kb(t) + c_2\beta}{c_1}, \quad h^*(t) = 1 \quad t \in [t_j, t_k] \\ u^*(t) = 0, \quad h^*(t) = 1, \quad t \in (t_k, t_l) \\ u^*(t) = 0, \quad h^*(t) = \dfrac{k(B - r(t))}{c_2(1-\beta)} \left(\dfrac{k(B - b(t))}{c_2\beta}\right), \\ \quad t \in [t_l, T]. \end{cases}$$

Since there is no chattering now, t_j, t_k, t_l can be determined numerically.

4. Output maximization with partial rechargeability

In this section, we extend our analysis to cases where rechargeability is not always available; in particular, we consider cases where a battery may be recharged only over a given interval $[a_1, a_2] \subset [0, T]$. This arises, for instance, in cases where a device employs a solar cell and recharging is possible only during daylight or known intervals with expected light availability; similarly, electric cars may only be recharged during intervals where it is known that they are not needed for transportation. In general, $[0, T]$ can be partitioned into alternating availability and unavailability intervals. We will limit ourselves here to three such intervals; it will be clear that a generalization is conceptually straightforward. The problem we consider is formulated as follows:

$$\min_{(u(t),h(t))\in\mathfrak{U}} -q_T = -\int_0^T u(t)dt \tag{39}$$

$$\dot{r}(t) = -c_1u(t) + c_2\beta h(t) + k(b(t) - r(t)) \tag{40}$$

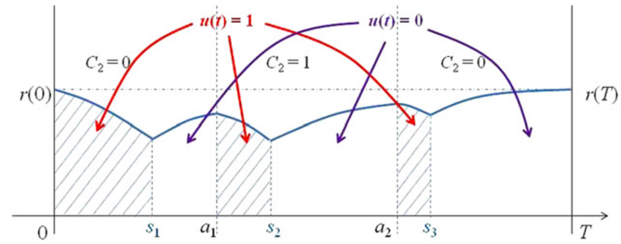

Fig. 3. Solution structure in partial rechargeability problem.

$$\dot{b}(t) = c_2(1-\beta)h(t) - k(b(t) - r(t)) \tag{41}$$

$$r(T) = r(0) \tag{42}$$

$$0 \leq r(t) \leq B, \quad 0 \leq b(t) \leq B, \quad t \in [0, T] \tag{43}$$

where $\mathfrak{U} = \{(u, h) \in \mathbb{R}^2 : 0 \leq u(t) \leq 1, u(t)h(t) = 0, t \in [0, T],$ $0 \leq h(t) \leq 1,\ t \in [a_1, a_2],\ h(t) = 0,\ t \in [0, a_1) \cup (a_2, T]\}$ and $c_1 > c_2$.

Compared to the fully available rechargeability case in (1)–(7), the only difference here is in the admissible control set. To solve it, we proceed by decomposing it into three subproblems:

$$\min_{(u(t),h(t))\in\mathfrak{U}_i} -q_i = -\int_{a_{i-1}}^{a_i} u(t)dt \tag{44}$$

$$\dot{r}(t) = -c_1u(t) + c_2\beta h(t) + k(b(t) - r(t))$$

$$\dot{b}(t) = c_2(1-\beta)h(t) - k(b(t) - r(t))$$

$$r(a_i) = \bar{r}_i$$

$$0 \leq r(t) \leq B, \quad 0 \leq b(t) \leq B, \quad t \in [a_{i-1}, a_i]$$

$$i = 1, 2, 3$$

where $\mathfrak{U}_1 = \mathfrak{U}_3 = \{(u, h) \in \mathbb{R}^2 : 0 \leq u(t) \leq 1,\ h(t) = 0\}$ and $\mathfrak{U}_2 = \{(u, h) \in \mathbb{R}^2 : 0 \leq u(t) \leq 1,\ 0 \leq h(t) \leq 1,\ u(t)h(t) = 0\}$; $a_0 = 0$, $a_3 = T$ and $\bar{r}_3 = r(T) = r(0)$. If $\bar{r}_1, \bar{r}_2$ are assumed known, then each subproblem is equivalent to our original problem (1)–(7) while cases $i = 1$ and 3 correspond to the simpler special case with $c_2 = 0$ which we discussed in Remarks 2 and 3. Therefore, the solution boils down to the determination of $\bar{r}_1, \bar{r}_2$. To avoid the complication brought about by chattering, we will consider the case where the constraint $r(t) \geq 0$ never becomes active. It will become clear that the possibility of chattering does not change the essence of the solutions obtained.

Based on the solution to problem (1)–(7) that we have obtained, the optimal solution structure corresponding to each subproblem (44) is known. In particular, for subproblem 1 and 3, Remark 2 applies to (19), i.e., for $t \in [a_{i-1}, a_i),\ i = 1, 3$

$$u^*(t) = \begin{cases} 1 & t \in [a_{i-1}, s_i] \\ 0 & t \in [s_i, a_i), \end{cases} \qquad h^*(t) = 0$$

where s_i are the switching times. For subproblem 2, the optimal controls are given by (19) with some switching time $s_2 \in [a_1, a_2)$. Fig. 3 illustrates this optimal solution structure for the problem. Therefore, the objective function in (39) can be written as

$$J(s_1, s_2, s_3) \equiv \sum_{i=1}^{3} \int_{a_{i-1}}^{a_i} u(t)dt = \sum_{i=1}^{3} s_i - a_1 - a_2 \tag{45}$$

since $u^*(t) = 0$ in the intervals $[s_1, a_1)$, $[s_2, a_2)$, and $[s_3, T)$. It follows that the determination of $\bar{r}_1, \bar{r}_2$ can be replaced by the determination of the optimal switching times s_1, s_2, s_3. Using (19) as the optimal solution in the state Eqs. (40) and (41) and solving these equations over an interval $[t_0, t_f]$ gives:

$$\begin{aligned} r(t_f) \ = \ & \frac{c_2}{2}(t_f - t_s) - \frac{b(t_0) - r(t_0) - \frac{c_1}{2k}}{2}(e^{-2k(t_f - t_0)} - 1) \\ & + r(t_0) - \frac{c_1}{2}(t_s - t_0) \\ & - \frac{c_1 - c_2(1 - 2\beta)}{4k}(e^{-2k(t_f - t_s)} - 1) \end{aligned} \tag{46}$$

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$$
\begin{aligned}
b(t_f) = {} & \frac{c_2}{2}(t_f - t_s) + \frac{b(t_0) - r(t_0) - \frac{c_1}{2k}}{2}(e^{-2k(t_f - t_0)} - 1) \\
& + b(t_0) - \frac{c_1}{2}(t_s - t_0) \\
& + \frac{c_1 - c_2(1 - 2\beta)}{4k}(e^{-2k(t_f - t_s)} - 1)
\end{aligned}
\tag{47}
$$

where setting $c_2 = 0$ recovers the cases corresponding to subproblems 1 and 3 as already discussed and t_s is the control switching time in $[t_0, t_f]$. Thus, over $[0, a_1)$, (46)–(47) apply with $t_0 = 0$, $t_f = a_1$, $t_s = s_1$, and $c_2 = 0$ and yield $r(a_1)$, $b(a_1)$ as a function of s_1 and the known initial conditions $r(0)$, $b(0)$. Similarly, over $[a_1, a_2)$, (46)–(47) apply with $t_0 = a_1$, $t_f = a_2$, $t_s = s_2$ and yield $r(a_2)$, $b(a_2)$ as a function of s_2 and $r(a_1)$, $b(a_1)$. Finally, over $[a_2, T)$, (46)–(47) apply with $t_0 = a_2$, $t_f = T$, $t_s = s_3$, and $c_2 = 0$ and yield $r(T) = r(0)$ and $b(T)$ as a function of s_3 and $r(a_2)$, $b(a_2)$. Focusing on $r(T)$ we can combine all these equations and chain them together to obtain a relationship that the switching times s_i, $i = 1, 2, 3$, must satisfy:

$$
\begin{aligned}
h(s_1, s_2, s_3) = {} & -\frac{b(0) - r(0)}{2} - \frac{c_2}{2}(a_2 - s_2) \\
& + \frac{c_1}{2}(s_1 + s_2 - a_1 + s_3 - a_2) + \frac{c_2(1 - 2\beta)}{4k} \\
& \times [e^{-2k(T - a_2)} - e^{-2k(T - s_2)}] \\
& + \frac{1}{2}\left[b(0) - r(0) - \frac{c_1}{2k}\right] e^{-2kT} \\
& + \frac{c_1}{4k}[e^{-2k(T - s_1)} - e^{-2k(T - a_1)} \\
& + e^{-2k(T - s_2)} - e^{-2k(T - a_2)} + e^{-2k(T - s_3)}]
\end{aligned}
\tag{48}
$$

where $r(0)$, $b(0)$ are known. In addition, the variables s_i satisfy:

$$
0 \le s_1 \le a_1, \qquad a_1 \le s_2 \le a_2, \qquad a_2 \le s_3 \le T. \tag{49}
$$

We can now see that problem (39)–(43) reduces to the minimization of $-J(s_1, s_2, s_3)$ in (45) subject to the constraints above, i.e.,

$$
\min_{s_i} -J(s_1, s_2, s_3) = -\sum_{i=1}^{3} s_i + a_1 + a_2 \quad \text{s.t. (48) and (49).} \tag{50}
$$

To solve this nonlinear optimization problem, the Kuhn–Tucker conditions help us narrow the optimal solution down to the cases where (i) $s_1^* \in (0, a_1)$, $s_2^* = a_1$, $s_3^* = a_2$ and (ii) $s_1^* = a_1$, with s_2^*, s_3^* limited to six other possible cases. We omit the derivation details, which can be found in Wang and Cassandras (2011). To summarize, this procedure allows us to obtain the optimal solution (s_1^*, s_2^*, s_3^*) from which the optimal control of the original problem is fully specified. Fig. 4 shows an example of an optimal solution with $s_1^* \in (0, a_1)$, $s_2^* = a_1$, $s_3^* = a_2$ and the associated parameter settings. In this case, following the solution procedure above we obtain $s_1^* = 5.473$, $s_2^* = 10$, and $s_3^* = 30$, shown as blue lines in the figure. Thus, the battery is initially discharged at its maximal rate until $t = 5.473$, and then idles until $t = a_1 = 10$. Since this is also the optimal switching time to recharge, the battery is fully recharged until $t = a_2 = 30$ and then idles for the remainder of $[0, T]$. The period over which the battery is recharging is identified by two red lines.

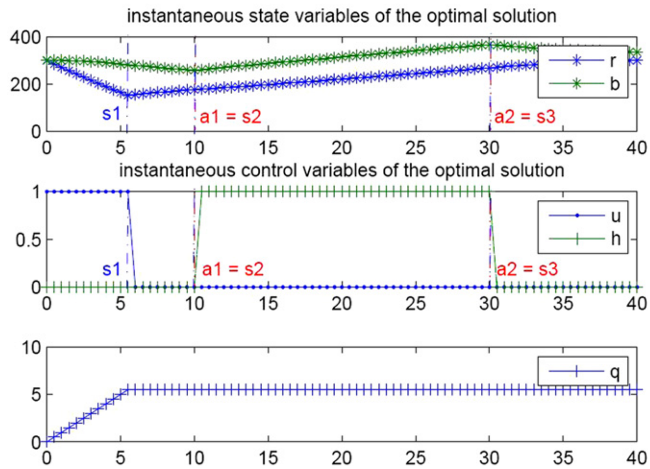

Fig. 4. Optimal solution of partial rechargeability problem with $r(0) = b(0) = 300$, $c_1 = 30$, $c_2 = 10$, $k = 0.05$, $\beta = 0$, $a_1 = 10$, $a_2 = 30$ and $T = 40$.

5. Conclusions and future work

We have used a Kinetic Battery Model (KBM) to study the problem of optimally controlling how to discharge and recharge a non-ideal battery so as to maximize the work it can perform over a given time period $[0, T]$ and still maintain a desired final energy level. Under the assumption that the battery can be recharged at any time in $[0, T]$, the solution to this problem is shown to be of bang–bang type with the battery always in recharging mode during the last part of the interval. When $T > T^*$, where T^* is some critical value we can explicitly determine, the optimal policy was shown to include chattering, unless we relax the constraint that recharging is only possible when discharging is inactive. When rechargeability is only feasible at certain intervals within $[0, T]$, we have studied a three-interval optimal control problem and shown that it can be transformed into a nonlinear optimization problem we can explicitly solve.

This line of research opens up a number of questions and open problems. First of all, finding a good modeling framework for non-ideal batteries remains a challenge. Although the KBM is an attractive model, alternatives to it have also been proposed, and it is fair to ask whether our optimal control analysis applies to such models. To address this issue, we have used the model proposed in Zhang and Shi (2009) and compared the optimal control policy we have derived in (19) to various alternatives. We have found that (19) outperforms these alternatives for a number of examples, but a rigorous analysis is still lacking. Future work involves extending the partial rechargeability case in Section 4 to settings where the recharge-feasible intervals are stochastic in nature (e.g., for solar recharging). Finally, we are exploring systems (e.g., wireless networks) whose components are battery-based and we can control their local discharging and recharging patterns with the goal of optimizing some system-wide objective.

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