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Optimal dwell times for persistent monitoring of a finite set of targets

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Abstract— **This paper considers the design of a periodic schedule for one or more agents moving around a finite number of targets, repeatedly visiting them to collect information and reduce uncertainty about the target. This data collection takes time and thus the design involves both the sequence of targets to be visited by each of the agents and the amount of time each agent should spend at each target. For a given visiting sequence, the problem is translated into a discrete-time dynamic system with the targets' sampled uncertainty level as the state vector and the dwell time at each target as the input vector. In the one-agent case we show that under a mild assumption and with a constant input this discrete-time system converges to an asymptotically stable steady state and that the underlying continuous dynamics converge to a periodic cycle with a fixed period. We show further that the sampled uncertainty, the peak uncertainty, and the period are all minimized under the policy that the agent switches to the next target in its sequence as soon as the uncertainty of the current target is reduced to zero. Finally, we show that if the average uncertainty over the steady state period is taken as a measure of performance, then the same policy is optimal under the additional assumption that the targets are homogeneous.**

I. INTRODUCTION

Monitoring a finite number of targets distributed throughout a finite domain using one or a collection of mobile agents is a problem that maps to many interesting application domains. The paradigm can describe surveillance systems, such as when a team must monitor large regions for changes, intrusions, or other dynamic events, or when it is tasked with sampling and monitoring environmental parameters such as temperature [1], [2]. The paradigm also finds use in single particle tracking in molecular biology where the goal is to track multiple individual biological macromolecules to understand their dynamics and their interactions [3], [4].

From an abstract point of view, the problem of persistent monitoring can be cast as one of a collection of agents moving between possibly mobile targets, collecting information from each to reduce uncertainty about that target. The uncertainty at each target evolves in time, increasing when not being visited and decreasing when it is being attending by one or multiple agents. Given a measure of performance, the key problems are determining for each agent the sequence of target visits and the associated dwell time at each target such that the overall cost function is optimized [5]. This description translates the persistent monitoring problem to the (optimal) control of a hybrid automaton [6]–[8].

When considering an infinite horizon problem, it makes intuitive sense to focus on periodic schedules as these are simpler to implement than aperiodic ones. It has been shown that under certain conditions, periodic schedules can be designed that ensure the entire system, from the point of view of managing the uncertainty levels of the targets, remains controllable despite the delays incurred by traveling from one target to the next [9], [10]. In addition, given a particular sequence it is clear that the uncertainty levels at the targets are a function of the dwell times. The dwell time can thus be viewed as a control input that can be optimized to determine the best cost of the given sequence. This can then be followed by an evaluation of a collection of such sequences to find one that achieves the best overall performance.

The enumeration of sequences can be viewed in a graph theoretic way, with targets being nodes and the goal being to determine the optimal path through the graph. The problem is clearly related to Traveling Salesmen Problems (TSPs) and Vehicle Routing Problems (VRPs). TSPs and VRPs have a long and rich literature and, while they are NP hard, there are many sub-optimal approximations that have been developed that permit solutions to be found rapidly [11], [12]. However, these problems have well-defined edge costs that do not depend on the visiting sequence; approximate solutions rely on this property. In our persistent monitoring problem, a change at any part of the sequence alters the optimization problem that must be solved to find the dwell times at each target and thus the cost of the sequence.

There are greedy approaches that can be applied, such as the use of Infinitesimal Perturbation Analysis (IPA) to perform gradient descent online and move towards a locally optimal solution [13], [14]. However, these are not guaranteed to find a globally optimal solution. While exactly solving the full graph-theoretic problem is not feasible in real time for problems of even moderate size, it can be solved off line as a means of testing approximate solutions found through other means. In this paper we focus on efficient solutions of the dwell time optimization, both as a means of reducing the overall computation time and as a launch point for developing approximations to the full problem that have better computational properties.

In this paper we focus on an one agent case and the sequence of target visits for this agent containing a fixed number of visits are enumerated. With the idea that the

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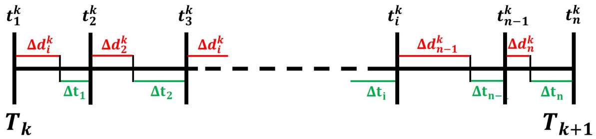
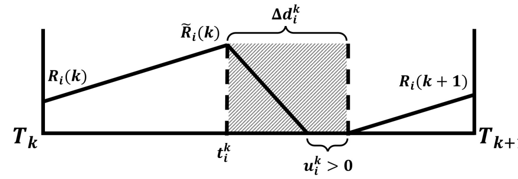
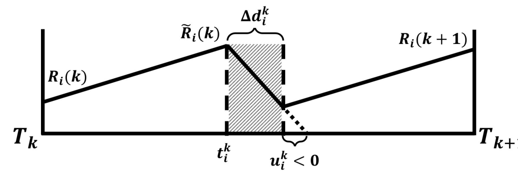


Fig. 1: The time line of an agent. Δt_i is the transit time to the next target while Δd_i is the dwell time at that target.



(a) $u_i^k > 0$.



(b) $u_i^k < 0$.

Fig. 2: Evolution of the uncertainty of target i in period k. The shaded section indicates the time an agent is visiting the target. (a) A positive value of u_i^k implies the agent remains with the target even after the uncertainty has been driven to zero while (b) a negative value implies the agent departs while the uncertainty is still positive.

sequence will be repeated for persistent monitoring over the infinite time horizon, we consider all sequences where each target is visited exactly once per cycle with the agent returning to the first target at the end of the cycle to prepare for the next. We then solve the optimization problem defining the dwell time at each of the targets in a sequence. In Sec. II, the continuous scheduling problem is translated into a discrete one where the control input is the switching condition defining the uncertainty value at which an agent should depart the current target. Then we show that under mild assumptions, any constant policy stabilizes the system to a steady state depending on that policy, and determine the optimal policy with respect to the specified performance measures. With this in place, the cost of the given sequence can be easily calculated, eliminating a computationally expensive step in the evaluation of the visiting sequences.

II. Discrete Dynamics

Consider a collection of n targets and a single agent. A sequence is given such that every target is visited exactly once. We label the targets as $i = 1, ..., n$; a loop through the targets is a cycle. The location of target i is denoted as s_i and the uncertainty related to it at time t is denoted as $r_i(t)$. The agent can move between targets with an average speed $\bar{v}$. We construct a discrete-time system by abstracting a cycle of visits completed by the agent into one step of a discrete-time system. The time when the agent begins its k^{th} visit at the first target is defined to be the beginning of the k^{th} step and is denoted T_k. The sampled information is denoted $R_i(k) = r_i(T_k)$. The duration of the k^{th} step is given by

$$T_{k+1} - T_k = \sum_{i=1}^{n} (\Delta d_i^k + \Delta t_i), \tag{1}$$

where Δd_i^k is the dwell time at the i^{th} target and Δt_i is the travel time from target i to $i+1$ with Δt_n the time to travel from target n back to target 1. The travel times are given by

$$\Delta t_i = \frac{\|s_{i+1} - s_i\|}{\bar{v}}, \quad \forall i = 1, ..., n, \quad s_{n+1} = s_1, \tag{2}$$

and the time of arrival of the agent to the i^{th} target in the k^{th} step is given by

$$t_i^k = T_k + \sum_{q=1}^{i-1} (\Delta d_q^k + \Delta t_q). \tag{3}$$

A graphical depiction of times in a cycle is shown in Fig.1.

The information state of target i is bounded below by 0, increases monotonically when it is not being visited, and decreases monotonically when the target is being attended to by an agent. A natural model for this is

$$\dot{r}_i(t) = \begin{cases} -b_i & \text{if } \exists k \text{ s.t. } t - t_i^k \in [0, \Delta d_i^k); r_i(t) > 0, \\ 0 & \text{if } \exists k \text{ s.t. } t - t_i^k \in [0, \Delta d_i^k); r_i(t) = 0, \\ a_i & \text{otherwise,} \end{cases} \tag{4}$$

where a_i and b_i are positive scalars.

The dwell time Δd_i^k is determined by $r_i(t_i^k)$ and the switching condition provided for target i. The condition is defined such that the switching to the next target should be made at a specific time relative to the point at which the uncertainty drops to zero. This condition for target i in the k^{th} step is denoted as u_i^k, and its range is defined as

$$u_i^k \in \left[-\frac{r_i(t_i^k)}{b_i}, \infty \right). \tag{5}$$

The physical meaning of negative u_i^k is that switching should be made when $r_i = (-b_i) \cdot u_i^k$. Fig.2 illustrates the dynamics of r_i in step k for both positive and negative values of u_i^k.

Consider the uncertainty related to target i in this subsystem. From the beginning of step k it starts growing with rate a_i. Its value when the agent arrives at the target is given by

$$r_i(t_i^k) = R_i(k) + a_i \sum_{q=1}^{i-1} (\Delta d_q^k + \Delta t_q) \triangleq \tilde{R}_i(k) \tag{6}$$

where $R_i(k)$ is the initial value at the beginning of the period. We denote this value as $\tilde{R}_i(k)$. Note that it is the peak value for r_i during the agent's k^{th} visit of agent i.

Once the agent arrives, the uncertainty of the target decreases with a rate of $-b_i$ until the switching condition u_i^k is reached. The agent's actual dwell time at target i can be written as

$$\Delta d_i^k = \frac{\tilde{R}_i(k)}{b_i} + u_i^k. \tag{7}$$

Δd_i^k is guaranteed to be non-negative though (5).

When the agent departs the target, the uncertainty is

$$r_i(t_i^k + \Delta d_i^k) = (-b_i)\bar{u}_i^k, \tag{8}$$

where we define $\bar{u}_i^k = \min(u_i^k, 0)$, and $\underline{u}_i^k = \max(u_i^k, 0)$. Note that this uncertainty level depends only on u_i^k and in particular is independent of both the initial value $R_i(k)$ and the peak value $\tilde{R}_i(k)$.

The uncertainty of target i at the beginning of next step is

$$\begin{aligned} R_i(k+1) =& (-b_i)\bar{u}_i^k + a_i\Delta t_i + a_i \sum_{q=i+1}^{n} \left(\Delta t_q + \Delta d_q^k\right) \\ =& (-b_i)\bar{u}_i^k + a_i\Delta t_i + a_i \sum_{q=i+1}^{n} \left(\Delta t_q + u_q^k + \frac{R_q(k)}{b_q}\right) \\ &+ a_i \sum_{q=i+1}^{n} \frac{a_q}{b_q} \sum_{l=1}^{q-1} \left(\Delta t_l + u_l^k + \frac{R_l(k)}{b_l}\right) \\ &+ a_i \sum_{q=i+1}^{n} \frac{a_q}{b_q} \sum_{l=1}^{q-1} \frac{a_l}{b_l} \sum_{m=1}^{l-1} \left(\Delta t_m + u_m^k + \frac{R_m(k)}{b_m}\right) \\ &+ \cdots . \end{aligned} \tag{9}$$

Combining this for all the targets, the discrete dynamics of the uncertainty can be expresed as

$$\begin{aligned} R(k+1) = BHB^{-1}R(k) + BHU(k) - B\bar{U}(k) + \\ (BH + A)D \end{aligned} \tag{10}$$

where

$$\begin{aligned} &B = \textbf{diag}[b_1, ...b_n], \quad A = \textbf{diag}[a_1, ...a_n], \\ &D = [\Delta t_1, ...\Delta t_n]^T, \quad R(k) = [R_1(k), ..., R_n(k)]^T, \\ &U(k) = [u_1^k, ..., u_n^k]^T, \end{aligned}$$

and $H \in \mathbb{R}^{n\times n}$ is defined by

$$h_{ij} = \frac{a_i}{b_i}\left(\prod_{q=j+1}^{n}(\frac{a_q}{b_q}+1) - \prod_{q=j+1}^{i}(\frac{a_q}{b_q}+1)\right),$$

where, with a slight abuse of notation, $\prod_q^p(\cdot)$ is defined to be 1 when $p = q-1$ and 0 when $p < q-1$. As an illustration, when $n = 2$, we have

$$H_2 = \begin{bmatrix} \frac{a_1 a_2}{b_1 b_2} & \frac{a_1}{b_1} \\ 0 & 0 \end{bmatrix},$$

and when $n = 3$,

$$H_3 = \begin{bmatrix} \frac{a_1a_2}{b_1b_2} + \frac{a_1a_3}{b_1b_3} + \frac{a_1a_2a_3}{b_1b_2b_3} & \frac{a_1}{b_1} + \frac{a_1a_3}{b_1b_3} & \frac{a_1}{b_1} \\ \frac{a_2a_3}{b_2b_3} + \frac{a_2^2a_3}{b_2^2b_3} & \frac{a_2a_3}{b_2b_3} & \frac{a_2}{b_2} \\ 0 & 0 & 0 \end{bmatrix}.$$

III. Steady state analysis

Sec. II defined the discrete dynamics of the uncertainty of the targets. We now analyze the system in (10) and show that for any given choice of constant control, there is an asymptotically stable equilibrium, beginning with the existence of the equilibrium.

Proposition 1: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$. If $b_i > (n-1)a_i,\ \forall i$, then there exists a steady state solution $\bar{R} = [\bar{R}_1, \bar{R}_2, ...\bar{R}_n]$ that is independent of the initial states. Furthermore, every entry in $B^{-1}\bar{R} + U$ is non-negative.

Proof: Let $R(k+1) = R(k) = \bar{R}$. From (10) we have

$$B^{-1}\bar{R} + U = (I-H)^{-1}\left(\underline{U} + (H + B^{-1}A)D\right). \tag{11}$$

Prop.1 holds if $(I-H)$ is invertible and $(I-H)^{-1}$ contains no negative entry. It can be shown (omitted for space reasons) that the proposition holds if

$$\sum_{p=2}^{n-l+1}(p-1)\mathcal{S}_{l,n}^p < \prod_{i=l}^{n} b_i, \qquad \forall l = 1, ...n-1, \tag{12}$$

with $\mathcal{S}_{l,n}^p$ defined as

$$\mathcal{S}_{l,n}^p = \left(\prod_{i=l}^{n} b_i\right) \sum_{j_1 \neq \ldots \neq j_p} \left(\prod_{q=1}^{p} \frac{a_{j_q}}{b_{j_q}}\right). \tag{13}$$

Here $j_{q,\ q=1,...p}$ are p different indexes with $l \le j_q \le n$. For example, when $n = 2$, Prop.1 holds if

$$\sum_{p=2}^{2}(p-1)\mathcal{S}_{1,2}^2 < \prod_{i=l}^{2} b_i, \quad \mathcal{S}_{1,2}^2 = \left(\prod_{i=1}^{2} b_i\right)\cdot\frac{a_1a_2}{b_1b_2},$$

which simplifies to $a_1a_2 < b_1b_2$. For $n = 3$ Prop.1 holds if

$$\begin{aligned} l = 1: \quad & \mathcal{S}_{1,3}^2 + 2\mathcal{S}_{1,3}^3 < b_1b_2b_3, \text{ where} \\ & \mathcal{S}_{1,3}^2 = a_1a_2b_3 + a_1b_2a_3 + b_1a_2a_3, \\ & \mathcal{S}_{1,3}^3 = a_1a_2a_3; \\ l = 2: \quad & \mathcal{S}_{2,3}^2 < b_2b_3 \quad \text{ with } \mathcal{S}_{2,3}^2 = a_2a_3. \end{aligned}$$

By assumption $b_i > (n-1)a_i,\ \forall i$. Thus

$$\mathcal{S}_{l,n}^p < \binom{n-l+1}{p}\frac{1}{(n-1)^p}\left(\prod_{i=l}^{n} b_i\right)$$

and therefore

$$\sum_{p=2}^{n-l+1}(p-1)\mathcal{S}_{l,n}^p < \left(\prod_{i=l}^{n} b_i\right)\sum_{p=2}^{n-l+1}\binom{n-l+1}{p}\frac{p-1}{(n-1)^p}.$$

We have

$$\sum_{p=2}^{n-l+1}\binom{n-l+1}{p}\frac{p-1}{(n-1)^p} \le \sum_{p=2}^{n-l+1}\binom{n-l+1}{p}\frac{p-1}{(n-l)^p}.$$

The final expression equals to 1 and therefore (12) holds. ■

Notice that the stability condition that $b_i > (n-1)a_i$ in this proposition echoes a basic stability condition in queueing theory that the service rate should be greater than the arrival rate at each target.

The next proposition establishes the asymptotic stability of the equilibrium.

Proposition 2: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$. If $b_i > (n-1)a_i,\ \forall i$, then $\lim_{k\to\infty} R = \bar{R}$.

Proof: Let $E(k) = R(k) - \bar{R}$ be the error of $R(k)$ from the steady state $\bar{R}$ at step k. Then, from (10),

$$\begin{aligned} &B^{-1}E(k+1) = B^{-1}R(k+1) - B^{-1}\bar{R} \\ &= HB^{-1}(\bar{R} + E(k)) + HU(k) - \bar{U}(k) + (H + B^{-1}A)D \\ &= HB^{-1}E(k). \end{aligned}$$

Since B^{-1} is a diagonal matrix with its diagonal entries positive, $E(k)$ converges if and only if the eigenvalues of H satisfy $|\lambda_{H,i}| < 1, \forall i$. Establishing this bound on the eigenvalues is similar to the analysis in the proof of Prop.1 and is omitted for space reasons. ∎

Next we establish that the peak values of the uncertainties also converge to a steady state.

Proposition 3: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$. If $b_i > (n-1)a_i, \ \forall i$, then the peak value $\tilde{R}(k)$ converges to a fixed set of value $\tilde{R} = [\tilde{R}_1, \tilde{R}_2, ...\tilde{R}_n]$ that is independent of the initial states.

Proof: According to (6) we have

$$\tilde{R}_i(k) = R_i(k) + a_i \sum_{q=1}^{i-1} \left(\Delta t_q + u_q + \frac{\tilde{R}_q(k)}{b_q} \right) \quad (14)$$

$$\begin{aligned} &= R_i(k) + a_i \sum_{q=1}^{i-1} \left(\Delta t_q + u_q + \frac{R_q(k)}{b_q} \right) \\ &\quad + a_i \sum_{q=1}^{i-1} \frac{a_q}{b_q} \sum_{l=1}^{q-1} \left(\Delta t_l + u_l + \frac{R_l(k)}{b_l} \right) \\ &\quad + \cdots . \end{aligned} \quad (15)$$

Prop.2 states that $R_i(k) \to \bar{R}_i$ with $k \to \infty$. Thus $\tilde{R}_i(k)$ converges to a constant value $\tilde{R}_i$. ∎

The value of the steady state for the peak values can be calculated as follows. First, from (9) we have

$$\begin{aligned} \frac{\tilde{R}_i}{b_i} = &- \left(\frac{a_i \tilde{R}_i}{b_i^2} + \frac{a_i u_i}{b_i} + \bar{u}_i \right) + \frac{a_i}{b_i} \sum_{q=1}^{n} \left(\Delta t_q + u_q^k + \frac{\bar{R}_q}{b_q} \right) \\ &+ \frac{a_i}{b_i} \sum_{q=1}^{n} \frac{a_q}{b_q} \sum_{l=1}^{q-1} \left(\Delta t_l + u_l^k + \frac{\bar{R}_l}{b_l} \right) + \cdots . \end{aligned}$$

Then

$$\begin{aligned} \tilde{R}_i = &- \frac{b_i(a_i u_i + b_i \bar{u}_i)}{a_i + b_i} + \frac{a_i}{a_i + b_i} \sum_{q=1}^{n} \left(\Delta t_q + u_q^k + \frac{\bar{R}_q}{b_q} \right) \\ &+ \frac{a_i}{a_i + b_i} \sum_{q=1}^{n} \frac{a_q}{b_q} \sum_{l=1}^{q-1} \left(\Delta t_l + u_l^k + \frac{\bar{R}_l}{b_l} \right) + \cdots . \end{aligned}$$

In addition to the uncertainty states and their peak values, the duration of the time steps also converges.

Proposition 4: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$. If $b_i > (n-1)a_i, \ \forall i$, then the period length $T_{k+1} - T_k$ converges to a fixed amount of time which is independent of the initial states.

Proof: The period is given by

$$T_{k+1} - T_k = \sum_{i=1}^{n} \left(\frac{\tilde{R}_i}{b_i} + u_i + \Delta t_i \right). \quad (16)$$

Since u_i and Δt_i are constant values and $\frac{\tilde{R}_i}{b_i}$ converges $\forall i$, $T_{k+1} - T_k$ converges to a constant value. ∎

The final proposition in this section establishes that the average value of the uncertainty also converges. While this is perhaps clear from the previous results, the proof establishes the relationship between this average value and several other parameters.

Proposition 5: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$. If $b_i > (n-1)a_i, \ \forall i$, then the average value $\langle R_i \rangle = \frac{1}{T_{k+1} - T_k} \int_{T_k}^{T_{k+1}} r_i^k(t)dt$ converges to a fixed value that is independent of the initial states.

Proof: From (4), we have

$$\int_{T_k}^{T_{k+1}} r_i^k(t)dt = \frac{\tilde{R}_i^2(k)}{2a_i b_i} - (T_{k+1} - T_k)\, b_i \bar{u}_i. \quad (17)$$

The proposition then follows from Props. 3 and 4. ∎

IV. Optimized Switching Conditions

In this section we search for the optimal switching policy U^* for the given sequence. There are several natural choices of performance, including the uncertainty values at the start of each period, the peak value during the period, the average value over the period, and the length of the period. Since our primary concern is with the infinite time horizon, we consider the steady state values of these terms. The previous section established that each of these converges to a steady value if the control is constant.

A. When $U \equiv 0$ is optimal.

The next proposition shows that for all but one of these performance measures, the control $U = 0$ is optimal.

Proposition 6: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$. If $b_i > (n-1)a_i, \ \forall i$, then the sum of the steady state uncertainties, $\sum_{i=1}^{n} \bar{R}_i$, the sum of the peak steady state uncertainties, $\sum_{i=1}^{n} \tilde{R}_i$, and the period length $T_{k+1} - T_k$ are all minimized when $u_i \equiv 0$.

Proof: From the proofs of Props 1, 3 and 4, the values of $\bar{R}_i$ and $\tilde{R}_i$ are monotonic functions of $|u_i|$. Therefore all these values reach their lowest point when $U(k) \equiv [0]^{n \times 1}$. Thus these are all minimized using the zero control.

Now consider the period length $T_{k+1} - T_k$ given in (16). Its value depends on u_i both explicitly and through $\tilde{R}_i$. Clearly, for positive u_i, the period length increases. For negative u_i, the contribution from the explicit term decreases but the contribution from $\tilde{R}_i$ increases by the same amount. Thus the period is minimized when $u_i \leq 0$ for all i. In particular it is minimized with the zero control. ∎

Thus the optimal choice of the switching condition is for the agent to leave its current target as soon as the uncertainty reaches zero. This result relies only on the fact that the uncertainty converges, as established in Sec. III. In particular, it does not depend on the distance between the targets, their layout, the sequence of visiting them, or even their particular dynamics (outside of the convergence criterion). In the context of the larger problem of finding the best sequence, this result establishes that evaluating the cost of a given sequence is the calculation of a straightforward analytical expression and no longer requires the solution of a nonlinear optimization problem.

B. Optimal switching condition for $\langle R_i \rangle$.

The switching condition that optimizes the average value of the uncertainty is not as simple to determine. In general, the optimizing choice depends on the details of the targets and there is not a single result. However, the next proposition establishes that under the special condition of homogenous targets, the zero control is once again optimal.

Proposition 7: Consider the discrete system (10) with a constant input $U(k) \equiv [u_1, u_2, ..., u_n]$ and assume that $a_i = a,\ b_i = b\ \forall i$. If $b > (n-1)a$, then the sum of the average values over the steady state condition, $\sum_{i=1}^{n} \langle R_i \rangle$, is minimized when $u_i \equiv 0$.

Proof: From Prop. 3 we have that

$$\tilde{R}_i = a_i \left(\sum_{q=1}^{n} (\Delta t_q + u_q + \frac{\tilde{R}_q}{b_q}) - u_i - \frac{\tilde{R}_i}{b_i} \right) - b_i \bar{u}_i.$$

Rearranging we get

$$\begin{aligned} \sum_{i=1}^{n} \frac{\tilde{R}_i}{b_i} = & \sum_{i=1}^{n} \frac{a_i}{b_i} \cdot \sum_{i=1}^{n} (\Delta t_i + u_i + \frac{\tilde{R}_i}{b_i}) \\ & - \sum_{i=1}^{n} \frac{a_i u_i}{b_i} - \sum_{i=1}^{n} \frac{a_i \tilde{R}_i}{b_i^2} - \sum_{i=1}^{n} \bar{u}_i. \end{aligned} \tag{18}$$

Since $a_i = a,\ b_i = b$, (18) can be simplified to

$$\sum_{i=1}^{n} \frac{\tilde{R}_i}{b_i} = \frac{na \sum_{i=1}^{n} \Delta t_i + (n-1)a \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} \bar{u}_i}{b - (n-1)a}.$$

Therefore,

$$\begin{aligned} \tilde{R}_i = & a \left(\sum_{q=1}^{n} (\Delta t_q + u_q) - u_i - \frac{\tilde{R}_i}{b} \right) - b \bar{u}_i \\ & + \frac{a^2 \sum_{q=1}^{n} (n \Delta t_q + (n-1) u_q) - ab \sum_{q=1}^{n} \bar{u}_q}{b - (n-1)a}. \end{aligned}$$

From this we get

$$\begin{aligned} \frac{\tilde{R}_i}{b} = & \frac{a}{a+b-na} \sum_{q=1}^{n} \left(\Delta t_q + \frac{b}{(a+b)} \underline{u}_q \right) \\ & - \frac{a u_i + b \bar{u}_i}{a+b}. \end{aligned} \tag{19}$$

We denote the peak uncertainty when the inputs are set to zero for all targets as $\tilde{R}_i^0$. According to (19) we have

$$\frac{\tilde{R}_i^0}{b} = \frac{a}{a+b-na} \sum_{q=1}^{n} \Delta t_q, \tag{20}$$

and according to (17) the average uncertainty over time subject to this set of inputs is

$$\langle R_i^0 \rangle = \frac{1}{2} \tilde{R}_i^0 = \frac{ab}{2(a+b-na)} \sum_{q=1}^{n} \Delta t_q. \tag{21}$$

Now consider an input sequence that is not identically zero. We start with analyzing one target among the group. For this selected target i, if we have $u_i = 0$, then regardless of the other inputs, we will always have

$$\langle R_i \rangle = \frac{1}{2} \tilde{R}_i = \frac{ab}{2(a+b-na)} \sum_{q=1}^{n} \left(\Delta t_q + \frac{b}{a+b} \underline{u}_q \right)$$

so that

$$\langle R_i \rangle - \langle R_i^0 \rangle = \frac{ab^2}{2(a+b)(a+b-na)} \sum_{q=1}^{n} \underline{u}_q \geq 0. \tag{22}$$

Similarly, if $u_i < 0$, we have $\bar{u}_i = u_i$ and

$$\begin{aligned} \langle R_i \rangle = & \frac{1}{2} \tilde{R}_i - b \bar{u}_i \\ = & \frac{ab}{2(a+b-na)} \sum_{q=1}^{n} \left(\Delta t_q + \frac{b}{(a+b)} \underline{u}_q \right) - \frac{3 b u_i}{2} \\ \geq & \langle R_i^0 \rangle + \frac{ab^2}{2(a+b)(a+b-na)} \sum_{q=1}^{n} \underline{u}_q \geq \langle R_i^0 \rangle \end{aligned} \tag{23}$$

regardless of the other inputs. The above results indicate that, for any target with a non-positive switching condition, no matter what value the other inputs shall be, the agent's performance with this specific target is no better than with the all-zero inputs. Furthermore, if a set of inputs contains no positive entries, we will have $\langle R_i \rangle \geq \langle R_i^0 \rangle$ for every target, and thus $\sum_{i=1}^{n} \langle R_i \rangle \geq \sum_{i=1}^{n} \langle R_i^0 \rangle$.

When an input set contains at least one positive entry, the situation becomes more complicated. We first consider a set of inputs with exactly one positive input, that is $u_i > 0$. For this specific target i we have

$$\begin{aligned} \langle R_i \rangle - \langle R_i^0 \rangle = & \frac{(a+b) \tilde{R}_i^2}{2(a+b) \tilde{R}_i + 2abu_i} - \frac{1}{2} \tilde{R}_i^0 \\ = & \frac{(a+b) \tilde{R}_i (\tilde{R}_i - \tilde{R}_i^0) - abu_i \tilde{R}_i^0}{2(a+b) \tilde{R}_i + 2abu_i}. \end{aligned} \tag{24}$$

Consider now a different target, l. Since by assumption the i^{th} target is the only with a positive value of the switching condition, we have $u_l \leq 0$. Then

$$\langle R_l \rangle = \langle R_l^0 \rangle + \frac{ab^2}{2(a+b)(a+b-na)} \sum_{q=1}^{n} \underline{u}_q.$$

Under this scenario, $\sum_{q=1}^{n} \underline{u}_q = u_i$. Then

$$\begin{aligned} & \left(\langle R_i \rangle - \langle R_i^0 \rangle \right) + \left(\langle R_l \rangle - \langle R_l^0 \rangle \right) > 0 \\ \Leftrightarrow & \ ab^2 u_i \left((a+b) \sum_{q=1}^{n} \Delta t_q + b u_i \right) \\ & + ab \left(b u_i - (a+b-na) u_i \right)^2 \\ & + ab(a+b) \sum_{q=1}^{n} \Delta t_q \left(b u_i - 2(a+b-na) u_i \right) > 0 \\ \Leftrightarrow & \ b^2 u_i^2 + \left(b u_i - (a+b-na) u_i \right)^2 + (n-1) a u_i > 0 \end{aligned}$$

and this last holds for $n > 1$. As stated above, (22) and (23) guarantee that $\langle R_q \rangle \geq \langle R_q^0 \rangle$ for all other targets, $q \neq i, l$.

Thus when $u_i > 0$ and $u_q \leq 0$ for all $q \neq i$ we have

$$\sum_{q=1}^{n} \langle R_q \rangle > \sum_{q=1}^{n} \langle R_q^0 \rangle, \text{ s.t. } u_i > 0, \text{ and } u_q \leq 0, \; \forall q \neq i.$$

Finally we consider an input set with $z \geq 2$ positive entries. Without loss of generality we let $u_1, ..., u_z > 0$ and $u_{z+1}, ..., u_n \leq 0$. According to (24) we have

$$\sum_{i=1}^{z} \left(\langle R_i \rangle - \langle R_i^0 \rangle \right) \geq 0 \qquad \Leftrightarrow$$
$$(a+b) \sum_{q=1}^{n} \Delta t_q \sum_{i=1}^{z} \left(b \sum_{q=1}^{n} \underline{u}_q - 2(a+b-na)u_i \right) \geq 0. \tag{25}$$

Since

$$\sum_{q=1}^{n} \underline{u}_q = \sum_{i=1}^{z} u_i,$$

(25) turns into

$$(a+b) \sum_{q=1}^{n} \Delta t_q \left(zb \sum_{i=1}^{z} u_i - 2(a+b-na) \sum_{i=1}^{z} u_i \right) \geq 0$$

which holds for $z \geq 2$. The remaining targets all have non-positive switching conditions so that their average uncertainty exceeds the value under an all-zero input set. Now we have shown that Prop. 7 holds for a set of inputs that contains multiple positive entries. Together with the conclusions above we have established Prop. 7.

■

Notice that under the zero policy, (21) leads to

$$\sum_{i=1}^{n} \langle R_i \rangle^* = \frac{nab}{2(a+b-na)} \sum_{q=1}^{n} \Delta t_q. \tag{26}$$

This in turn implies that for a single agent assigned to a set of homogeneous targets, the average uncertainty is minimized with the total traveling time, $\sum_{q=1}^{n} \Delta t_q$. The problem is then reduced exactly to a TSP. It is important to note that this does not hold when the targets are heterogeneous nor does it imply anything about the multi-agent problem. In fact, (26) suggests that the number of targets assigned to an agent affects the cost of the sequences and therefore the weight of the edges in the graph formulation. However, in the multiagent case with homogeneous targets, the result in (26) can be used to determine the optimal partitioning of the targets to the agents.

Finally, we note that the constraint of homogeneous targets is sufficient but not necessary. Following the proof of Prop.7 it can be shown that the same result holds if the homogeneity constraint is replaced by $\frac{b_i}{a_i} = k > n-1, \; \forall i$. However, the optimality of the zero switching condition may not hold if the targets are strongly heterogeneous. The optimal switching conditions can still be determined on a case by case basis through numerical solution of the optimization problem.

V. CONCLUSION

In this paper we considered the problem of determining the optimal dwell time of an agent moving between multiple targets while seeking to minimize some function of the targets' uncertainty states. The problem was abstracted into a discrete system and we showed the existence of stable solutions to the dynamics of the uncertainty under certain conditions. We further showed that the simple choice of staying with a target until its uncertainty reaches zero and then switching to another is optimal for minimizing the steady state uncertainty, peak uncertainty, and period length. In addition, when the targets have homogeneous dynamics in their uncertainty, the zero policy also minimizes the steady state average uncertainty and, under this special condition, the problem is in fact reduced to a TSP.

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