On The Calibration of Competitive Industry Dynamics Models*

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Abstract
We show that, in the context of the neoclassical model of investment with mean-reverting and log-normally distributed productivity shocks, information on the asymptotic distribution of the investment rate does not identify the parameters of the stochastic process. This likely explains why a variety of recent models with firm-level heterogeneity – both in macroeconomics and finance – are able to generate sensible cross-sectional investment rate moments in spite of assuming radically different values for the persistence and volatility of the shocks. Information on investment rates does entail a restriction on the two parameters, which in turn implies that – contingent on the curvature of the production function – not all recent estimates of the parameters will be consistent with empirically plausible cross-sectional investment rate moments.

Key words: Neoclassical Investment Model, Firm Heterogeneity.
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*All remaining errors are our own responsibility.
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1 Introduction

Over the last fifteen years or so, the fields of macroeconomics and finance have witnessed a surge in the number of theoretical studies that explicitly model firm– and plant–level heterogeneity.\(^1\) While entertaining different research questions, many of such papers share two features: (i) The cross–sectional distribution of investment rates is central to the analysis and (ii) the process of firm–level investment arises from a variant of the neoclassical model of investment with mean–reverting and log-normally distributed multiplicative shocks to the production function.\(^2\)

In spite of these commonalities, those studies often make radically different assumptions about the parameters governing the stochastic process driving the shocks.\(^3\) Given the relevance that the cross–sectional distribution of investment rates has for the conclusions drawn by that research, one has to wonder whether the different parameter choices produce similar implications for key moments of the distribution. Necessary, although not sufficient condition for this to be the case is that the distribution of investment rates falls short of identifying the parameters of the stochastic process.

In Section 2, we show indeed that in the simplest variant of the neoclassical model, one with first–order auto-regressive productivity shocks and no adjustment costs, the asymptotic distribution of the investment rate does not identify the two parameters of the stochastic process. There exists an uncountable set of parameter values that produce exactly the same distribution.

In Section 3, we show by means of numerical simulations that the property just described is robust to a relevant generalization of the model, i.e. one in which convex as well as non–convex capital adjustment costs are added to the picture. Radically different parameter choices produce the same values for the set of investment rate moments emphasized in the studies mentioned above.

However, high levels of persistence are compatible with the evidence on investment rates only if the volatility of innovations is very low. Interestingly, recent production function estimation exercises generated similar estimates for the volatility of the innovation, but very different ones for the autocorrelation coefficient. Our analysis implies that, for given production function curvature, these estimates will necessarily produce different

\(^1\)Since none of these studies differentiates between single– and multi–plant firms, in the remainder of this paper we will use the words plant and firm interchangeably.

\(^2\)With no ambition to be exhaustive, given the very long list of papers that fit our categorization, we recall Gomes (2001), Zhang (2005), Khan and Thomas (2008), Eisfeldt and Muir (2015), Winberry (2015), and Clementi and Palazzo (2016).

\(^3\)We are not the first to point this out. Khan and Thomas (2008) and Lee and Mukoyama (2015), for example, take notice of the disagreement about the persistence of the idiosyncratic shock process.
cross-sectional investment rate moments.

It is obvious that further theoretical restrictions can be brought to bear in order to identify both $\rho$ and $\sigma$. In Section 4, we show that in our scenario the simulated standard deviation of firm-level capital-output ratio varies substantially across parameterizations that generate similar investment rate distributions.

2 A simple model of industry dynamics

Time is discrete and is indexed by $t = 0, 1, 2, \ldots$. The horizon is infinite. A continuum of ex-ante identical firms produce one homogenous good by means of the production function

$$y_t = e^{s_t}k_t^{\alpha_k}l_t^{\alpha_l},$$

where $\alpha_k, \alpha_l \in (0, 1)$, $\alpha_k + \alpha_l < 1$, $k_t \geq 0$ denotes installed capital and $l_t \geq 0$ is labor. With $s_t$ we denote a firm-specific, idiosyncratic random productivity disturbance distributed according to the process

$$s_t = \rho s_{t-1} + \sigma \varepsilon_t,$$

where $\rho \in (0, 1)$, $\sigma > 0$, and $\varepsilon_t \sim N(0, 1)$.

Firms are price takers in both input and output markets and prices are constant. Labor is hired on the spot market at the wage $w$, while the investment undertaken at time $t$ – denoted as $x_t$ – becomes productive in the next period. Capital depreciates at the rate $\delta \in (0, 1)$ and the discount factor is $\beta \in (0, 1)$. At time $t = 0$, given initial conditions $k_0$ and $s_0$, each firm’s problem is to choose the contingent sequences of investment and labor $\{x_t, l_t\}_{t=0}^{\infty}$ that solve the following problem:

$$\max E_0 \left[ \sum_{t=0}^{\infty} \beta^t (y_t - w l_t - x_t) \right]$$

s.t. $k_{t+1} = (1 - \delta) k_t + x_t$,

(1), (2).

2.1 Optimal labor and investment decisions

The first-order conditions, which are both necessary and sufficient for optimality, are

$$\alpha_l e^{s_t}k_t^{\alpha_k}l_t^{\alpha_l-1} = w,$$

$$\alpha_k E_t \left[ e^{s_{t+1}}k_{t+1}^{\alpha_k-1}l_{t+1}^{\alpha_l} \right] = \frac{1}{\beta} - (1 - \delta).$$

(3)
Solving (3) for $l_t$ and substituting in (4) yields

$$k_{t+1} = \left[ Ae^{\frac{\rho s_t}{1-\alpha_l} + \frac{\sigma^2}{2(1-\alpha_l)^2}} \right]^\theta,$$

where $A \equiv \alpha_k \left( \frac{\alpha_l}{w} \right)^{\alpha_l/(1-\alpha_l)}$ and $\theta \equiv (1-\alpha_l)/(1-\alpha_l - \alpha_k)$. It follows that

$$\ln k_{t+1} = -\theta \ln \left[ \frac{1}{\beta} - 1 + \delta \right] + \theta \ln A + \theta \left[ \frac{\rho s_t}{1-\alpha_l} + \frac{\sigma^2}{2(1-\alpha_l)^2} \right].$$

(6)

### 2.2 The stationary cross-sectional distribution of the investment rate

An individual firm’s gross investment rate at time $t$ is defined as

$$x_t = \frac{k_{t+1}}{k_t} - (1 - \delta).$$

(7)

Taking the natural logarithm of (5) and first-differencing yields the following expression for the continuously compounded growth rate of capital at time $t + 1$, conditional on realizations of $s_{t-1}$ and $\varepsilon_t$:

$$\ln \left( \frac{k_{t+1}}{k_t} \right) = \frac{\rho(1-\rho)s_{t-1}}{\alpha_l + \alpha_k - 1} - \frac{\rho \sigma \varepsilon_t}{\alpha_l + \alpha_k - 1}. \quad (8)$$

Since $s_{t-1}$ and $\varepsilon_t$ are independently distributed, the unconditional distribution of $\ln(k_{t+1}/k_t)$ converges to a Normal with mean 0 and variance $\sigma^2_{x/k} = \frac{\rho^2 \sigma^2}{(\alpha_l + \alpha_k - 1)^2} \left( \frac{2}{1+\rho} \right)$. Then, we can express the first two unconditional moments of the investment rate as

$$\mu_{x/k} \equiv E \left( \frac{x_t}{k_t} \right) = e^\frac{\frac{1}{2} \sigma^2_{y/k}}{e^{\frac{1}{2} \sigma^2_{y/k}} - (1 - \delta)}.$$

(9)

$$\sigma^2_{x/k} \equiv Var \left( \frac{x_t}{k_t} \right) = \left( e^{\sigma^2_{y/k}} - 1 \right) e^{\sigma^2_{y/k}}. \quad (10)$$

Notice that any pair $\{\rho, \sigma\}$ that yields the same value of $\sigma^2_{y/k}$ will also yield the same first and second moment of the investment rate.

Equations (9) and (10) also imply that, given parameters $\{\alpha_k, \alpha_l, \rho, \sigma\}$, the set of pairs $\{\mu_{x/k}, \sigma^2_{x/k}\}$ the model is capable of reproducing is given by the locus

$$\sigma^2_{x/k} = \left\{ \left[ \mu_{x/k} + (1 - \delta) \right]^2 - 1 \right\} \left[ \mu_{x/k} + (1 - \delta) \right]^2. \quad (11)$$

Then, we can state the following result.

**Lemma 1** Given values for $\{\alpha_k, \alpha_l, \delta\}$, consider any non-negative vector $\{\mu_{x/k}, \sigma^2_{x/k}\}$ that satisfies (11). Then,
1. the set of pairs \( \{ \rho, \sigma \} \in [0,1] \times \mathbb{R}^+ \), which is consistent with \( \{ \mu_{x/k}, \sigma^2_{x/k} \} \), is uncountable;

2. the distribution of the investment rate is the same for all such pairs \( \{ \rho, \sigma \} \).

A corollary of Lemma 1 is that the unconditional moments of the investment rate do not identify the parameters of the process driving idiosyncratic productivity. Figure 1 effectively illustrates our findings. For \( \delta = 0.1 \), the left panel reproduces the restriction on \( \mu_{x/k} \) and \( \sigma_{x/k} \) implied by (11). The right panel depicts the locus of values for \( \rho \) and \( \sigma \) consistent with \( \mu_{x/k} = 0.147 \) and \( \sigma_{x/k} = 0.326 \) – values close to those reported in the empirical study by Cooper and Haltiwanger (2006).\(^4\)

![Figure 1: Model–implied restrictions.](image)

3 A model with capital adjustment costs

The simple model considered above is handy, as it lends itself to an analytical characterization. Our next goal is to investigate whether the result obtained in that framework extends to a scenario with capital adjustment costs commonly assumed in macroeconomics. To this end, we assume that a gross investment \( x \) when capital in place is \( k \) requires firms to incur a cost

\[
g(x, k) = Ic_0k + c_1 \left( \frac{x}{k} \right)^2 k, \quad c_0, c_1 \geq 0,
\]

where \( I = 0 \) for \( x = 0 \), and \( I = 1 \) otherwise. This is the specification adopted in Clementi and Palazzo (2016) and most other competitive industry models.

\(^4\)The values of \( \alpha_k \) and \( \alpha_I \) are listed in Table 1.
3.1 The firm’s optimization program

Letting

$$\pi(k, s) = \max_l e^{s_l} k^{\alpha_k} l^{\alpha_l} - w_l,$$

denoting the conditional distribution of the firm–level productivity shock as $H(s'|s)$ and firm value as $V(k, s)$, we can cast the firm’s optimization problem in a recursive fashion:

$$V(k, s) = \pi(k, s) + \max_x \left[ -x - g(x, k) + \beta \int_{\mathbb{R}} V(k', s') dH(s'|s) \right],$$

s.t. $k' = k(1 - \delta) + x$.

Since the model is not amenable to an analytical characterization, we resort to numerical methods in order to approximate the policy function for investment and compute the moments of the stationary distribution of the investment rate.\(^5\)

3.2 Calibration

One period corresponds to one year. We set all parameters except those shaping idiosyncratic shocks and adjustment costs to values that are standards in the macroeconomics literature. See Table 1.

<table>
<thead>
<tr>
<th>Description</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capital share</td>
<td>$\alpha_k$</td>
<td>0.24</td>
</tr>
<tr>
<td>Labor share</td>
<td>$\alpha_l$</td>
<td>0.56</td>
</tr>
<tr>
<td>Depreciation rate</td>
<td>$\delta$</td>
<td>0.1</td>
</tr>
<tr>
<td>Discount factor</td>
<td>$\beta$</td>
<td>0.9615</td>
</tr>
</tbody>
</table>

Table 1: Parameter values borrowed from other studies.

The parameters \{$\rho, \sigma, c_0, c_1$\} are chosen in such a way that four moments of the limiting distribution of the investment rate are close to their empirical counterparts. Besides the mean and standard deviation of the investment rate, we also consider its autocorrelation and the inaction rate, defined as the fraction of firms that adjust their capital stock by less than 1%. The targets are the moments computed by Cooper and Haltiwanger (2006) using a balanced panel from the LRD from 1972 to 1988.

Consistent with the results obtained in Section 2, we find that the four moments of the investment rate distribution do not uniquely identify a vector of parameter values. In table 2 we show four sets of parameters that yield the same simulated moments.

\(^5\)The computer codes are available from the authors upon request.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Data</th>
<th>Targets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{x/k}$</td>
<td>0.122</td>
<td>0.146</td>
</tr>
<tr>
<td>$\sigma_{x/k}$</td>
<td>0.337</td>
<td>0.314</td>
</tr>
<tr>
<td>$I_{</td>
<td>x</td>
<td>&lt;0.01}$</td>
</tr>
<tr>
<td>$\rho_{x/k}$</td>
<td>0.058</td>
<td>0.059</td>
</tr>
<tr>
<td>$\sigma_{k/y}$</td>
<td>1.966</td>
<td>1.192</td>
</tr>
</tbody>
</table>

Table 2: Parameter values and simulated moments.

Notice that we are unable to exactly match the targets. Similarly to Section 2, where equation (11) implies a restriction on $\{\mu_{x/k}, \sigma_{x/k}\}$, the model constrains the set of attainable investment rate moments.

Once again, we conclude that moments of the cross-sectional distribution of investment rates do not identify the parameters of the stochastic process.

4 Adding theoretical restrictions

An obvious solution to the lack of identification is to consider further restrictions on the data. As an example, consider that for the simple model considered in Section 2, the capital to output ratio is

$$\frac{k_t}{y_t} = \frac{\alpha_k e^{\frac{-2(1-\alpha_l)^2}{\sigma^2}} e^{-\frac{\sigma}{1-\alpha_l} \varepsilon_t}}{\frac{1}{\sigma} - 1 + \delta}.$$  

Since the ratio does not depend on $\rho$, the moments of its limiting distribution will also be independent of that parameter. It follows that one can use estimates for the unconditional standard deviation of the capital–output ratio, say, to pin down $\sigma$, and then recover $\rho$ from equation (10).

The bottom row in Table 2 shows that even in the more general model of which in Section 3, the standard deviation of the capital–output ratio exhibits substantial variation across parameterizations that produce the same values for the moments of the investment rate.

5 Conclusion

In this paper, we have characterized the mapping between the parameters of the stochastic process for productivity and the cross-sectional distribution of the investment rates...
implied by a model of investment that has become the workhorse of the macroeconomics and finance literature. We find that moments of the distribution do not identify the parameters.

Very different pairs of \((\rho, \sigma)\) imply exactly the same statistics. This likely explains how recent studies can claim success in matching the distribution of investment rates while positing different values for the two parameters.

A second result is that the model does impose a restriction on the two parameters. In particular, we find that high persistence is consistent with the evidence on the investment rates only if the standard deviation of innovations is very small. This conclusion is relevant for the still small but fast–growing literature that obtains values for \(\rho\) and \(\sigma\) from production function estimation exercises.

According to Lee and Mukoyama (2015), available estimates for \(\rho\) at annual frequency are as high as their own of 0.969 and as low as those reported by Abraham and White (2006) and Castro, Clementi, and Lee (2015), both smaller than 0.5. Interestingly, the range of estimates for the volatility of the innovation is much narrower. Abraham and White (2006) find \(\sigma = 0.15\), while Castro, Clementi, and Lee (2015) and Lee and Mukoyama (2015) report 0.2 and 0.28, respectively. Foster, Haltiwanger, and Syverson (2008) find that \(\rho\) can be as high as 0.8, while \(\sigma\) is about 0.2.

The analysis conducted above rules out that, for given curvature of the production function, these estimates produce the same moments for the investment rate.

References


