Representing Preferences with a Unique Subjective State Space: Corrigendum[1]

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1 Introduction

Dekel, Lipman, and Rustichini [2001] (henceforth DLR) claim two results which are false without stronger assumptions than are given in the paper. Specifically, as shown in Section 2 below, Theorem 3.A of DLR requires a stronger version of independence, one which holds not only for strict comparisons, as assumed, but also for indifference. As discussed in Section 3 below, Theorem 4.A of DLR requires an additional axiom which yields Lipschitz continuity. The supplementary appendix contains a complete proof of a correct version of DLR's Theorem 4.A.

DLR consider a preference relation $\succ$ over the set of nonempty subsets of $\Delta(B)$, endowed with the Hausdorff topology, where $\Delta(B)$ is the set of probability distributions over a finite set B. The two representations discussed here each consist of three objects: a (nonempty) state space S, a state–dependent utility function $U : \Delta(B) \times S \to \mathbf{R}$, and an aggregator $u : \mathbf{R}^S \to \mathbf{R}$. Each representation must satisfy two properties. First, the function V defined by

$$V(x) = u\left(\left(\sup_{\beta \in x} U(\beta, s)\right)_{s \in S}\right) \tag{1}$$

is continuous and represents $\succ$. Second, each $U(\cdot, s)$ is an expected–utility function in the sense that for every $\beta \in \Delta(B)$, $U(\beta, s) = \sum_{b \in B} \beta(b) U(b, s)$. DLR interpret $\left(\sup_{\beta \in x} U(\beta, s)\right)_{s \in S}$ as the vector of ex post utilities from x.[1]

2 Existence of an Ordinal EU Representation

An ordinal EU representation adds to the properties discussed above the requirement that u is strictly increasing on $\{(\sup_{\beta \in x} U(\beta, s))_{s \in S} \mid x \subseteq \Delta(B)\}$.

We thank Jacob Sagi for the following example. Let $B = \{a, b, c\}$. Define expected utility preferences U_1 and U_2 by

	U_1	U_2
a	0	0
b	0	1
c	2	2

Define a preference over menus by $V(x) = [\max_{\beta \in x} U_1(\beta)] [\max_{\beta \in x} U_2(\beta)]$. This preference satisfies the axioms DLR state in Theorem 3.A, but the conclusion of that theorem does not hold. Note that $V(\{a\}) = V(\{a, b\}) = 0$ even though the menu $\{a, b\}$ yields strictly higher ex post utility in subjective state U_2. Hence there is no representation of

[1]DLR also require some nonredundancy conditions not relevant to this corrigendum.

this preference which aggregates the ex post utilities with a strictly increasing u.[2] Hence this preference does not have an ordinal EU representation.

The problem is the definition of weak independence, defined by DLR as

Axiom 1 (Weak Independence — Original.) *If $x' \subset x$ and $x \succ x'$, then for all $\lambda \in (0, 1]$ and all $\bar{x}$,*

$$\lambda x + (1-\lambda)\bar{x} \succ \lambda x' + (1-\lambda)\bar{x}.$$

DLR's footnote 31 states that this implies the usual "indifference version" of independence (see statement below). Sagi's example shows that this claim is incorrect. For example, we obtain a contradiction for $x' = \{a\}$, $x = \{a, b\}$, $\bar{x} = \{c\}$, and $\lambda = 1/2$. Because the indifference version is necessary for an ordinal EU representation, DLR's Theorem 3.A holds as stated if we strengthen weak independence to

Axiom 2 (Weak Independence — New.) *If $x' \subset x$, then for all $\lambda \in (0, 1]$ and all $\bar{x}$,*

$$x \succ x' \implies \lambda x + (1-\lambda)\bar{x} \succ \lambda x' + (1-\lambda)\bar{x}$$
$$x \sim x' \implies \lambda x + (1-\lambda)\bar{x} \sim \lambda x' + (1-\lambda)\bar{x}.$$

3 Existence of an Additive EU Representation

An additive EU representation adds to the properties in Section 1 the requirement that there is a finitely additive[3] measure μ on S such that, for all $x \subseteq \Delta(B)$,

$$u((\sup_{\beta \in x} U(\beta, s))_{s \in S}) = \int_S \sup_{\beta \in x} U(\beta, s)\mu(ds).$$

In their existence proof, DLR construct an affine function V on X, the closed and convex subsets of $\Delta(B)$, which represents $\succ$. They define a space S^K and show that X is one–to–one with a certain set of functions, C, mapping S^K to $\mathbf{R}$. The set x is mapped to its *support function*, the function $\sigma_x(s)$ defined by $\sigma_x(s) = \max_{\beta \in x} \beta \cdot s$. Since C is one–to–one with X, we can define $W : C \to \mathbf{R}$ by $W(\sigma_x) = V(x)$. DLR extend W to

[2]The uniqueness of the subjective state space shown by DLR implies that we cannot escape this conclusion by using some other set of possible utility functions to represent this preference.

[3]While DLR require only a finitely additive measure, the supplementary appendix shows that the results do not change if we require a countably additive measure.

a space H^*. We use the following facts about W and H^* below. First, for all $f \in H^*$, there exists $\sigma^1, \sigma^2 \in C$ and $r > 0$ such that $f = r(\sigma^1 - \sigma^2)$. Second, W is linear in the sense that $W(r_1 f - r_2 g) = r_1 W(f) - r_2 W(g)$. Finally, $W(\mathbf{0}) = 0$ where $\mathbf{0}$ denotes the zero function.

DLR's Lemma 12 claims that W is bounded on H^*. However, the proof assumes that W is continuous on H^*, a hypothesis which is not justified. We thank Christopher Chambers for first pointing out this error. In the appendix, we show that the claim is false by giving an example of a preference satisfying the DLR axioms which does not have an additive EU representation.

Boundedness of W on H^* requires W to be Lipschitz continuous on C. Since W on C is essentially equivalent to V, DLR require V to be Lipschitz continuous. Given sets x and y, let $d_h(x, y)$ denote the Hausdorff distance between x and y.

Definition 1 *$V : X \to \mathbf{R}$ is* Lipschitz continuous *if there is a $\bar{N}$ such that*

$$V(y) - V(x) \leq \bar{N} d_h(x, y), \quad \forall x, y.$$

The following axiom yields this property.[4]

Axiom 3 (L–Continuity) *There exists nonempty sets $x^*, x_* \subseteq \Delta(B)$ and a $N > 0$ such that for every $\varepsilon \in (0, 1/N)$, for every x and y with $d_h(x, y) \leq \varepsilon$,*

$$(1 - N\varepsilon)x + N\varepsilon x^* \succeq (1 - N\varepsilon)y + N\varepsilon x_*.$$

For intuition, note that unless $x \sim y$ for all sets x and y, this axiom and independence require $x^* \succ x_*$. Given continuity, independence, and $x^* \succ x_*$, for any x and y with $x \prec y$, there is a largest $\lambda \in (0, 1)$ such that $\lambda x + (1 - \lambda)x^* \succeq \lambda y + (1 - \lambda)x_*$. L–continuity requires this largest λ to converge smoothly to 1 as $d_h(x, y)$ converges to 0.

Even if we assume independence, neither continuity nor L–continuity implies the other. We give an example in the appendix of a preference satisfying independence and continuity but not L–continuity. One can construct a lexicographic version of the additive EU representation to generate a preference which satisfies independence and L–continuity but not continuity.

[4] It is natural to interpret x^* and x_* as the best and worst sets respectively since the existence of such sets follows from our other axioms.

Before showing that L–continuity implies Lipschitz continuity, we comment on a minor separate error in DLR. DLR define S^K, the domain for their support functions, to be $\{(s_1, \ldots, s_K) \in \mathbf{R}^K \mid \sum_i s_i = 0, \ \sum_i |s_i| = 1\}$. DLR's Lemma 8 says that

$$d_h(x, y) = \| \sigma_x - \sigma_y \| \tag{2}$$

where $\| \cdot \|$ denotes sup norm. This is not correct. The error is easily fixed by redefining S^K by replacing $\sum_i |s_i| = 1$ with $\sum_i s_i^2 = 1$ (with some minor and obvious changes in proof details). (For proof of (2) with this correction, see Theorem 1.8.11 in Schneider [1993], page 53.)

Lemma 1 *Assume $\succ$ has an affine representation V. Then V is Lipschitz continuous if and only if $\succ$ satisfies L–continuity.*

Proof. Suppose $\succ$ satisfies L–continuity. Fix the N, x^*, and x_* of the axiom, any $D \in (0, 1/N)$, and any x and y with $d_h(x, y) \leq D$. Let $\delta = d_h(x, y)$. If $\delta = 0$, then x and y closed implies that $x = y$, in which case the conclusion required for V to be Lipschitz continuous obviously holds. So suppose $\delta > 0$. Then L–continuity implies

$$(1 - N\delta)x + N\delta x^* \succeq (1 - N\delta)y + N\delta x_*.$$

Using the affine representation, this implies

$$V(y) - V(x) \leq \frac{N}{1 - N\delta}[V(x^*) - V(x_*)]d_h(x, y).$$

Since $N\delta = Nd_h(x, y) \leq ND < 1$, we have $N/(1 - N\delta) \leq N/(1 - ND) < \infty$. Let $\bar{N} = [N/(1 - ND)][V(x^*) - V(x_*)]$. Then for any x and y with $d_h(x, y) \leq D$, we have

$$V(y) - V(x) \leq \bar{N} d_h(x, y).$$

To complete the proof, we show the same for arbitrary x and y. Fix any x and y and any sequence $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_M < \lambda_{M+1} = 1$ such that $(\lambda_{m+1} - \lambda_m)d_h(x, y) \leq D$. Let $x_m = \lambda_m x + (1 - \lambda_m)y$. Then

$$\begin{aligned} d_h(x_{m+1}, x_m) &= \| \sigma_{x_{m+1}} - \sigma_{x_m} \| \\ &= (\lambda_{m+1} - \lambda_m) \| \sigma_x - \sigma_y \| \\ &= (\lambda_{m+1} - \lambda_m) d_h(x, y). \end{aligned}$$

Hence from the previous part, we see that

$$V(x_{m+1}) - V(x_m) \leq \bar{N}(\lambda_{m+1} - \lambda_m)d_h(x, y).$$

Summing both sides over m from $m = 0$ to $m = M$ gives $V(y) - V(x) \leq \bar{N} d_h(x, y)$, so V is Lipschitz continuous.

For the converse, suppose there is a $\bar{N}$ such that $V(y) - V(x) \leq \bar{N} d_h(x, y)$ for all x and y. If $x \sim y$ for all x and y, then $\succ$ is trivially L–continuous. So suppose there exist sets x^* and x_* with $x^* \succ x_*$. Let $N = \bar{N}/[V(x^*) - V(x_*)]$. So for all x and y, we have

$$V(y) - V(x) \leq N[V(x^*) - V(x_*)] d_h(x, y).$$

So for all x and y with $d_h(x, y) < 1/N$,

$$V(y) - V(x) \leq \frac{N d_h(x, y)}{1 - N d_h(x, y)} [V(x^*) - V(x_*)].$$

So for every $\varepsilon \in [d_h(x, y), 1/N)$,

$$V(y) - V(x) \leq \frac{N\varepsilon}{1 - N\varepsilon} [V(x^*) - V(x_*)].$$

Rearranging by reversing the steps above, we see that $\succ$ is L–continuous. ▌

We obtain the following corrected version of DLR's Theorem 4.A.

Theorem 1 *The ex ante preference $\succ$ has an additive EU representation if and only if it satisfies weak order, continuity,[5] nontriviality, independence, and L–continuity.*

Proof. Necessity of the first four axioms is obvious. For L–continuity, fix an additive EU representation V. Since S^K includes every expected utility preference, there exists $f : S^K \to \mathbf{R}_{++}$ and $g : S^K \to \mathbf{R}$ such that $V(x) = \int_{S^K} [f\sigma_x + g] \, \mu(ds)$. We can write μ as $\mu^+ - \mu^-$ where both of these measures are positive. Let $N = \int_{S^K} f \, \mu^+(ds) + \int_{S^K} f \, \mu^-(ds)$. Note that N is finite.[6] Then

$$\begin{aligned} V(y) - V(x) &\leq |\textstyle\int_{S^K} f(\sigma_y - \sigma_x) \, \mu(ds)| \\ &= |\textstyle\int_{S^K} (\sigma_y - \sigma_x) f \, \mu^+(ds) - \int_{S^K} (\sigma_y - \sigma_x) f \, \mu^-(ds)| \\ &\leq |\textstyle\int_{S^K} (\sigma_y - \sigma_x) f \, \mu^+(ds)| + |\int_{S^K} (\sigma_y - \sigma_x) f \, \mu^-(ds)| \\ &\leq N \parallel \sigma_y - \sigma_x \parallel \\ &= N d_h(x, y). \end{aligned}$$

[5]It is worth noting that we can replace continuity with von Neumann–Morgenstern continuity in this theorem. See the supplemental appendix for details.

[6]Proof: By Lemma 3 of Sarver [2005], there exist $x, y \subseteq \Delta(B)$ such that $\sigma_x(s) = 0$ for all s and $\sigma_y(s) = c > 0$ for all s. Then

$$V(y) - V(x) = c \int_{S^K} f \, \mu(ds) = c \left[\int_{S^K} f \, \mu^+(ds) - \int_{S^K} f \, \mu^-(ds) \right].$$

Since V is real–valued, $V(y) - V(x)$ must be real–valued, so $\int_{S^K} f \, \mu^+(ds)$ and $\int_{S^K} f \, \mu^-(ds)$ are finite.

Thus V is Lipschitz continuous. Since it is affine, $\succ$ satisfies L–continuity by Lemma 1.[7]

For sufficiency, we complete DLR's proof by showing there is a κ such that for all $f \in H^*$, $W(f) \leq \kappa \parallel f \parallel$. For any $f \in H^*$, there exists $\sigma^1, \sigma^2 \in C$ and a number $r > 0$ such that $f = r(\sigma^1 - \sigma^2)$. By linearity of W and L–continuity, there exists N such that

$$W(f) \leq |W(f)| = r|W(\sigma^1) - W(\sigma^2)| \leq Nr \parallel \sigma^1 - \sigma^2 \parallel \quad = N \parallel f \parallel .$$

Setting $\kappa = N$, we have the required bound. ▌

While we require an axiom to ensure Lipschitz continuity in general, we note two cases of interest where such an axiom is not needed. First, when the state space is finite, DLR's continuity axiom is sufficient. The state space is subjective in DLR, so whether it is finite depends on the preference. Dekel, Lipman, and Rustichini [2006] give an axiom which is necessary and sufficient for the state space to be finite and show that L–continuity is not required when this axiom holds.[8]

L–continuity is also not needed when the preference is monotonic.

Theorem 2 $\succ$ *has an additive EU representation with a positive measure* μ *iff it satisfies weak order, continuity, nontriviality, independence, and monotonicity.*

Proof. Necessity is straightforward. For sufficiency, note that monotonicity implies that W is increasing in the pointwise order on C. So consider $f, g \in H^*$ with $f - g \geq \mathbf{0}$. Since H^* is a vector subspace $f - g \in H^*$. Hence there exists $\sigma^1, \sigma^2 \in C$ and $r > 0$ such that $r[\sigma^1 - \sigma^2] = f - g \geq \mathbf{0}$. So $\sigma^1 \geq \sigma^2$. Hence $W(\sigma^1) \geq W(\sigma^2)$, so $W(r[\sigma^1 - \sigma^2]) \geq 0$ implying $W(f - g) \geq 0$ or $W(f) \geq W(g)$. Hence W is increasing on H^*.

For any $f \in H^*$, $f \leq \parallel f \parallel \mathbf{1}$, where $\mathbf{1}$ is the function identically equal to 1, so $W(f) \leq \parallel f \parallel W(\mathbf{1})$. Letting $\kappa = W(\mathbf{1})$, we have the bound needed to complete DLR's proof. ▌

[7]To be precise, this proof shows only that $\succ$ is L–continuous on X in the sense that it holds for any two sets which are closed and convex. To extend this to all menus, first note that weak order, continuity, and independence are necessary conditions. DLR showed that these properties imply that for any $x \subseteq \Delta(B)$, we have $x \sim \mathrm{cl}(x)$ and $x \sim \mathrm{conv}(x)$. (Our supplemental appendix contains a proof that, in fact, continuity is not needed for this conclusion.) It is not hard to show that $d_h(x, y) \geq d_h(\mathrm{conv}(x), \mathrm{conv}(y))$ and that $d_h(x, y) = d_h(\mathrm{cl}(x), \mathrm{cl}(y))$. Using these facts, it is easy to show that, given weak order, continuity, and independence, L–continuity on X implies L–continuity on the whole domain.

[8]Gul and Pesendorfer's [2001] set betweenness axiom implies this new axiom; this explains why their representation does not require a separate assumption of L–continuity.

A Appendix

We give an example of a preference satisfying the axioms DLR state for Theorem 4.A but which violates L–continuity and hence does not have an additive EU representation. We define the preference by constructing a function W on C, the space of support functions. This induces a preference on the closed, convex sets by $x \succ y$ iff $W(\sigma_x) > W(\sigma_y)$ which is sufficient to define a preference over all menus. This preference will satisfy independence iff W is affine, continuity iff W is continuous with respect to the sup norm, and L–continuity iff W is Lipschitz continuous.

Let $\| \cdot \|_E$ denote the Euclidean norm. Assume B has at least three elements. To construct W, first choose an arbitrary $s^* \in S^K$ and a sequence $\{s^n\} \subset S^K$ such that $\| s^n - s^* \|_E = \frac{1}{n^2}$ for $n = 1, 2, \ldots$. Since B has at least three elements, it is easily verified that such a sequence exists. Define $W : C \to \mathbf{R}$ by

$$W(\sigma) = \sum_{n=1}^{\infty} [\sigma(s^n) - \sigma(s^*)].$$

It is not hard to show that for any $\sigma \in C$ and $s, s' \in S^K$,[9]

$$|\sigma(s) - \sigma(s')| \leq \| s - s' \|_E \, .$$

Using this, we see that W is well–defined since for any $\sigma \in C$,

$$W(\sigma) \leq \sum_{n=1}^{\infty} |\sigma(s^n) - \sigma(s^*)| \leq \sum_{n=1}^{\infty} \| s^n - s^* \|_E = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

W is clearly affine in σ, so the preference satisfies independence.

We show that W is continuous on C. It is not hard to show that for any $\sigma_x, \sigma_y \in C$,

$$\begin{array}{rl} |(\sigma_x(s^n) - \sigma_y(s^n)) - (\sigma_x(s^*) - \sigma_y(s^*))| & \leq \min\{2 \| \sigma_x - \sigma_y \|, 2 \| s^n - s^* \|_E\} \\ & = 2 \min\{\| \sigma_x - \sigma_y \|, 1/n^2\} \end{array} .$$

Hence

$$\begin{array}{rcl} |W(\sigma_x) - W(\sigma_y)| & \leq & \sum_{n=1}^{\infty} |(\sigma_x(s^n) - \sigma_y(s^n)) - (\sigma_x(s^*) - \sigma_y(s^*))| \\ & \leq & 2 \sum_{n=1}^{\infty} \min\{\| \sigma_x - \sigma_y \|, 1/n^2\} \\ & = & 2 \sum_{n \leq \| \sigma_x - \sigma_y \|^{-1/2}} \| \sigma_x - \sigma_y \| + \sum_{n > \| \sigma_x - \sigma_y \|^{-1/2}} \frac{2}{n^2} \\ & \leq & 2 \| \sigma_x - \sigma_y \|^{1/2} + \sum_{n > \| \sigma_x - \sigma_y \|^{-1/2}} \frac{2}{n^2} \end{array}$$

Both terms converge to 0 as $\| \sigma_x - \sigma_y \|$ converges to 0. Thus W is continuous.

[9] Proof: Fix $\beta \in x$ with $\sigma_x(s) = \beta \cdot s$. Then $\sigma_x(s) - \sigma_x(s') \leq \beta \cdot s - \beta \cdot s' \leq \| \beta \|_E \cdot \| s - s' \|_E$. Since $\| \beta \|_E \leq 1$, the last term is less than $\| s - s' \|_E$. Reversing the role of s and s' completes the proof.

Finally, we show that W is not Lipschitz continuous. First, we note two useful facts. For any $s, s' \in S^K$, $s \cdot s' \leq 1$. Also, rearranging $\| s^n - s^* \|_E = 1/n^2$ shows that $s^n \cdot s^* = 1 - 1/2n^4$.

Let $x = \{\beta \mid \beta \cdot s^* \leq 0\}$. Let $\beta^* = (1/K, \ldots, 1/K)$ where K is the cardinality of B. $\sum_k s_k = 0$ for all $s \in S^K$ implies $\beta^* \in x$. For any $\varepsilon \in (0, 1/K)$, define $x(\varepsilon) = \text{conv}(x \cup \{\beta^* + \varepsilon s^*\})$. It is not hard to show $\beta^* + \varepsilon s^* \in \Delta(B)$ for all $\varepsilon \in (0, 1/K)$.

It is easy to see that

$$\sigma_{x(\varepsilon)}(s) = \max\{(\beta^* + \varepsilon s^*) \cdot s, \sigma_x(s)\} = \max\{\beta^* \cdot s + \varepsilon s^* \cdot s, \sigma_x(s)\}.$$

For $s = s^*$,

$$\sigma_{x(\varepsilon)}(s^*) = \max\{\beta^* \cdot s^* + \varepsilon s^* \cdot s^*, \sigma_x(s^*)\} = \max\{0 + \varepsilon, 0\} = \varepsilon,$$

where the second equality follows from $\sum_k s_k^* = 0$, $\sum_k (s_k^*)^2 = 1$, and the definition of x. Hence $\sigma_{x(\varepsilon)}(s^*) = \varepsilon = \varepsilon + \sigma_x(s^*)$.

For any $s \in S^K$, $\beta^* \in x$ implies $\beta^* \cdot s \leq \sigma_x(s)$. Also, from the second fact above, $s^* \cdot s \leq 1$. Hence

$$\beta^* \cdot s + \varepsilon s^* \cdot s \leq \sigma_x(s) + \varepsilon,$$

implying $\sigma_x(s) \leq \sigma_{x(\varepsilon)}(s) \leq \sigma_x(s) + \varepsilon$.

We now show that the first inequality holds with equality at $s = s^n$ for all $n \leq (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}$. To see this, for each n, let $\beta_n = \beta^* + \frac{1}{K}(s^n - s^*)$. It is not hard to show that $\beta_n \in \Delta(B)$ for all n. In addition,

$$\beta_n \cdot s^* = \frac{1}{K}(s^n - s^*) \cdot s^* = 0 + \frac{1}{K}(1 - 1/2n^4 - 1) < 0.$$

Hence $\beta_n \in x$ for all n. Finally, note that

$$\begin{array}{rll} \beta_n \cdot s^n \geq (\beta^* + \varepsilon s^*) \cdot s^n & \text{iff} & \frac{1}{K}(s^n \cdot s^n - s^* \cdot s^n) \geq \varepsilon s^* \cdot s^n \\ & \text{iff} & \frac{1}{K}\left(\frac{1}{2n^4}\right) \geq \varepsilon \left(1 - \frac{1}{2n^4}\right) \end{array}$$

which holds iff $n \leq (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}$. Hence when this inequality holds, $\sigma_x(s^n) = \sigma_{x(\varepsilon)}(s^n)$.

Then, we have

$$\begin{array}{rcl} W(\sigma_x) - W(\sigma_{x(\varepsilon)}) & = & \sum_{n=1}^{\infty} \left[(\sigma_x(s^n) - \sigma_{x(\varepsilon)}(s^n)) - (\sigma_x(s^*) - \sigma_{x(\varepsilon)}(s^*))\right] \\ & = & \sum_{n=1}^{\infty} \left[(\sigma_x(s^n) - \sigma_{x(\varepsilon)}(s^n)) - (-\varepsilon)\right] \\ & = & \sum_{n \leq (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}} [0 + \varepsilon] + \sum_{n > (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}} \left[(\sigma_x(s^n) - \sigma_{x(\varepsilon)}(s^n)) + \varepsilon\right] \\ & \geq & \left[\left(\frac{1}{2} + \frac{1}{2K\varepsilon}\right)^{1/4} - 1\right] \varepsilon + \sum_{n > (\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4}} [(-\varepsilon) + \varepsilon] \\ & = & \left[\left(\frac{1}{2} + \frac{1}{2K\varepsilon}\right)^{1/4} - 1\right] \| \sigma_x - \sigma_{x(\varepsilon)} \| \end{array}$$

Since $(\frac{1}{2} + \frac{1}{2K\varepsilon})^{1/4} \to \infty$ as $\varepsilon \to 0$, W cannot be Lipschitz continuous on C.

References

[1] Dekel, E., B. Lipman, and A. Rustichini, "Representing Preferences with a Unique Subjective State Space," *Econometrica*, **69**, July 2001, 891–934.

[2] Dekel, E., B. Lipman, and A. Rustichini, "Temptation–Driven Preferences," working paper, April 2006.

[3] Gul, F., and W. Pesendorfer, "Temptation and Self–Control," *Econometrica*, **69**, November 2001, 1403–1435.

[4] Sarver, T., "Anticipating Regret: Why Fewer Options May Be Better," Boston University working paper, November 2005.

[5] Schneider, R., *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge University Press, 1993.