Answers to Problem Set 5

Economics 703

Spring 2015

1. (a) Fix a bidder i with valuation \( \theta_i \) and assume all other bidders use a strategy of the form \( \sigma_j(\theta_j) = a\theta_j \). Then i’s expected payoff to a bid of \( b \) is

\[
(\theta_i - b)^\alpha \left( \frac{b}{a} \right)^{I-1}.
\]

The first–order condition is

\[
-\alpha(\theta_i - b)^{\alpha - 1} \left( \frac{b}{a} \right)^{I-1} + (I - 1)(\theta_i - b)^\alpha \left( \frac{b^{I-2}}{a^{I-1}} \right) = 0.
\]

Rearranging, we get

\[
b = \frac{I - 1}{I - 1 + \alpha} \theta_i.
\]

So there is an equilibrium where everyone bids \( a\theta_i \) for \( a = (I - 1)/(I - 1 + \alpha) \).

(b) It’s still a dominant strategy to bid \( \theta_i \). The dominant strategy argument has nothing to do with risk attitudes since it says you get a higher payoff for every possible bids by the other players.

(c) Since the equilibrium for the second price auction hasn’t changed, the expected revenue is still \( (I - 1)/(I + 1) \). For the first price auction, the expected revenue is \( (I - 1)/(I - 1 + \alpha) \) times the expected value of the first order statistic. That expected value is

\[
\int_0^1 \theta I \theta^{I-1} d\theta = \frac{I}{I + 1}.
\]

So the expected revenue in the first price auction is higher if

\[
\frac{I(I - 1)}{(I + 1)(I - 1 + \alpha)} > \frac{I - 1}{I + 1}
\]

or \( \alpha < 1 \) which holds by assumption. Expected revenue in the second price auction is independent of \( \alpha \), while expected revenue in the first price auction is decreasing in \( \alpha \).
2. Whoever wins the first auction certainly loses the second. So if A wins the first auction, B will win the second one and pay A’s valuation for a second unit, namely 10. If B wins the first auction, A wins the second and pays 5.

Given this, if A gets the first object, his payoff for the two auctions together is \( v_A^1 \) minus his payment. If he loses the first auction, his payoff is \( v_A^1 - 5 \). Hence the value to him of winning the first auction is 5 so this is what he bids. Similarly, B bids 10. Hence B wins the first auction paying 5 (the second highest bid), so A wins the second one, also paying 5. The seller’s revenue then is 10, regardless of the values of \( v_A^1 \) and \( v_B^1 \).

3. a) Suppose the seller knows the buyer is following the strategy of offering \( b_2(\theta_2) = \alpha_2 + \beta_2 \theta_2 \). If the seller offers \( b_1 \), then he earns zero if \( b_1 < \alpha_2 + \beta_2 \theta_2 \) and earns

\[
\frac{b_1 + \alpha_2 + \beta_2 \theta_2}{2} - \theta_1
\]

if \( \alpha_2 + \beta_2 \theta_2 \geq b_1 \). Hence, assuming \( \beta_2 > 0 \) (which we’ll see it will be), the seller’s expected payoff is

\[
\int_{(b_1 - \alpha_2)/\beta_2}^1 \left[ \frac{b_1 + \alpha_2 + \beta_2 \theta_2}{2} - \theta_1 \right] d\theta_2
\]

if \( (b_1 - \alpha_2)/\beta_2 \in [0, 1] \). For the moment, assume \( b_1 \) is in this range. Hence we can do the integration to rewrite the seller’s expected payoff as

\[
\left\{ \frac{1}{2} [b_1 + \alpha_2] - \theta_1 \right\} \left[ 1 - \frac{b_1 - \alpha_2}{\beta_2} \right] + \frac{\beta_2}{4} \left[ 1 - \left( \frac{b_1 - \alpha_2}{\beta_2} \right)^2 \right].
\]

Note that this is just a quadratic in \( b_1 \). I’ll omit the algebra, but if you maximize this with respect to \( b_1 \), you get

\[
b_1 = \frac{2}{3} \theta_1 + \frac{1}{3} (\alpha_2 + \beta_2).
\]

This only applies when this gives us a \( b_1 \) with \( (b_1 - \alpha_2)/\beta_2 \in [0, 1] \) — that is, \( b_1 \geq \alpha_2 \) and \( b_1 \leq \alpha_2 + \beta_2 \). As we’ll see momentarily, the first of these inequalities will always be satisfied. The second is trickier though. Intuitively, if the second one is violated, so \( b_1 > \alpha_2 + \beta_2 \), this says that the seller is offering a price that is above every offer the buyer will make and so is certain to not be accepted. Note that the seller’s offer is a convex combination of \( \theta_1 \) and \( \alpha_2 + \beta_2 \). So if his bid ends up above \( \alpha_2 + \beta_2 \), it must be true that his bid is below \( \theta_1 \). That is, he is offering to trade at a price below his costs. Hence his optimal bid in this case is anything that isn’t accepted and \( \alpha_2 + \beta_2 \) is good enough.

So now consider the buyer. He knows that the seller’s bid will take the form \( b_1 = \alpha_1 + \beta_1 \theta_1 \). Again assuming \( \beta_1 > 0 \), this means that his expected payoff can be written as

\[
\int_0^{(b_2 - \alpha_1)/\beta_1} \left[ \theta_2 - \frac{1}{2} (b_2 + \alpha_1 + \beta_1 \theta_1) \right] d\theta_1
\]

assuming \((b_2 - \alpha_1)/\beta_1 \in [0, 1]\). Performing the integration, we get

\[
[\theta_2 - \frac{1}{2}\alpha_1 - \frac{1}{2}b_2] - \frac{\beta_1}{\beta_1} - \frac{\beta_1}{4} \left( \frac{b_2 - \alpha_1}{\beta_1} \right)^2.
\]

Again, this is a quadratic in 2’s strategy. Maximizing with respect to \(b_2\) yields

\[
b_2 = \frac{2}{3}\theta_2 + \frac{1}{3}\alpha_1.
\]

Analogously to the above, this only applies if \(b_2\) is in the right range. For the same reasons as above, though, we don’t need to worry about these conditions.

Summarizing, we get \(b_1 = \beta_1\theta_1 + \alpha_1\) where \(\beta_1 = 2/3\) and \(\alpha_1 = (1/3)(\alpha_2 + \beta_2)\). Also, \(b_2 = \beta_2\theta_2 + \alpha_2\) where \(\beta_2 = 2/3\) and \(\alpha_2 = (1/3)\alpha_1\). Solving these equations simultaneously, we see that \(\beta_1 = \beta_2 = 2/3\), \(\alpha_1 = 1/4\), and \(\alpha_2 = 1/12\).

To see what social choice function this implements, note that it implements \(q = y = 0\) if \(b_1 > b_2\) or

\[
\frac{2}{3}\theta_1 + \frac{1}{4} > \frac{2}{3}\theta_2 + \frac{1}{12}.
\]

That is, we implement \(q = y = 0\) if

\[
\theta_2 - \theta_1 < \frac{1}{4}.
\]

If \(\theta_2 - \theta_1 > \frac{1}{4}\), we implement \(q = 1\) and

\[
y = \frac{b_1 + b_2}{2} = \frac{1}{3}(\theta_1 + \theta_2) + \frac{1}{6}.
\]

This social choice function is not ex post efficient, since ex post efficiency would call for \(q = 1\) whenever \(\theta_2 > \theta_1\), not just when \(\theta_2 > \theta_1 + (1/4)\).

b) Take the direct mechanism where seller and buyer both report their \(\theta\)'s. Let \(\hat{\theta}_i\) denote the reported value of \(\theta_i\) (since this might differ from the true value). The seller’s payoff as a function of his true \(\theta_i\) and his report is

\[
\int_{\hat{\theta}_i + (1/4)}^{1} \left[ \frac{1}{3} \left( \hat{\theta}_1 + \theta_2 \right) + \frac{1}{6} - \theta_1 \right] d\theta_2.
\]

Ignoring (as before) the implicit constraint that this only applies when \(\hat{\theta}_i + (1/4) \in [0, 1]\), if we perform the integration, we get

\[
\left[ \frac{1}{3} \hat{\theta}_1 + \frac{1}{6} - \theta_1 \right] \left[ \frac{3}{4} - \hat{\theta}_1 \right] + \left( \frac{1}{6} \right) \left[ 1 - \left( \hat{\theta}_1 + \frac{1}{4} \right)^2 \right].
\]
This is a quadratic in $\hat{\theta}_1$. The first order condition for maximizing this with respect to $\hat{\theta}_1$ is

$$\frac{1}{4} - \frac{1}{6} + \theta_1 - \frac{2}{3} \hat{\theta}_1 - \frac{1}{3} \left( \hat{\theta}_1 + \frac{1}{4} \right) = 0$$

which is easily rearranged to $\hat{\theta}_1 = \theta_1$.

Similarly, the buyer’s payoff as a function of his true $\theta_2$ and his report is

$$\int_0^{\theta_2^{-1/4}} \left[ \theta_2 - \frac{1}{3} (\theta_1 + \hat{\theta}_2) - \frac{1}{6} \right] d\theta_1.$$ Performing the integration gives

$$\left[ \theta_2 - \frac{1}{3} \hat{\theta}_2 - \frac{1}{6} \right] \left[ \hat{\theta}_2 - \frac{1}{4} \right] - \frac{1}{6} \left( \hat{\theta}_2 - \frac{1}{4} \right)^2.$$ Again, the first order condition for the optimal choice of $\hat{\theta}_2$ reduces to $\hat{\theta}_2 = \theta_2$.

4. (a) For concreteness, consider player 1. (Of course, the game is symmetric, so we would get the same analysis if we focus on player 2.) Fix a particular value of $\theta_1$ and a particular bid $b$. Given that player 2 is bidding $\alpha \theta_2$, 1 knows that he wins with the bid $b$ iff $b \geq \alpha \theta_2$ (where we may as well use a weak inequality here since the probability of a tie is zero) or $\theta_2 \leq b/\alpha$. Thus 1’s expected payoff to bidding $b$ is

$$\int_0^{b/\alpha} \left[ 2 \theta_1 + \theta_2 - b \right] d\theta_2 = \frac{b}{\alpha} (2\theta_1 - b) + \frac{1}{2} \left( \frac{b}{\alpha} \right)^2.$$ (This assumes $0 \leq b/\alpha \leq 1$, but it’s not hard to show that the optimal $b$ will satisfy this.) So 1 should choose $b$ to maximize this. The first-order condition is

$$\frac{2\theta_1}{\alpha} - \frac{2b}{\alpha} + \frac{b}{\alpha^2} = 0.$$ Rearranging:

$$2\alpha \theta_1 = b(2\alpha - 1)$$
or

$$b = \frac{2\alpha}{2\alpha - 1} \theta_1.$$ So the optimal strategy for 1 is to bid $\alpha \theta_1$ if

$$\alpha = \frac{2\alpha}{2\alpha - 1}$$
or $2\alpha = 3$ or $\alpha = 3/2$. 

4
So the seller’s expected revenue is

\[
\frac{3}{2} E [\max\{\theta_1, \theta_2\}] = \frac{3}{2} \int_0^1 2k^2 \, dk = 1.
\]

(b) Again, consider player 1. Now his payoff as a function of \( \theta_1 \) and \( b \) is

\[
\int_0^{b/\alpha} [2\theta_1 + \theta_2 - \alpha \theta_2] \, d\theta_2 = \frac{b}{\alpha} 2\theta_1 + \frac{1 - \alpha}{2} \left( \frac{b}{\alpha} \right)^2.
\]

The first–order condition is

\[
\frac{2}{\alpha} \theta_1 + \frac{1 - \alpha}{\alpha^2} b = 0
\]

so

\[
b = \frac{2\alpha}{\alpha - 1} \theta_1.
\]

So now we have

\[
\alpha = \frac{2\alpha}{\alpha - 1}
\]

so \( 2 = \alpha - 1 \) or \( \alpha = 3 \).

So the seller’s expected revenue is

\[
3 E [\min\{\theta_1, \theta_2\}] = 3 \int_0^1 2[1 - k]k \, dk
\]

\[
= 6 \left[ \int_0^1 k \, dk - \int_0^1 k^2 \, dk \right]
\]

\[
= 6 \left[ \frac{1}{2} - \frac{1}{3} \right] = 3 - 2 = 1.
\]