Answers to Problem Set 4

Economics 703

Spring 2015

1. The objective function is now

\[ \lambda_H [\pi_H(e_H) - w_H] + \lambda_M [\pi_M(e_M) - w_M] + \lambda_L [\pi_L(e_L) - w_L]. \]

I won’t write out all the constraints. There are three individual rationality constraints and six incentive compatibility constraints (each of three types has to prefer the truth to each of two possible lies).

Essentially the same argument as the one in class shows that the individual rationality constraints for the middle and high types can’t bind. We see that incentive compatibility implies

\[ w_i - g(e_i, \theta_i) \geq w_L - g(e_L, \theta_i), \quad i = M, H \]

so

\[ w_i - g_i(e_i, \theta_i) \geq w_L - g(e_L, \theta_L) + g(e_L, \theta_L) - g(e_L, \theta_i). \]

Since \( g(e_L, \theta_L) \geq g(e_L, \theta_i) \), this implies

\[ w_i - g_i(e_i, \theta_i) \geq w_L - g(e_L, \theta_L) \geq v^{-1}(\bar{u}), \]

where the last inequality comes from individual rationality for the low type. Hence \( w_i - g_i(e_i, \theta_i) \geq v^{-1}(\bar{u}) \) for \( i = M, H \). Therefore, we can ignore the IR constraints for the middle and high type since they will hold automatically.

This implies that individual rationality for the low type must bind. Otherwise, we could lower all three wages by \( \varepsilon \), not affecting incentive compatibility, and improve the principal’s payoff. So \( w_L = g(e_L, \theta_L) + v^{-1}(\bar{u}) \).

A similar argument shows that the constraint that \( \theta_H \) not take the contract for \( \theta_L \) cannot bind. To see this, note that the constraint that \( \theta_H \) not take the \( \theta_M \) contract says

\[ w_H - g(e_H, \theta_H) \geq w_M - g(e_M, \theta_H) = w_M - g(e_M, \theta_M) + g(e_M, \theta_M) - g(e_M, \theta_H). \]
Using the constraint that $\theta_M$ not take the contract for $\theta_L$, we get

$$w_M - g(e_M, \theta_M) + g(e_M, \theta_M) - g(e_M, \theta_H) \geq w_L - g(e_L, \theta_M) + g(e_M, \theta_M) - g(e_M, \theta_H)$$

$$= w_L - g(e_L, \theta_H) + [g(e_M, \theta_M) - g(e_L, \theta_M)] - [g(e_M, \theta_H) - g(e_L, \theta_H)].$$

The first term in brackets on the right is the marginal cost to the $\theta_M$ type of increasing effort from $e_L$ to $e_M$. The term in brackets we’re subtracting from that is the marginal cost to the $\theta_H$ type of the same increase in effort. Since the marginal cost is decreasing in type, this difference must be positive. Hence the last term is at least $w_L - g(e_L, \theta_H)$. Hence

$$w_H - g(e_H, \theta_H) \geq w_L - g(e_L, \theta_H).$$

So we can ignore the constraint that $\theta_H$ not take the contract for $\theta_L$ and be sure this will hold anyway.

This tells us that the constraint that $\theta_H$ not take the $\theta_M$ contract must bind. This is now the only constraint left which says $w_H$ must be bigger than something, so if it holds with a strict inequality, we can lower $w_H$, improve the principal’s payoff, and not mess up any of the other constraints. Hence

$$w_H = g(e_H, \theta_H) + w_M - g(e_L, \theta_H).$$

Next, consider the constraint

$$w_M - g(e_M, \theta_M) \geq w_H - g(e_H, \theta_M).$$

Substituting from the above for $w_H$, we see that this holds iff

$$w_M - g(e_M, \theta_M) \geq g(e_H, \theta_H) + w_M - g(e_M, \theta_H) - g(e_H, \theta_M)$$

or

$$g(e_H, \theta_M) - g(e_M, \theta_M) \geq g(e_H, \theta_H) - g(e_M, \theta_H)$$

which holds iff $e_H \geq e_M$. As in class, it is natural to conjecture that this is not binding and to use “ignore and verify” on it. So I’ll ignore this constraint for now, verifying it holds at the end.

This means the constraint that $\theta_M$ not take the $\theta_L$ contract must bind since this is the only constraint left which says $w_M$ must be greater than something. Hence

$$w_M = g(e_M, \theta_M) + w_L - g(e_L, \theta_M).$$

Just as in class, one can show that the incentive compatibility conditions that $\theta_L$ not take one of the wrong contracts reduce to $e_H \geq e_L$ and $e_M \geq e_L$. Again, let use “ignore and verify” on these (although we’ll be a bit lax on the “verify” part).
So we’ve now eliminated all the constraints and determined that

\[ w_L = g(e_L, \theta_L) + v^{-1}(\bar{u}) \]
\[ w_M = g(e_M, \theta_M) + w_L - g(e_L, \theta_M) \]
\[ = g(e_M, \theta_M) + g(e_L, \theta_L) - g(e_L, \theta_M) + v^{-1}(\bar{u}) \]
\[ w_H = g(e_H, \theta_H) + w_M - g(e_M, \theta_H) \]
\[ = g(e_H, \theta_H) + g(e_M, \theta_M) - g(e_M, \theta_H) + g(e_L, \theta_L) - g(e_L, \theta_M) + v^{-1}(\bar{u}) \]

Substituting into the objective function, we get

\[
\lambda_H[\pi(e_H) - g(e_H, \theta_H)] + \lambda_M[\pi(e_M) - g(e_M, \theta_M)] + \lambda_L[\pi(e_L) - g(e_L, \theta_L)] \\
- \lambda_H[g(e_M, \theta_M) - g(e_M, \theta_H) + g(e_L, \theta_L) - g(e_L, \theta_M)] \\
- \lambda_M[g(e_L, \theta_L) - g(e_L, \theta_M)] - v^{-1}(\bar{u}).
\]

The first line gives the objective function from the first–best. The second line gives the cost of the rents we have to give the high type. Note that if he imitates the middle type, he gets, in effect, the rents the middle type would get plus more due to his own cost advantage. Finally, the last line (ignoring the \(v^{-1}(\bar{u})\)) gives the rents for the middle type.

The first order condition for \(e_H\) is \(\pi'(e_H) = g_e(e_H, \theta_H)\). So, just as in the simpler model from class, the effort choice for the high type is first best.

The first order condition for \(e_M\) also looks like what we saw in class for the low type, namely,

\[
\lambda_M[\pi'(e_M) - g_e(e_M, \theta_M)] = \lambda_H[g_e(e_M, \theta_M) - g_e(e_M, \theta_H)].
\]

Thus, as in class, we see that the second–best effort choice for the middle type is below the first–best. Since the first–best effort choice for the middle type is below that of the high type, we see that our “ignore and verify” worked on the constraint that \(e_M \leq e_H\).

Finally, the first order condition for \(e_L\) is slightly messier. We have

\[
\lambda_L[\pi'(e_L) - g_e(e_L, \theta_L)] = (\lambda_H + \lambda_M)[g_e(e_L, \theta_L) - g_e(e_L, \theta_M)].
\]

Again, this implies that the low type’s second–best effort is below the first–best and hence is below that of the high type.

The last constraint we should verify is that \(e_L \leq e_M\). This is messier and so I will omit it. To see the issue, think of \(\lambda_M\) as very small. Intuitively, as it converges to zero, we’re converging to the two–type case we discussed in class. But the first–order condition for \(e_M\) above says that our choice of \(e_M\) will be getting small — converging to zero in general. Thus if \(\lambda_M\) is too small, we will end up with \(e_M\) below \(e_L\) and will have to explicitly impose the constraint that this does not occur. That is, the constraint will bind and we’ll have to set \(e_L = e_M\), changing the first–order conditions.
2. (a) If the owner can observe $\theta$, he will pay the worker $e/\theta$ and choose $e$ to maximize $2\sqrt{e} - (e/\theta)$. The first–order condition is $e^{-1/2} = \theta^{-1}$ so $e = \theta^2$. So the optimal contract is $e_j = \theta_j^2$ and $w_j = \theta_j$ for $j = L, H$. The owner’s expected profits are $p\theta_H + (1 - p)\theta_L$.

(b) If the owner cannot observe $\theta$, he will choose $e_H, w_H, e_L, \text{and } w_L$ to maximize his expected profits subject to the individual rationality and incentive compatibility constraints:

$$w_j - \frac{e_j}{\theta_j} \geq 0, \quad j = L, H$$

$$w_L - \frac{e_L}{\theta_L} \geq w_H - \frac{e_H}{\theta_L}$$

$$w_H - \frac{e_H}{\theta_H} \geq w_L - \frac{e_L}{\theta_H}.$$  

As usual, the only binding constraints are the individual rationality constraint for type $\theta_L$ and the incentive compatibility constraint for type $\theta_H$. Hence

$$w_L = \frac{e_L}{\theta_L}$$

$$w_H = \frac{e_H}{\theta_H} + \frac{e_L}{\theta_L} - \frac{e_L}{\theta_H}.$$  

Substituting these results into the objective function, the owner will choose $e_L$ and $e_H$ to maximize

$$p \left[ 2\sqrt{e_H} - \frac{e_H}{\theta_H} - \frac{e_L}{\theta_L} + \frac{e_L}{\theta_H} \right] + (1 - p) \left[ 2\sqrt{e_L} - \frac{e_L}{\theta_L} \right].$$

The first–order condition for $e_H$ is the same as in the first–best (as usual), so $e_H = \theta_H^2$.

The first–order condition for $e_L$ is

$$(1 - p) \left[ \frac{1}{\sqrt{e_L}} - \frac{1}{\theta_L} \right] = p \left[ \frac{1}{\theta_L} - \frac{1}{\theta_H} \right].$$

Rearranging gives $e_L = [(1 - p)\theta_H \theta_L/(\theta_H - p\theta_L)]^2$. The wages are obtained by substituting for $e_L$ and $e_H$ into the equations above.

(c) When there is a probability that the owner observes the manager’s type, the problem changes dramatically. Recall that a contract can depend on anything mutually observed. Hence a contract can depend on the observation made by the owner, if any, as well as the reported type by the manager. Suppose the contract says that if the owner observes that the manager’s type is not what he claimed, then the manager is paid $-\infty$. Then for any strictly positive probability that his type is observed, the manager will not lie. Hence we are back into the first–best world and the contracts are as in (a) with the provision that the manager receives a bad payoff if he is caught lying.

3. The IRS chooses $T_h, T_\ell, A_h$, and $A_\ell$ to maximize

$$\lambda(T_h - c(A_h)) + (1 - \lambda)(T_\ell - c(A_\ell)).$$
The constraints: First, we have IR constraints $I_h \geq T_h$ and $I_\ell \geq T_\ell$. Second, we have incentive compatibility. If you tell the truth, you don’t care if you’re audited so your payoff is your income minus your taxes. If you lie, then there’s a chance that you lose all your income. So your payoff is the probability you aren’t audited times income minus taxes. In short, the incentive compatibility constraints are

$$I_h - T_h \geq (1 - A_\ell)(I_h - T_\ell)$$

and

$$I_\ell - T_\ell \geq (1 - A_h)(I_\ell - T_\ell).$$

Now let’s determine which constraints bind. First, note that the IR constraint for the high type cannot bind. Reason: Incentive compatibility says that $I_h - T_h \geq (1 - A_\ell)(I_h - T_\ell)$. But $I_h > I_\ell \geq T_\ell$ where the last inequality uses IR for the low type. Hence $(1 - A_\ell)(I_h - T_\ell) \geq 0$. So IC for the high type implies IR for the high type.

There are several ways to solve from here. My choice: Consider various relationships between $T_\ell$ and $T_h$. First, suppose that $T_\ell > T_h$ (implausible, of course). In this case, the high type has no incentive to imitate the low type so that incentive compatibility constraint won’t bind. But then the only remaining constraint with $T_h$ is the incentive compatibility condition for the low type and increasing $T_h$ helps with that one. Hence the IRS would gain by increasing $T_h$, a contradiction.

Second, suppose $T_h = T_\ell$. In this case, neither incentive compatibility constraint binds. Clearly, the IRS will set $A_h = A_\ell = 0$ and will set the highest possible $T_\ell$, which is $I_\ell$. In this case, the IRS’s payoff is $I_\ell$. Below, we’ll check to see if this is optimal.

Finally, then, suppose $T_h > T_\ell$. In this case, the low type has no incentive to imitate the high type, so we can drop that constraint. Given this, the IR constraint for the low type must bind — otherwise, the IRS could increase $T_\ell$, help on the incentive constraint for the high type, and not violate IR for the low type. So $T_\ell = I_\ell$. If we substitute this into the incentive compatibility constraint for the high type and rearrange, we get

$$T_h \leq A_\ell I_h + (1 - A_\ell)I_\ell.$$  

Clearly, this constraint must bind — otherwise, the IRS could increase $T_h$ and be better off. Hence this equation holds with equality. Substituting, then, we see that the IRS will choose $A_h$ and $A_\ell$ to maximize

$$\lambda [A_\ell I_h + (1 - A_\ell)I_\ell - c(A_h)] + (1 - \lambda)[I_\ell - c(A_\ell)].$$

$A_h$ has no effect other than to add to costs, so $A_h = 0$. The first order condition for the optimal $A_\ell$ is

$$\lambda[I_h - I_\ell] - (1 - \lambda)c'(A_\ell) = 0.$$  \hspace{1cm} (1)

Our conditions on $c$ ensure that there is an $A_\ell \in (0, 1)$ satisfying this equation.
Is this better than $T_h = T_\ell = I_\ell$ and $A_h = A_\ell = 0$? Yes, it must be since this is what we’d get in the last step above if we chose $A_\ell = 0$. Hence the solution is $T_\ell = I_\ell$, $A_h = 0$, $T_h = A_\ell I_h + (1 - A_\ell) I_\ell$ and $A_\ell$ given as the solution to equation (1).

4. What we need to show is that bidding $\theta_i$ (weakly) dominates any other bid for player $i$. First, consider a bid $b'$ below $\theta_i$. If both of these bids lose (i.e., there are two bidders above $\theta_i$), then $i$ is indifferent between them. Suppose both win. Then there is at most one of the other bidders whose bid is above $b'$. (If two others were above $b'$, $b'$ would not be a winning bid.) Hence $i$ wins with either bid and pays the same amount either way — the second highest bid among the other bidders. So in this case, again, $i$ is indifferent between the two bids. Since $\theta_i > b'$, if one bid wins and the other doesn’t, the winner is $\theta_i$. So suppose the other people’s bids are such that $i$ wins if he bids $\theta_i$ and loses if he bids $b'$. The amount he pays if he wins must be below his bid, so his payoff from winning must be positive. Hence this must be better than losing. So in all cases, bidding $\theta_i$ is at least weakly better than bidding $b'$ and sometimes is strictly better.

The argument for $b'$ above $\theta_i$ is similar. Here the only way there can be a difference between the two is when $\theta_i$ loses and $b'$ wins. Since $\theta_i$ loses, there must be two bids above this. Hence if $b'$ wins, the amount $i$ will have to pay must be above $\theta_i$. Hence $i$ is worse off bidding $b'$ than he would be if he bids $\theta_i$.

5. Suppose bidders $j \neq i$ use the strategy $\sigma(\theta_j) = a\theta_j$. What is $i$’s payoff as a function of $\theta_i$ and his bid $b$? It is

$$
\left(\theta_i - \frac{1}{2} b\right) \Pr[\max_{j \neq i} \theta_j < b/a] - \frac{1}{2} a \int_0^{b/a} \theta (I - 1) \theta^{I-2} d\theta
$$

or

$$
\theta_i \left(\frac{b}{a}\right)^{I-1} - \frac{1}{2} a^I - \frac{1}{2} a^{I-1} \frac{I - 1}{I}.
$$

So the first–order condition (after a little rearranging) is

$$
(I - 1) \theta_i - \frac{I}{2} b - \frac{I - 1}{2} b = 0.
$$

So $b = [2(I - 1)/(2I - 1)] \theta_i$. That is, it is an equilibrium for every bidder to bid $a\theta$ where $a = 2(I - 1)/(2I - 1)$. 

6