Answers to Problem Set 6

Economics 703
Spring 2017

1. Compute the $J_i$ functions. By definition,

$$J_i(\theta_i) = \theta_i - \frac{1 - \Phi_i(\theta_i)}{\varphi_i(\theta_i)}.$$ 

So

$$J_1(\theta_1) = \theta_1 - \frac{1 - \theta_1}{1} = 2\theta_1 - 1.$$ 

$$J_2(\theta_2) = \theta_2 - \frac{e^{-\theta_2}}{e^{-\theta_2}} = \theta_2 - 1.$$ 

From this, the optimal direct mechanism is clear. The seller asks the types of the buyers. If $\theta_1 < 1/2$ and $\theta_2 < 1$, the seller keeps the good and transfers are zero. Otherwise, the seller gives the good to 1 if $2\theta_1 > \theta_2$ and to 2 otherwise. The simplest specification of the transfers is that

$$t_i(\theta_1, \theta_2) = \bar{t}_i(\theta_i) = -\bar{y}_i(\theta_i)\theta_i + \int_0^{\theta_i} \bar{y}_i(s) \, ds$$

where $\bar{y}_i(\theta_i)$ is determined by the specification above. Specifically,

$$\bar{y}_1(\theta_1) = \begin{cases} 
0, & \text{if } \theta_1 < 1/2 \\
\Pr[\theta_2 < 2\theta_1] = 1 - e^{-2\theta_1}, & \text{otherwise,}
\end{cases}$$

and

$$\bar{y}_2(\theta_2) = \begin{cases} 
0, & \text{if } \theta_2 < 1 \\
\Pr[2\theta_1 < \theta_2] = \theta_2/2, & \text{if } \theta_2 \in [1, 2] \\
\Pr[2\theta_1 < \theta_2] = 1, & \text{if } \theta_2 > 2
\end{cases}$$

You can compute the integrals, etc., but this is all I was trying to ask for.

(b) There are surely many ways to answer this question. The simplest thing I can see is to have a second price auction with the following two twists. First, there’s a reserve price of 1. Second, bidder 1’s bids count “double.” More specifically, the auction is as
follows. Letting the bids be \((b_1, b_2)\), neither agent receives the object if \(2b_1 < 1 \) and \(b_2 < 1\). Otherwise, 1 receives the object if \(2b_1 > b_2\) and 2 receives it if \(b_2 > 2b_1\). If 1 wins, he pays the larger of \(1/2\) and \(b_2/2\). If 2 wins, he pays the larger of 1 and \(2b_1\).

It’s easy to see that for 2, the analysis is essentially the same as for the usual second price auction. So it’s optimal for 2 to bid \(\theta_2\). It’s a little less immediate for 1, but the same conclusion works. To see it, suppose \(\theta_1 > 1/2\). (Otherwise, 1 can’t win at a price worth paying and so may as well bid \(\theta_1\), knowing he’ll lose.) If 1 bids \(\theta_1 + \varepsilon\), then if 2 bids below \(2\theta_1\) or above \(2(\theta_1 + \varepsilon)\), 1 is indifferent between this bid and bidding \(\theta_1\). If 2 bids in between \(2\theta_1\) and \(2(\theta_1 + \varepsilon)\), 1 wins but pays more than the object is worth — he’d be better off bidding \(\theta_1\). Similar reasoning applies for bidding below \(\theta_1\).

Given this, we get the same \(y_i(\theta_1, \theta_2)\) as in the optimal mechanism. From revenue equivalence, we know we have the same revenue if the lowest types of players 1 and 2 get the same payoff as in the optimal mechanism. In the optimal mechanism, these types get payoffs of \(0\) as they do here.

2. (a) Compute the \(J\) functions. It’s easy to see that

\[
J_1(\theta_1) = \theta_1 - \frac{1 - \theta_1}{1} = 2\theta_1 - 1
\]

and

\[
J_2(\theta_2) = \theta_2 - \frac{1 - (\theta_2)/2}{1/2} = 2\theta_2 - 2.
\]

So the optimal allocation for the seller is to set \(y_1(\theta_1, \theta_2) = 1\) whenever \(2\theta_1 - 1 > 0\) and \(2\theta_1 - 1 > 2\theta_2 - 2\), to set \(y_2(\theta_1, \theta_2) = 1\) whenever \(2\theta_2 - 2 > 0\) and \(2\theta_2 - 2 > 2\theta_1 - 1\), and to set \(y_0(\theta_1, \theta_2) = 1\) otherwise. In other words, 1 receives the good if \(\theta_1 > 1/2\) and \(\theta_1 + 1/2 > \theta_2\), 2 receives the good if \(\theta_2 > 1\) and \(\theta_2 > \theta_1 + 1/2\), and neither agent gets the good otherwise. Note that if \(\theta_1 > 1/2\) and \(\theta_2 < 1\), then \(\theta_1 + 1/2 > \theta_2\). Similarly, if \(\theta_1 < 1/2\) and \(\theta_2 > 1\), we have \(\theta_1 + 1/2 < \theta_2\). Thus the case where neither gets the good can be restated as \(\theta_1 < 1/2\) and \(\theta_2 < 1\).

As always, there are many transfer rules that would “work” with this. Perhaps the simplest is to set

\[
t_i(\theta_1, \theta_2) = \mathcal{U}_i(\theta_i) - \bar{y}_i(\theta_i)\theta_i
\]

or

\[
t_i(\theta_1, \theta_2) = \int_0^{\theta_i} \bar{y}_i(s) \, ds - \bar{y}_i(\theta_i)\theta_i
\]

where I’m using the fact that \(\mathcal{U}_i(\theta_i) = 0\). For agent 1, we have

\[
\bar{y}_1(\theta_1) = \begin{cases} 
0, & \text{if } \theta_1 < 1/2; \\
\Pr[\theta_2 \leq \theta_1 + 1/2] = (1/2)[\theta_1 + 1/2], & \text{otherwise.}
\end{cases}
\]
After a bit of algebra, we get
\[ t_1(\theta_1, \theta_2) = \begin{cases} 0, & \text{if } \theta_1 < 1/2 \\ -\frac{\theta_1^2}{4} - \frac{3}{16}, & \text{if } \theta_1 > 1/2 \end{cases} \]

For agent 2, we have
\[ \bar{y}_2(\theta_2) = \begin{cases} 0, & \text{if } \theta_2 < 1; \\ 1, & \text{if } \theta_2 > 3/2; \\ \Pr[\theta_1 + 1/2 \leq \theta_2] = \theta_2 - 1/2, & \text{otherwise}. \end{cases} \]

Plugging this into the formula above and doing some algebra gives
\[ t_2(\theta_1, \theta_2) = \begin{cases} 0, & \text{if } \theta_2 < 1 \\ -9/8, & \text{if } \theta_2 > 3/2 \\ -\frac{\theta_2^2}{2}, & \text{otherwise}. \end{cases} \]

From these calculations, we can easily get the seller’s expected revenue. The amount of money the seller expects to make from agent 1 is \(-E_{\theta} t_1(\theta)\) which is
\[
\int_{1/2}^{1} \left[ \frac{\theta_1^2}{4} + \frac{3}{16} \right] d\theta_1 = \frac{1}{6}.
\]

Similarly, the money the seller expects to earn from agent 2 is
\[
\int_{1}^{3/2} \frac{\theta_2^2}{2} - \frac{1}{2} \theta_2 + \frac{9}{8} = \frac{23}{48}.
\]

So the seller’s expected revenue is the sum of these numbers which is 31/48.

The allocation rule is not \textit{ex post} efficient. First, it is possible that the seller does not allocate the object to anyone even though the valuations are strictly positive. In this case, there is a social loss. Second, even ignoring this issue, we see that we have inefficiencies even in some cases where the seller doesn’t keep the good. In particular, if \(\theta_1 + 1/2 > \theta_2 > \theta_1 > 1/2\), agent 1 gets the object even though agent 2 values it more.

(b) Again, the first step is to compute the \(J\) functions. Of course, \(J_1\) is the same as in (a). Now we have
\[ J_2(\theta_2) = \theta_2 - \frac{1 - (\theta_2 - 1)}{1} = 2\theta_2 - 2. \]

In a sense, this is a different function from (a). Yes, the formula is the same, but the range of \(\theta_2\)’s — that is, the domain of the function — is not the same. This means that the statement of the optimal allocation doesn’t change, except that we have fewer cases to consider. Specifically, it is now impossible to have \(\theta_2 < 1\), so we don’t have to discuss the allocation for such cases.
This means that the optimal allocation for the seller can now be stated as follows. If \( \theta_1 < 1/2 \), give the good to agent 2. If \( \theta_1 > 1/2 \), then give the good to 1 if \( \theta_1 + 1/2 > \theta_2 \) and to 2 otherwise. (We could really simplify the statement to just the last sentence since if \( \theta_1 < 1/2 \), we must have \( \theta_1 + 1/2 < \theta_2 \).

We can follow the same approach as in (a) to get one transfer function that works. I’ll omit the algebra, but you get

\[
t_1(\theta_1, \theta_2) = \begin{cases} 
0, & \text{if } \theta_1 < 1/2, \\
-\theta_2^2/2 + (1/8), & \text{otherwise;}
\end{cases}
\]

and

\[
t_2(\theta_1, \theta_2) = \begin{cases} 
-\theta_2^2/2, & \text{if } \theta_2 < 3/2, \\
-9/8, & \text{otherwise;}
\end{cases}
\]

Computing the seller’s expected revenue from agent 1, we get

\[
\int_{1/2}^{1} \left( \frac{\theta_2^2}{2} - \frac{1}{8} \right) d\theta_1 = \frac{1}{12}.
\]

The seller’s expected revenue from agent 2 is

\[
\int_{1}^{3/2} \frac{\theta_2^2}{2} d\theta_2 + \frac{9}{8} \frac{1}{2} = \frac{23}{24},
\]

so the seller’s expected revenue in total is 25/24.

Just as in (a), the outcome is not always \textit{ex post} efficient. Here the seller does always allocate the good, so the inefficiency associated with reserve prices is not the problem. However, just as in (a), if \( \theta_1 + 1/2 > \theta_2 \), agent 1 gets the good even though it is \textit{always} true that 2 values the good more than 1.

(c) The simplest way I found to do this is to start with a second price auction and then do a bit of adjusting. Specifically, for the setup in (a), the seller runs a second price auction with a reserve price of 1, but with the twist that if 1 wins, the seller gives both the object and 1/2 in money to agent 1. (Agent 1 still pays for this as in the usual case.) To see that this works, let’s find an equilibrium of this game. Note that the auction effectively makes the value of winning equal to \( \theta_1 + 1/2 \) for agent 1. So it is optimal for him to bid this. Similarly, the usual argument shows that it is optimal for 2 to bid \( \theta_2 \).

If \( \theta_1 < 1/2 \) and \( \theta_2 < 1 \), both agents will be below the reserve price, so the seller will keep the good. Otherwise, 1 wins if \( \theta_1 + 1/2 > \theta_2 \) and 2 wins if the inequality goes the other way. Thus we get exactly the same allocation as we computed in (a). Note that the lowest types of each agent lose for sure and so get payoffs of zero. Hence by the revenue equivalence theorem, we know that this equilibrium of this auction yields the seller exactly the revenue we computed in (a).

The setup in (b) is simpler in some ways but more complex in others. The aspect which is simpler is that we no longer need a reserve price since the seller is always
allocating the good to one of the agents. From this, you might be tempted to do exactly
the same thing as above but without the reserve price. The same reasoning as above
shows that it’s optimal for agent 1 to bid $\theta_1 + 1/2$ and for agent 2 to bid $\theta_2$. Thus the
allocation gives the good to 1 if $\theta_1 + 1/2 > \theta_2$ and 2 otherwise, exactly the allocation we
computed in (b). As above, the lowest type of agent 1 gets a payoff of 0 since he cannot
possibly win the auction.

On the other hand, note that the lowest type of agent 2 gets a strictly positive payoff
in this equilibrium of this auction. To see this, note that he wins if $\theta_1 < 1/2$ which
happens with probability 1/2. His expected payoff is

$$\int_0^{1/2} \left[ 1 - \theta_1 - \frac{1}{2} \right] d\theta_1 = \frac{1}{4} - \int_0^{1/2} \theta_1 d\theta_1 = \frac{1}{8}. $$

So this auction isn’t optimal as the lowest type of agent 2 is getting a strictly positive
payoff.

We can fix this easily enough though: Require agent 2 to pay 1/8 to participate in
the auction. Every type of agent 2 gets a payoff of at least 1/8, so it is optimal for 2 to
enter the auction. The bidding strategies and thus the allocation do not change, so we
get the allocation of the good computed in (b). Now the lowest types of both agents earn
0, so revenue equivalence tells us that this auction yields the same expected revenues as
what we computed in (b).