1. Suppose we have two agents, 1 and 2, and each owns one apple. The value to $i$ per apple is $\theta_i$ where $\theta_i \sim U[0,1]$ and $\theta_1$ and $\theta_2$ are independent. Consider the following mechanism. The agents simultaneously choose bids. If $i$’s bid is strictly larger than $j$’s, then $i$ pays his bid to $j$ and consumes both apples. If the bids are tied, each agent just consumes his own apple and no payments are made.

(a) Show that there is a Nash equilibrium where $i$’s strategy is $\sigma_i(\theta_i) = \alpha \theta_i$ and find $\alpha$.

(b) Compute the interim payoffs. Does each agent always prefer to participate in the mechanism (assuming that he can simply consume his own apple if he doesn’t participate)? Is the outcome ex post efficient?

(c) How does this example relate to the Myerson–Satterthwaite Theorem?

2. Consider the following two–player auction. There is one unit of a good. The value to bidder 1 is $2\theta_1 + \theta_2$, while the value to bidder 2 is $2\theta_2 + \theta_1$. Player $i$ knows only $\theta_i$. The common prior is that the $\theta_i$’s are independently distributed uniformly on $[0,1]$.

(a) Suppose we have a first–price auction to allocate the good. Show that there is a Nash equilibrium where bidder $i$’s strategy is $\sigma_i(\theta_i) = \alpha \theta_i$ and find $\alpha$. What is the seller’s expected revenue?

(b) Suppose instead that we have a second–price auction to allocate the good. Show that there is again a Nash equilibrium where bidder $i$’s strategy is $\sigma_i(\theta_i) = \alpha \theta_i$ and find this $\alpha$. What is the seller’s expected revenue?

3. A seller has a single unit of an indivisible good and has two possible buyers. Both buyers are risk neutral where bidder $i$’s utility is $\theta_i + t_i$ if he receives the object and receives transfer $t_i$ from the seller (equivalently, pays $-t_i$ to the seller) and has utility $t_i$ if he does not receive the object and receives transfer $t_i$ from the seller (equivalently, pays $-t_i$ to the seller). Assume $\theta_1 \sim U[0,1]$ while the cumulative distribution function for $\theta_2$ is $\Phi_2(\theta_2) = 1 - e^{-\theta_2}$.

(a) Find an optimal direct mechanism for the seller. (You don’t need to calculate the transfers but explain how they can be calculated.)
(b) Give a mechanism which is a variation on a second–price auction which has an equilibrium yielding the seller the same expected revenue as the mechanism in (a).

4. Suppose we have an auction with two bidders, 1 and 2. Assume $\theta_1$ and $\theta_2$ are independent random variables but are not necessarily identically distributed.

(a) Suppose $\theta_1 \sim U[0, 1]$ and $\theta_2 \sim U[0, 2]$. Find an optimal allocation and transfer rule for the seller. Compute the seller’s expected revenue. Is this allocation rule \textit{ex post} efficient? If we focus only on those type realizations for which the seller does not keep the good, is the allocation \textit{ex post} efficient?

(b) Now suppose $\theta_1 \sim U[0, 1]$ and $\theta_2 \sim U[1, 2]$. Answer the same questions as for (a). How does your answer change from (a)?

(c) For each of the two cases above, give a variation on a second price auction with reserve price which has an equilibrium implementing the allocation rule and revenue which is optimal for the seller. (Hint. There’s surely many ways to do this. Options to think about: (1) Try giving 1 some money if he wins. (2) Try charging 2 an entry fee.)