1. A seller has a single unit of an indivisible good and has two possible buyers. Both
buyers are risk neutral where bidder $i$’s utility is $\theta_i + t_i$ if he receives the object and
receives transfer $t_i$ from the seller (equivalently, pays $-t_i$ to the seller) and has utility
$t_i$ if he does not receive the object and receives transfer $t_i$ from the seller (equivalently,
pays $-t_i$ to the seller). Assume $\theta_1 \sim U[0, 1]$ while the cumulative distribution function
for $\theta_2$ is $\Phi_2(\theta_2) = 1 - e^{-\theta_2}$.

(a) Find an optimal direct mechanism for the seller. (You don’t need to calculate the
transfers but explain how they can be calculated.)

(b) Give a mechanism which is a variation on a second–price auction which has an
equilibrium yielding the seller the same expected revenue as the mechanism in (a).

2. Suppose we have two agents, 1 and 2, and each owns one apple. The value to $i$ per
apple is $\theta_i$ where $\theta_i \sim U[0, 1]$ and $\theta_1$ and $\theta_2$ are independent. Consider the following
mechanism. The agents simultaneously choose bids. If $i$’s bid is strictly larger than $j$’s,
then $i$ pays his bid to $j$ and consumes both apples. If the bids are tied, each agent just
consumes his own apple and no payments are made.

(a) Show that there is a Nash equilibrium where $i$’s strategy is $\sigma_i(\theta_i) = \alpha \theta_i$ and find $\alpha$.

(b) Compute the interim payoffs. Does each agent always prefer to participate in the
mechanism (assuming that he can simply consume his own apple if he doesn’t partici-
patate)? Is the outcome ex post efficient?

(c) How does this example relate to the Myerson–Satterthwaite Theorem?

3. Suppose we have an auction with two bidders, 1 and 2. Assume $\theta_1$ and $\theta_2$ are
independent random variables but are not necessarily identically distributed.

(a) Suppose $\theta_1 \sim U[0, 1]$ and $\theta_2 \sim U[0, 2]$. Find an optimal allocation and transfer rule
for the seller. Compute the seller’s expected revenue. Is this allocation rule ex post
efficient? If we focus only on those type realizations for which the seller does not keep
the good, is the allocation ex post efficient?
(b) Now suppose $\theta_1 \sim U[0, 1]$ and $\theta_2 \sim U[1, 2]$. Answer the same questions as for (a). How does your answer change from (a)?

(c) For each of the two cases above, give a variation on a second price auction with reserve price which has an equilibrium implementing the allocation rule and revenue which is optimal for the seller. (Hint. There’s surely many ways to do this. Options to think about: (1) Try giving 1 some money if he wins. (2) Try charging 2 an entry fee.)