1. (a) Fix a bidder \(i\) with valuation \(\theta_i\) and assume all other bidders use a strategy of the form \(\sigma_j(\theta_j) = a\theta_j\). Then \(i\)'s expected payoff to a bid of \(b\) is

\[
(\theta_i - b)^\alpha \left( \frac{b}{a} \right)^{I-1}.
\]

The first–order condition is

\[
-\alpha(\theta_i - b)^{\alpha - 1} \left( \frac{b}{a} \right)^{I-1} + (I - 1)(\theta_i - b)^\alpha \left( \frac{b^{I-2}}{a^{I-1}} \right) = 0.
\]

Rearranging, we get

\[
b = \frac{I - 1}{I - 1 + \alpha} \theta_i.
\]

So there is an equilibrium where everyone bids \(a\theta_i\) for \(a = (I - 1)/(I - 1 + \alpha)\).

(b) It’s still a dominant strategy to bid \(\theta_i\). The dominant strategy argument has nothing to do with risk attitudes since it says you get a higher payoff for every possible bids by the other players.

(c) Since the equilibrium for the second price auction hasn’t changed, the expected revenue is still \((I - 1)/(I + 1)\). For the first price auction, the expected revenue is \((I - 1)/(I + 1 + \alpha)\) times the expected value of the first order statistic. That expected value is

\[
\int_0^1 \theta I \theta^{I-1} d\theta = \frac{I}{I + 1}.
\]

So the expected revenue in the first price auction is higher if

\[
\frac{I(I - 1)}{(I + 1)(I - 1 + \alpha)} > \frac{I - 1}{I + 1}
\]

or \(\alpha < 1\) which holds by assumption. Expected revenue in the second price auction is independent of \(\alpha\), while expected revenue in the first price auction is decreasing in \(\alpha\).
2. Consider the payoff to bidder $i$ with value $\theta_i$ from bidding $b$ when for all $j \neq i$, $j$ follows the strategy of bidding $\alpha \theta_j^\beta$. The payoff is 

$$\theta_i \Pr[\alpha \theta_j^\beta \leq b, \forall j \neq i] - b$$

since $i$ wins iff his bid is larger than everyone else’s (and ties are probability zero) but pays his bid with probability 1. We can rewrite this as

$$\theta_i \left( \Pr \left[ \theta_j \leq \left( \frac{b}{\alpha} \right)^{\frac{1}{\beta}} \right] \right)^{I-1} - b = \theta_i \left( \frac{b}{\alpha} \right)^{\frac{I-1}{\beta}} - b.$$

The first–order condition for the optimal $b$ is

$$\frac{I-1}{\beta} \frac{\theta_i}{\alpha^{(I-1)/\beta}} b^{\frac{I-1}{\beta}} - 1 = 0.$$ 

Rearranging:

$$b = \theta^{\frac{\beta}{\beta - I + 1}} \left( \frac{I-1}{\beta} \right)^{\frac{\beta}{\beta - I + 1}} \alpha^{-\frac{I-1}{\beta - I + 1}}.$$

To be consistent with our hypothesis, we must have

$$\beta = \frac{\beta}{\beta - I + 1}$$

or $\beta = I$. Substituting, this gives 

$$b = \theta^I \left( \frac{I-1}{I} \right)^I \alpha^{-(I-1)}.$$

So 

$$\alpha = \left( \frac{I-1}{I} \right)^I \alpha^{-(I-1)}$$

implying $\alpha^I = [(I-1)/I]^I$ or $\alpha = (I-1)/I$.

Hence we have an equilibrium where $\sigma_i(\theta_i) = (I-1)\theta_i^I/I$.

The seller’s expected revenue is

$$\sum_i E \left( \frac{(I-1)\theta_i^I}{I} \right) = (I-1)E(\theta_i^I) = (I-1) \int_0^1 \theta^I d\theta = \frac{I-1}{I+1}.$$

We know that this must be the same as the expected revenue in the first price auction, at least if we focus on the equilibrium we computed earlier for that auction. Reason: Note that the bidder with the highest type always gets the object in this auction, just
as in the first price auction. Also, the lowest type of a bidder gets a payoff of 0 since he bids 0 and gets the object with probability 0. Again, this is the same as the payoff to the lowest type of a bidder in the first price auction. Hence the revenue equivalence theorem tells us that the revenue in the all pay auction must also be \((I - 1)/(I + 1)\).

3. a) Suppose the seller knows the buyer is following the strategy of offering \(b_2(\theta_2) = \alpha_2 + \beta_2\theta_2\). If the seller offers \(b_1\), then he earns zero if \(b_1 < \alpha_2 + \beta_2\theta_2\) and earns

\[
\frac{b_1 + \alpha_2 + \beta_2\theta_2}{2} - \theta_1
\]

if \(\alpha_2 + \beta_2\theta_2 \geq b_1\). Hence, assuming \(\beta_2 > 0\) (which we’ll see it will be), the seller’s expected payoff is

\[
\int_{(b_1 - \alpha_2)/\beta_2}^{1} \left[ \frac{b_1 + \alpha_2 + \beta_2\theta_2}{2} - \theta_1 \right] d\theta_2
\]

if \((b_1 - \alpha_2)/\beta_2 \in [0, 1]\). For the moment, assume \(b_1\) is in this range. Hence we can do the integration to rewrite the seller’s expected payoff as

\[
\left\{ \frac{1}{2}[b_1 + \alpha_2] - \theta_1 \right\} \left[ 1 - \frac{b_1 - \alpha_2}{\beta_2} \right] + \frac{\beta_2}{4} \left[ 1 - \left( \frac{b_1 - \alpha_2}{\beta_2} \right)^2 \right].
\]

Note that this is just a quadratic in \(b_1\). I’ll omit the algebra, but if you maximize this with respect to \(b_1\), you get

\[
b_1 = \frac{2}{3} \theta_1 + \frac{1}{3} (\alpha_2 + \beta_2).
\]

This only applies when this gives us a \(b_1\) with \((b_1 - \alpha_2)/\beta_2 \in [0, 1]\) — that is, \(b_1 \geq \alpha_2\) and \(b_1 \leq \alpha_2 + \beta_2\). As we’ll see momentarily, the first of these inequalities will always be satisfied. The second is trickier though. Intuitively, if the second one is violated, so \(b_1 > \alpha_2 + \beta_2\), this says that the seller is offering a price that is above every offer the buyer will make and so is certain to not be accepted. Note that the seller’s offer is a convex combination of \(\theta_1\) and \(\alpha_2 + \beta_2\). So if his bid ends up above \(\alpha_2 + \beta_2\), it must be true that his bid is below \(\theta_1\). That is, he is offering to trade at a price below his costs. Hence his optimal bid in this case is anything that isn’t accepted and \(\alpha_2 + \beta_2\) is good enough.

So now consider the buyer. He knows that the seller’s bid will take the form \(b_1 = \alpha_1 + \beta_1\theta_1\). Again assuming \(\beta_1 > 0\), this means that his expected payoff can be written as

\[
\int_{0}^{(b_2 - \alpha_1)/\beta_1} \left[ \theta_2 - \frac{1}{2} (b_2 + \alpha_1 + \beta_1\theta_1) \right] d\theta_1
\]

assuming \((b_2 - \alpha_1)/\beta_1 \in [0, 1]\). Performing the integration, we get

\[
[\theta_2 - \frac{1}{2} \alpha_1 - \frac{1}{2} b_2] \frac{b_2 - \alpha_1}{\beta_1} - \frac{\beta_1}{4} \left( \frac{b_2 - \alpha_1}{\beta_1} \right)^2.
\]
Again, this is a quadratic in 2’s strategy. Maximizing with respect to $b_2$ yields

$$b_2 = \frac{2}{3} \theta_2 + \frac{1}{3} \alpha_1.$$  

Analogously to the above, this only applies if $b_2$ is in the right range. For the same reasons as above, though, we don’t need to worry about these conditions.

Summarizing, we get $b_1 = \beta_1 \theta_1 + \alpha_1$ where $\beta_1 = 2/3$ and $\alpha_1 = (1/3)(\alpha_2 + \beta_2)$. Also, $b_2 = \beta_2 \theta_2 + \alpha_2$ where $\beta_2 = 2/3$ and $\alpha_2 = (1/3)\alpha_1$. Solving these equations simultaneously, we see that $\beta_1 = \beta_2 = 2/3$, $\alpha_1 = 1/4$, and $\alpha_2 = 1/12$.

To see what social choice function this implements, note that it implements $q = y = 0$ if $b_1 > b_2$ or

$$\frac{2}{3} \theta_1 + \frac{1}{4} > \frac{2}{3} \theta_2 + \frac{1}{12}.$$  

That is, we implement $q = y = 0$ if

$$\theta_2 - \theta_1 < \frac{1}{4}.$$  

If $\theta_2 - \theta_1 > \frac{1}{4}$, we implement $q = 1$ and

$$y = \frac{b_1 + b_2}{2} = \frac{1}{3}(\theta_1 + \theta_2) + \frac{1}{6}.$$  

This social choice function is not *ex post* efficient, since *ex post* efficiency would call for $q = 1$ whenever $\theta_2 > \theta_1$, not just when $\theta_2 > \theta_1 + (1/4)$.

b) Take the direct mechanism where seller and buyer both report their $\theta$’s. Let $\hat{\theta}_i$ denote the reported value of $\theta_i$ (since this might differ from the true value). The seller’s payoff as a function of his true $\theta_1$ and his report is

$$\int_{\hat{\theta}_1+(1/4)}^{1} \left[ \frac{1}{3} \left[ \hat{\theta}_1 + \theta_2 \right] + \frac{1}{6} - \theta_1 \right] d\theta_2.$$  

Ignoring (as before) the implicit constraint that this only applies when $\hat{\theta}_1 + (1/4) \in [0, 1]$, if we perform the integration, we get

$$\left[ \frac{1}{3} \hat{\theta}_1 + \frac{1}{6} - \theta_1 \right] \left[ \frac{3}{4} - \hat{\theta}_1 \right] + \left( \frac{1}{6} \right) \left[ 1 - \left( \hat{\theta}_1 + \frac{1}{4} \right)^2 \right].$$  

This is a quadratic in $\hat{\theta}_1$. The first order condition for maximizing this with respect to $\hat{\theta}_1$ is

$$\frac{1}{4} - \frac{1}{6} + \theta_1 - \frac{2}{3} \hat{\theta}_1 - \frac{1}{3} \left( \hat{\theta}_1 + \frac{1}{4} \right) = 0$$  

which is easily rearranged to $\hat{\theta}_1 = \theta_1$. 

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Similarly, the buyer’s payoff as a function of his true $\theta_2$ and his report is

$$\int_0^{\hat{\theta}_2^{(1/4)}} \left[ \theta_2 - \frac{1}{3} (\theta_1 + \hat{\theta}_2) - \frac{1}{6} \right] d\theta_1.$$  

Performing the integration gives

$$\left[ \theta_2 - \frac{1}{3} \hat{\theta}_2 - \frac{1}{6} \right] \left[ \hat{\theta}_2 - \frac{1}{4} \right] - \frac{1}{6} \left( \hat{\theta}_2 - \frac{1}{4} \right)^2.$$  

Again, the first order condition for the optimal choice of $\hat{\theta}_2$ reduces to $\hat{\theta}_2 = \theta_2$.

4. We know that we must have

$$\mathcal{U}_i(\theta_i) = \mathcal{U}_i(0) + \int_0^{\theta_i} \bar{y}_i(s) \, ds.$$  

Since the good is always allocated to the bidder with the highest type, $\bar{y}_i(s)$ is the probability that the other $I - 1$ bidders have types below $s$. Hence $\bar{y}_i(s) = s^{I-1}$. So

$$\mathcal{U}_i(\theta_i) = \mathcal{U}_i(0) + \int_0^{\theta_i} s^{I-1} \, ds = \frac{\theta_i^I}{I},$$  

where I also use the assumption that $\mathcal{U}_i(0) = 0$. We also know that

$$-\bar{\ell}_i(\theta_i) = \theta_i \bar{y}_i(\theta_i) - \mathcal{U}_i(\theta) = \theta_i^I - \frac{\theta_i^I}{I},$$  

so the seller’s expected revenue is

$$I \int_0^1 \left( 1 - \frac{1}{I} \right) \theta^I \, d\theta = \frac{I - 1}{I + 1}.$$  

This is exactly what we computed for the first and second price auctions in class using the equilibrium strategies in these auctions.