Answers to Problem Set 5

Economics 703

Spring 2017

1. Consider the payoff to bidder $i$ with value $\theta_i$ from bidding $b$ when for all $j \neq i$, $j$ follows the strategy of bidding $\alpha \theta_j^\beta$. The payoff is

$$\theta_i \Pr[\alpha \theta_j^\beta \leq b, \forall j \neq i] - b$$

since $i$ wins iff his bid is larger than everyone else’s (and ties are probability zero) but pays his bid with probability 1. We can rewrite this as

$$\theta_i \left( \Pr \left[ \theta_j \leq \left( \frac{b}{\alpha} \right)^{1/\beta} \right] \right)^{I-1} - b = \theta_i \left( \frac{b}{\alpha} \right)^{\frac{I-1}{\beta}} - b.$$

The first–order condition for the optimal $b$ is

$$\frac{I - 1}{\beta} \frac{\theta_i}{\alpha^{(I-1)/\beta}} b^{\frac{I-1}{\beta}-1} - 1 = 0.$$

Rearranging:

$$b = \theta_i b^{\frac{\beta}{\beta-I+1}} \left( \frac{I-1}{\beta} \right)^{\frac{\beta}{\beta-I+1}} \alpha^{-\frac{I}{\beta-I+1}}.$$

To be consistent with our hypothesis, we must have

$$\beta = \frac{\beta}{\beta - I + 1}$$

or $\beta = I$. Substituting, this gives

$$b = \theta_i \left( \frac{I-1}{I} \right)^I \alpha^{-(I-1)}.$$

So

$$\alpha = \left( \frac{I-1}{I} \right)^I \alpha^{-(I-1)}$$
implying $\alpha^I = [(I - 1)/I]^I$ or $\alpha = (I - 1)/I$.

Hence we have an equilibrium where $\sigma_i(\theta_i) = (I - 1)\theta_i^I / I$.

The seller’s expected revenue is

$$\sum_i E \left( \frac{(I - 1)\theta_i^I}{I} \right) = (I - 1)E(\theta_i^I) = (I - 1) \int_0^1 \theta^I \, d\theta = \frac{I - 1}{I + 1}.$$ 

We know that this must be the same as the expected revenue in the first price auction, at least if we focus on the equilibrium we computed earlier for that auction. Reason: Note that the bidder with the highest type always gets the object in this auction, just as in the first price auction. Also, the lowest type of a bidder gets a payoff of 0 since he bids 0 and gets the object with probability 0. Again, this is the same as the payoff to the lowest type of a bidder in the first price auction. Hence the revenue equivalence theorem tells us that the revenue in the first price auction must also be $(I - 1)/(I + 1)$.

2. We know that we must have

$$U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} \bar{y}_i(s) \, ds.$$ 

Since the good is always allocated to the bidder with the highest type, $\bar{y}_i(s)$ is the probability that the other $I - 1$ bidders have types below $s$. Hence $\bar{y}_i(s) = s^{I - 1}$. So

$$U_i(\theta_i) = U_i(0) + \int_0^{\theta_i} s^{I - 1} \, ds = \frac{\theta_i^I}{I},$$

where I also use the assumption that $U_i(0) = 0$. We also know that

$$-\bar{t}_i(\theta_i) = \theta_i\bar{y}_i(\theta_i) - U_i(\theta) = \theta_i^I - \frac{\theta_i^I}{I},$$

so the seller’s expected revenue is

$$I \int_0^1 \left(1 - \frac{1}{I}\right) \theta^I \, d\theta = \frac{I - 1}{I + 1}.$$ 

This is exactly what we computed for the first and second price auctions in class using the equilibrium strategies in these auctions.

3. (a) Consider player $i$ of type $\theta_i$ and his payoff to bidding $b$ given that the opponent’s strategy is $\sigma_j(\theta_j) = \alpha \theta_j$. $i$ is the high bidder iff $\theta_j \leq b/\alpha$ and gets $2\theta_i - b$ in this case. Otherwise, his payoff is $j$’s bid or $\alpha \theta_j$. So $i$’s expected payoff is

$$\int_0^{b/\alpha} (2\theta_i - b) \, d\theta_j + \int_{b/\alpha}^1 \alpha \theta_j \, d\theta_j = \frac{\alpha}{2} + \frac{2\theta_i}{\alpha} b - \frac{3}{2\alpha} b^2.$$
The first-order condition for \( b \) is 
\[
\frac{2\theta_i}{\alpha} = \frac{3}{\alpha} b, 
\]
so \( b = (2/3)\theta_i \). Hence we have an equilibrium where \( \sigma_i(\theta_i) = (2/3)\theta_i \) for \( i = 1, 2 \). That is, \( \alpha = 2/3 \).

(b) Substituting into the above for \( \alpha \) and \( b \), we see that \( i \)'s interim payoff is
\[
\frac{1}{3} + 2\theta_i^2 - \theta_i^2 = \theta_i^2 + \frac{1}{3}.
\]
If \( i \) does not participate in the mechanism and simply consumes his own apple, his payoff is \( \theta_i \). Hence \( i \) prefers to participate iff
\[
\theta_i^2 + \frac{1}{3} \geq \theta_i 
\]
or
\[
\theta_i(1 - \theta_i) \leq \frac{1}{3}.
\]
It is not hard to show that \( \theta_i(1 - \theta_i) \) is maximized at \( \theta_i = 1/2 \) where this function equals 1/4. Since this is less than 1/3, we see that
\[
\theta_i(1 - \theta_i) < \frac{1}{3}, \quad \forall \theta_i.
\]
Hence each agent always strictly prefers participating in the mechanism.

The outcome is also ex post efficient since the agent with the higher valuation is always getting all the apples. To see that this property implies ex post efficiency, suppose the outcome isn’t ex post efficient. That is, suppose there is some \( \theta \) and some reallocation that makes at least one agent strictly better off and the other no worse off. Clearly, the reallocation cannot only change the transfers since this would have to make one agent worse off and the other better off (assuming there is a change). Similarly, the reallocation cannot just change who gets the apples since this too would have to hurt one of the agents. So we must reallocate both apples and money. But the agent with the higher value is getting both apples. The amount of extra money we’d have to give him to leave him at least indifferent would have to be larger than the amount the other agent would be willing to pay to get an apple, a contradiction. Hence the outcome is ex post efficient.

(c) At first glance, this seems to contradict Myerson–Satterthwaite. The outcome of the mechanism must be incentive compatible (since it couldn’t be an equilibrium otherwise). We showed in (b) that the outcome is ex post efficient and individually rational. The key: In effect, each agent owns half of the endowment here, while in Myerson–Satterthwaite, one agent owned all of the good. More to the point, suppose we redid the example we looked at in class but assumed that each agent owned half of the good at the outset. (Each having half of one apple is effectively the same as each having
one of the two apples — the rescaling doesn’t change anything.) Nothing would change in our calculation until the very last step. Instead of individual rationality implying that $U_1(1) \geq 1$, it would only imply that $U_1(1) \geq 1/2$. But then we would not have a contradiction. For more on this, see Cramton, Gibbons, and Klemperer, “Dissolving a Partnership Efficiently,” *Econometrica*, May 1987.