1. Suppose we have \( I \) bidders with independent private values distributed uniformly on the interval \([0, 1]\). Suppose these bidders are risk averse — more specifically, if bidder \( i \)'s valuation is \( \theta_i \) and he gets the good at price \( p_i \), his payoff is \((\theta_i - p_i)^\alpha\) where \( \alpha \in (0, 1) \).

(Note: This is not always well defined. For example, if \( \theta_i < p_i \) and \( \alpha = 1/2 \), utility is not a real number. In such cases, assume that when \( \theta_i < p_i \), utility is \(-(p_i - \theta_i)^\alpha\). Alternatively, ignore this possibility — the answer won’t be affected.)

(a) Show that there is an equilibrium in the first price auction where \( i \) bids \( a\theta_i \) for some constant \( a \). Find the constant.

(b) Find an equilibrium for the second price auction.

(c) Do the two auctions yield the same expected revenues? If not, which yields more?

2. We have \( I \) bidders in an independent private values auction. The value of the object to bidder \( i \) is \( \theta_i \sim U[0, 1] \). Suppose the seller uses an all–pay auction. That is, each bidder \( i \) puts in a bid \( b_i \in [0, \infty) \). If \( b_i > \max_{j \neq i} b_j \), then \( i \) receives the object. If we have a tie, we randomize uniformly over the bidders who bid the most to determine which gets the object. But in all cases, all bidders pay their bids.

Show that there is an equilibrium where bidder \( i \)'s strategy is \( \sigma_i(\theta_i) = \alpha \theta_i^\beta \) for some \( \alpha \) and \( \beta \) and find \( \alpha \) and \( \beta \). Calculate the seller’s expected revenue in this equilibrium. Without calculating the revenue for a first price auction, what can you say about how the revenue in the all–pay auction compares to revenue in the first price auction?

3. Suppose we have two agents and the \( \theta_i \)'s are independent distributed uniformly on \([0, 1]\). Agent 1 is a seller and agent 2 a buyer, where \( \theta_1 \) is 1’s cost of producing a good and \( \theta_2 \) is the buyer’s valuation for this good. More specifically, a social choice, \( x \), will be a pair \((q, y)\) where \( q \) is the quantity of the good sold by the seller to the buyer (where this must be either 0 or 1) and \( y \) is the payment made by the buyer to the seller. The
utility functions are

\[ u_1(q, y, \theta_1) = y - q\theta_1 \]

and

\[ u_2(q, y, \theta_2) = \theta_2q - y. \]

Consider the following mechanism. The players simultaneously choose bids, \( b_1 \) and \( b_2 \). Think of \( b_1 \) as the price proposed by the seller and \( b_2 \) as the price proposed by the buyer. If \( b_1 > b_2 \), the demands are incompatible, so we set \( q = y = 0 \). If \( b_1 \leq b_2 \), then \( q = 1 \) and we determine the price by splitting the difference — that is, \( y = (b_1 + b_2)/2 \).

(a) Find an equilibrium of this game. (Hint: Look for one where \( b_1^*(\theta_1) = \alpha_1 + \beta_1\theta_1 \) and \( b_2^*(\theta_2) = \alpha_2 + \beta_2\theta_2 \) for some constants \( \alpha_i \) and \( \beta_i \).) What is the social choice function this equilibrium of this mechanism implements? Is it \textit{ex post} efficient?

(b) Using the social choice function from (a), show that this function is incentive compatible. What is the direct mechanism?

4. Suppose we have \( I \) bidders in an independent private values auction. The value of the good to bidder \( i \), \( \theta_i \), is distributed uniformly on \([0, 1]\), iid across bidders. Without using things we know about first or second price auctions, compute the seller’s expected revenue for any auction with the property that the good always goes to the bidder with the highest value and any bidder whose value is 0 has an expected payoff of 0.