Econ 703: Problem Set 5

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Spring 2015

Due: Tuesday April 21

1. Suppose we have \( I \) bidders with independent private values distributed uniformly on the interval \([0, 1]\). Suppose these bidders are risk averse — more specifically, if bidder \( i \)'s valuation is \( \theta_i \) and he gets the good at price \( p_i \), his payoff is \((\theta_i - p_i)^\alpha\) where \( \alpha \in (0, 1) \).

(Note: This is not always well defined. For example, if \( \theta_i < p_i \) and \( \alpha = 1/2 \), utility is not a real number. In such cases, assume that when \( \theta_i < p_i \), utility is \(-(p_i - \theta_i)^\alpha\). Alternatively, ignore this possibility — the answer won’t be affected.)

(a) Show that there is an equilibrium in the first price auction where \( i \) bids \( a\theta_i \) for some constant \( a \). Find the constant.

(b) Find an equilibrium for the second price auction.

(c) Do the two auctions yield the same expected revenues? If not, which yields more?

2. Suppose we have two bidders, \( A \) and \( B \), each of whom may want to purchase up to two units of a good. The seller has exactly two units to sell. Each bidder has diminishing marginal utility, so if she buys two units, the second one purchased does not have as high a value to her as the first. More specifically, let the valuation of bidder \( A \) for the \( i^{th} \) unit be \( v^A_i \) and similarly for \( B \). It is common knowledge that \( v^A_2 = 10 \) and \( v^B_2 = 5 \), but \( v^A_1 \) and \( v^B_1 \) are private information, known only to the bidder. The beliefs are that these valuations are independently distributed uniformly on \([200, 2000]\).

Suppose the seller uses two second–price auctions to sell the goods. That is, he uses a second–price auction to sell one unit, then the winner receives her unit, and then we have a second–price auction to sell the remaining unit. If \( A \) wins the first auction, what happens in the second? If \( B \) wins the first auction, what happens in the second? Given this, what is the maximum \( A \) would be willing to bid in the first auction? What is the maximum \( B \) would be willing to bid? What is the outcome?

3. Suppose we have two agents and the \( \theta_i \)'s are independent distributed uniformly on \([0, 1]\). Agent 1 is a seller and agent 2 a buyer, where \( \theta_1 \) is 1’s cost of producing a good.
and $\theta_2$ is the buyer’s valuation for this good. More specifically, a social choice, $x$, will be a pair $(q, y)$ where $q$ is the quantity of the good sold by the seller to the buyer (where this must be either 0 or 1) and $y$ is the payment made by the buyer to the seller. The utility functions are

$$u_1(q, y, \theta_1) = y - q\theta_1$$

and

$$u_2(q, y, \theta_2) = \theta_2q - y.$$ 

Consider the following mechanism. The players simultaneously choose bids, $b_1$ and $b_2$. Think of $b_1$ as the price proposed by the seller and $b_2$ as the price proposed by the buyer. If $b_1 > b_2$, the demands are incompatible, so we set $q = y = 0$. If $b_1 \leq b_2$, then $q = 1$ and we determine the price by splitting the difference — that is, $y = (b_1 + b_2)/2$.

(a) Find an equilibrium of this game. (Hint: Look for one where $b_1^*(\theta_1) = \alpha_1 + \beta_1\theta_1$ and $b_2^*(\theta_2) = \alpha_2 + \beta_2\theta_2$ for some constants $\alpha_i$ and $\beta_i$.) What is the social choice function this equilibrium of this mechanism implements? Is it ex post efficient?

(b) Using the social choice function from (a), show that this function is incentive compatible. What is the direct mechanism?

4. Consider the following two–player auction. There is one unit of a good. The value to bidder 1 is $2\theta_1 + \theta_2$, while the value to bidder 2 is $2\theta_2 + \theta_1$. Player $i$ knows only $\theta_i$. The common prior is that the $\theta_i$’s are independently distributed uniformly on $[0, 1]$.

(a) Suppose we have a first–price auction to allocate the good. Show that there is a Nash equilibrium where bidder $i$’s strategy is $\sigma_i(\theta_i) = \alpha\theta_i$ and find $\alpha$. What is the seller’s expected revenue?

(b) Suppose instead that we have a second–price auction to allocate the good. Show that there is again a Nash equilibrium where bidder $i$’s strategy is $\sigma_i(\theta_i) = \alpha\theta_i$ and find this $\alpha$. What is the seller’s expected revenue?