1. (a) Fix a bidder $i$ with valuation $\theta_i$ and assume all other bidders use a strategy of the form $\sigma_j(\theta_j) = a\theta_j$. Then $i$’s expected payoff to a bid of $b$ is

$$(\theta_i - b)^\alpha \left(\frac{b}{a}\right)^{I-1}.$$ 

The first–order condition is

$$-\alpha(\theta_i - b)^{\alpha-1} \left(\frac{b}{a}\right)^{I-1} + (I - 1)(\theta_i - b)^\alpha \left(\frac{b^{I-2}}{a^{I-1}}\right) = 0.$$ 

Rearranging, we get

$$b = \frac{I - 1}{I - 1 + \alpha} \theta_i.$$ 

So there is an equilibrium where everyone bids $a\theta_i$ for $a = (I - 1)/(I - 1 + \alpha)$.

(b) It’s still a dominant strategy to bid $\theta_i$. The dominant strategy argument has nothing to do with risk attitudes since it says you get a higher payoff for every possible bids by the other players.

(c) Since the equilibrium for the second price auction hasn’t changed, the expected revenue is still $(I - 1)/(I + 1)$. For the first price auction, the expected revenue is $(I - 1)/(I - 1 + \alpha)$ times the expected value of the first order statistic. That expected value is

$$\int_0^1 \theta I\theta^{I-1} d\theta = \frac{I}{I + 1}.$$ 

So the expected revenue in the first price auction is higher if

$$\frac{I(I - 1)}{(I + 1)(I - 1 + \alpha)} > \frac{I - 1}{I + 1}$$

or $\alpha < 1$ which holds by assumption. Expected revenue in the second price auction is independent of $\alpha$, while expected revenue in the first price auction is decreasing in $\alpha$. 

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2. Whoever wins the first auction certainly loses the second. So if A wins the first auction, B will win the second one and pay A’s valuation for a second unit, namely 10. If B wins the first auction, A wins the second and pays 5.

Given this, if A gets the first object, his payoff for the two auctions together is $v_A^1$ minus his payment. If he loses the first auction, his payoff is $v_A^1 - 5$. Hence the value to him of winning the first auction is 5 so this is what he bids. Similarly, B bids 10. Hence B wins the first auction paying 5 (the second highest bid), so A wins the second one, also paying 5. The seller’s revenue then is 10, regardless of the values of $v_A^1$ and $v_B^1$.

3. a) Suppose the seller knows the buyer is following the strategy of offering $b_2(\theta_2) = \alpha_2 + \beta_2\theta_2$. If the seller offers $b_1$, then he earns zero if $b_1 < \alpha_2 + \beta_2\theta_2$ and earns

$$\frac{b_1 + \alpha_2 + \beta_2\theta_2}{2} - \theta_1$$

if $\alpha_2 + \beta_2\theta_2 \geq b_1$. Hence, assuming $\beta_2 > 0$ (which we’ll see it will be), the seller’s expected payoff is

$$\int_{(b_1 - \alpha_2)/\beta_2}^1 \left[ \frac{b_1 + \alpha_2 + \beta_2\theta_2}{2} - \theta_1 \right] d\theta_2$$

if $(b_1 - \alpha_2)/\beta_2 \in [0, 1]$. For the moment, assume $b_1$ is in this range. Hence we can do the integration to rewrite the seller’s expected payoff as

$$\left\{ \frac{1}{2} [b_1 + \alpha_2] - \theta_1 \right\} \left[ 1 - \frac{b_1 - \alpha_2}{\beta_2} \right] + \frac{\beta_2}{4} \left[ 1 - \left( \frac{b_1 - \alpha_2}{\beta_2} \right)^2 \right].$$

Note that this is just a quadratic in $b_1$. I’ll omit the algebra, but if you maximize this with respect to $b_1$, you get

$$b_1 = \frac{2}{3} \theta_1 + \frac{1}{3} (\alpha_2 + \beta_2).$$

This only applies when this gives us a $b_1$ with $(b_1 - \alpha_2)/\beta_2 \in [0, 1]$ — that is, $b_1 \geq \alpha_2$ and $b_1 \leq \alpha_2 + \beta_2$. As we’ll see momentarily, the first of these inequalities will always be satisfied. The second is trickier though. Intuitively, if the second one is violated, so $b_1 > \alpha_2 + \beta_2$, this says that the seller is offering a price that is above every offer the buyer will make and so is certain to not be accepted. Note that the seller’s offer is a convex combination of $\theta_1$ and $\alpha_2 + \beta_2$. So if his bid ends up above $\alpha_2 + \beta_2$, it must be true that his bid is below $\theta_1$. That is, he is offering to trade at a price below his costs. Hence his optimal bid in this case is anything that isn’t accepted and $\alpha_2 + \beta_2$ is good enough.

So now consider the buyer. He knows that the seller’s bid will take the form $b_1 = \alpha_1 + \beta_1\theta_1$. Again assuming $\beta_1 > 0$, this means that his expected payoff can be written as

$$\int_{0}^{(b_2 - \alpha_1)/\beta_1} \left[ \theta_2 - \frac{1}{2} (b_2 + \alpha_1 + \beta_1\theta_1) \right] d\theta_1$$

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assuming \((b_2 - \alpha_1)/\beta_1 \in [0, 1]\). Performing the integration, we get

\[
[\theta_2 - \frac{1}{2} \alpha_1 - \frac{1}{2} b_2, b_2 - \alpha_1] \frac{1}{\beta_1} - \frac{\beta_1}{4} \left( \frac{b_2 - \alpha_1}{\beta_1} \right)^2.
\]

Again, this is a quadratic in 2’s strategy. Maximizing with respect to \(b_2\) yields

\[
b_2 = \frac{2}{3} \theta_2 + \frac{1}{3} \alpha_1.
\]

Analogously to the above, this only applies if \(b_2\) is in the right range. For the same reasons as above, though, we don’t need to worry about these conditions.

Summarizing, we get \(b_1 = \beta_1 \theta_1 + \alpha_1\) where \(\beta_1 = 2/3\) and \(\alpha_1 = (1/3)(\alpha_2 + \beta_2)\). Also, \(b_2 = \beta_2 \theta_2 + \alpha_2\) where \(\beta_2 = 2/3\) and \(\alpha_2 = (1/3)\alpha_1\). Solving these equations simultaneously, we see that \(\beta_1 = \beta_2 = 2/3\), \(\alpha_1 = 1/4\), and \(\alpha_2 = 1/12\).

To see what social choice function this implements, note that it implements \(q = y = 0\) if \(b_1 > b_2\) or

\[
\frac{2}{3} \theta_1 + \frac{1}{4} > \frac{2}{3} \theta_2 + \frac{1}{12}.
\]

That is, we implement \(q = y = 0\) if

\[
\theta_2 - \theta_1 < \frac{1}{4}.
\]

If \(\theta_2 - \theta_1 > \frac{1}{4}\), we implement \(q = 1\) and

\[
y = \frac{b_1 + b_2}{2} = \frac{1}{3} (\theta_1 + \theta_2) + \frac{1}{6}.
\]

This social choice function is not \(ex\ post\) efficient, since \(ex\ post\) efficiency would call for \(q = 1\) whenever \(\theta_2 > \theta_1\), not just when \(\theta_2 > \theta_1 + (1/4)\).

b) Take the direct mechanism where seller and buyer both report their \(\theta\)’s. Let \(\hat{\theta}_i\) denote the reported value of \(\theta_i\) (since this might differ from the true value). The seller’s payoff as a function of his true \(\theta_1\) and his report is

\[
\int_{\theta_1+(1/4)}^{1} \frac{1}{3} \left[ \frac{1}{2} \left[ \hat{\theta}_1 + \theta_2 \right] + \frac{1}{6} - \theta_1 \right] d\theta_2.
\]

Ignoring (as before) the implicit constraint that this only applies when \(\hat{\theta}_1 + (1/4) \in [0, 1]\), if we perform the integration, we get

\[
\left[ \frac{1}{3} \hat{\theta}_1 + \frac{1}{6} - \theta_1 \right] \left[ \frac{3}{4} - \hat{\theta}_1 \right] + \left( \frac{1}{6} \right) \left[ 1 - \left( \hat{\theta}_1 + \frac{1}{4} \right)^2 \right].
\]
This is a quadratic in $\hat{\theta}_1$. The first order condition for maximizing this with respect to $\hat{\theta}_1$ is
\[
\frac{1}{4} - \frac{1}{6} + \theta_1 - \frac{2}{3} \hat{\theta}_1 - \frac{1}{3} \left( \hat{\theta}_1 + \frac{1}{4} \right) = 0
\]
which is easily rearranged to $\hat{\theta}_1 = \theta_1$.

Similarly, the buyer’s payoff as a function of his true $\theta_2$ and his report is
\[
\int_{0}^{\theta_2-(1/4)} \left[ \theta_2 - \frac{1}{3} (\theta_1 + \hat{\theta}_2) - \frac{1}{6} \right] d\theta_1.
\]
Performing the integration gives
\[
\left[ \theta_2 - \frac{1}{3} \hat{\theta}_2 - \frac{1}{6} \right] \left[ \hat{\theta}_2 - \frac{1}{4} \right] - \frac{1}{6} \left( \hat{\theta}_2 - \frac{1}{4} \right)^2.
\]
Again, the first order condition for the optimal choice of $\hat{\theta}_2$ reduces to $\hat{\theta}_2 = \theta_2$.

4. (a) For concreteness, consider player 1. (Of course, the game is symmetric, so we would get the same analysis if we focus on player 2.) Fix a particular value of $\theta_1$ and a particular bid $b$. Given that player 2 is bidding $\alpha \theta_2$, 1 knows that he wins with the bid $b$ iff $b \geq \alpha \theta_2$ (where we may as well use a weak inequality here since the probability of a tie is zero) or $\theta_2 \leq b/\alpha$. Thus 1’s expected payoff to bidding $b$ is
\[
\int_{0}^{b/\alpha} \left[ 2\theta_1 + \theta_2 - b \right] d\theta_2 = \frac{b}{\alpha} (2\theta_1 - b) + \frac{1}{2} \left( \frac{b}{\alpha} \right)^2.
\]
(This assumes $0 \leq b/\alpha \leq 1$, but it’s not hard to show that the optimal $b$ will satisfy this.) So 1 should choose $b$ to maximize this. The first–order condition is
\[
\frac{2\theta_1}{\alpha} - \frac{2b}{\alpha} + \frac{b}{\alpha^2} = 0.
\]
Rearranging:
\[
2\alpha \theta_1 = b(2\alpha - 1)
\]
or
\[
b = \frac{2\alpha}{2\alpha - 1} \theta_1.
\]
So the optimal strategy for 1 is to bid $\alpha \theta_1$ if
\[
\alpha = \frac{2\alpha}{2\alpha - 1}
\]
or $2\alpha = 3$ or $\alpha = 3/2$. 
So the seller’s expected revenue is
\[
\frac{3}{2} \mathbb{E} [\max \{\theta_1, \theta_2\}] = \frac{3}{2} \int_0^1 2k^2 \, dk = 1.
\]

(b) Again, consider player 1. Now his payoff as a function of \(\theta_1\) and \(b\) is
\[
\int_0^{b/\alpha} [2\theta_1 + \theta_2 - \alpha \theta_2] \, d\theta_2 = \frac{b}{\alpha} 2\theta_1 + \frac{1 - \alpha}{2} \left( \frac{b}{\alpha} \right)^2.
\]
The first-order condition is
\[
\frac{2}{\alpha} \theta_1 + \frac{1 - \alpha}{\alpha^2} b = 0
\]
so
\[
b = \frac{2\alpha}{\alpha - 1} \theta_1.
\]
So now we have
\[
\alpha = \frac{2\alpha}{\alpha - 1}
\]
so \(2 = \alpha - 1\) or \(\alpha = 3\).

So the seller’s expected revenue is
\[
3 \mathbb{E} [\min \{\theta_1, \theta_2\}] = 3 \int_0^1 2[1 - k]k \, dk
\]
\[
= 6 \left[ \int_0^1 k \, dk - \int_0^1 k^2 \, dk \right]
\]
\[
= 6 \left[ \frac{1}{2} - \frac{1}{3} \right] = 3 - 2 = 1.
\]