Answers to Problem Set 3

Economics 703

Spring 2018

1. Now the best way to induce effort $e$ is to use the wage function $w(\cdot)$ which maximizes

$$\int U(\pi - w(\pi)) f(\pi \mid e) \, d\pi$$

subject to

$$\int v(w(\pi)) f(\pi \mid e) \, d\pi \geq g(e) + \bar{u}$$

and

$$\int v(w(\pi)) f(\pi \mid e) \, d\pi - g(e) \geq \int v(w(\pi)) f(\pi \mid \hat{e}) \, d\pi - g(\hat{e})\).$$

The first order conditions can be written as

$$\frac{U'(\pi - w(\pi))}{v'(w(\pi))} = \gamma + \mu \left[ 1 - \frac{f(\pi \mid \hat{e})}{f(\pi \mid e)} \right].$$

Thus we get a distortion away from optimal risk sharing analogous to the distortion we saw in the case where the principal is risk neutral.

2. (a) When effort is not observable, it is impossible to induce the agent to choose effort level $e_2$. To see this, let’s suppose we can induce this effort level. Suppose $w(\cdot)$ is a wage contract which leads the agent to choose effort $e_2$. Following the hint, let $v_1 = v(w(\pi_H))$ and let $v_2 = v(w(\pi_L))$. Then if this induces $e_2$, it must be true that

$$\frac{2}{3} v_1 + \frac{1}{3} v_2 - g(e_1) \leq \frac{1}{2} v_1 + \frac{1}{2} v_2 - g(e_2)$$

and

$$\frac{1}{3} v_1 + \frac{2}{3} v_2 - g(e_3) \leq \frac{1}{2} v_1 + \frac{1}{2} v_2 - g(e_2).$$

Rearranging the first inequality yields

$$\frac{1}{6} (v_1 - v_2) \leq g(e_1) - g(e_2),$$
while the second one gives

\[
\frac{1}{6}(v_1 - v_2) \geq g(e_2) - g(e_3).
\]

(Intuitively, this just says that the marginal product of extra effort is below the marginal cost when moving from \(e_2\) up to \(e_1\) but is worth it in moving from \(e_3\) up to \(e_2\). The symmetry of the problem makes these marginal products the same in the two cases.) Clearly, then, we must have \(g(e_1) - g(e_2) \geq g(e_2) - g(e_3)\) or \(g(e_2) \leq (g(e_1) + g(e_3))/2\). But \((g(e_1) + g(e_3))/2 = 1.5\), while \(g(e_2) = 1.6\). Hence effort \(e_2\) cannot be induced. If \(g(e_2)\) were less than 1.5, then it could be implemented.

(b) Now that we know that \(e_2\) cannot be achieved, finding the optimal contract is a matter of characterizing the choice between \(e_1\) and \(e_3\). We know that a constant wage will lead the agent to choose \(e_3\), so the principal’s profits from inducing this effort level are the same as those calculated in (a). Hence we only need to consider the cost of inducing effort level \(e_1\). Using the same notation as in part (b), we see that we need to choose the wage so that

\[
\frac{2}{3}v_1 + \frac{1}{3}v_2 - g(e_1) \geq \frac{1}{2}v_1 + \frac{2}{3}v_2 - g(e_2)
\]

and

\[
\frac{2}{3}v_1 + \frac{1}{3}v_2 - g(e_1) \geq \frac{1}{3}v_1 + \frac{2}{3}v_2 - g(e_3).
\]

Rearranging the first inequality yields

\[
\frac{1}{6}(v_1 - v_2) \geq \frac{1}{15}
\]

or \(v_1 - v_2 \geq 2/5\). Rearranging the second one gives

\[
\frac{1}{3}(v_1 - v_2) \geq \frac{1}{3}
\]

or \(v_1 - v_2 \geq 1\). Because this constraint is more severe, we can ignore the first one and focus only on this one. This means that the incentive constraint that \(e_1\) be better than \(e_2\) cannot be binding. Hence we can treat this just like a two effort problem with just effort levels \(e_1\) and \(e_3\). Hence we can use the first order condition from class:

\[
\frac{1}{w'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi \mid e_3)}{f(\pi \mid e_1)}\right].
\]

Since \(v(w) = w^{1/2}\), \(1/v'(w) = 2w^{1/2}\). Hence, letting \(w_1 = w(\pi_H)\) and \(w_2 = w(\pi_L)\),

\[
2w_1^{1/2} = \gamma + \mu \left[1 - \frac{1/3}{2/3}\right]
\]

or

\[
w_1 = \left(\frac{\gamma}{2} + \frac{\mu}{4}\right)^2.
\]
Similar calculations yield
\[ w_2 = \left( \frac{\gamma - \mu}{2} \right)^2. \]
Recall that both constraints are binding. Hence \( v_1 - v_2 = 1 \). That is,
\[ \frac{\gamma}{2} + \frac{\mu}{4} - \frac{\gamma}{2} + \frac{\mu}{2} = 1 \]
so \( \mu = 4/3 \). The individual rationality constraint is
\[ \frac{2}{3}v_1 + \frac{1}{3}v_2 \geq \frac{5}{3}. \]
Since it must bind, \( 2v_1 + v_2 = 5 \). Hence
\[ 2\left( \frac{\gamma}{2} + \frac{\mu}{4} \right) + \frac{\gamma}{2} - \frac{\mu}{2} = 5 \]
So \( \gamma = 10/3 \). Hence \( w(\pi_H) = 4 \) and \( w(\pi_L) = 1 \).

Which effort should the principal induce? His profits to inducing \( e_3 \), as calculated above, are about 1.56. His profits to inducing \( e_1 \) are approximately 3.67, which is still better than the profits to inducing \( e_3 \).

3. With the limited liability constraint, the contract we used in the Problem Set 2 is no longer feasible. To see this, note that given the function we used, we have
\[ w(\pi_F) = \pi_F - \max\{\pi_S - c, p\pi_S + (1 - p)\pi_F\} \leq \pi_F - (p\pi_S + (1 - p)\pi_F) = p(\pi_F - \pi_S) < 0. \]
Let calculate the optimal contract by finding the best way to induce each level of effort and then find the best effort.

To induce the low effort, the principal can simply pay a flat wage of 0. This will induce the agent to choose low effort and leaves him indifferent between accepting and rejecting the contract. Clearly, the principal cannot induce \( e_L \) at a lower cost.

Let’s turn to the high effort then. So the principal chooses \( w_S \) (short for \( w(\pi_S) \)) and \( w_F \) (or \( w(\pi_F) \)) to minimize \( w_S \) subject to \( w_S \geq 0, w_F \geq 0 \),
\[ w_S - c \geq 0 \]
and
\[ w_S - c \geq pw_S + (1 - p)w_F \]
The non-negativity constraints together with the last constraint imply \( w_S - c \geq 0 \) so we can ignore this constraint. The last constraint says \( w_S \geq w_F + c/(1 - p) \). Since \( c/(1 - p) > 0 \) and we have to have \( w_F \geq 0 \), we see that the constraint that \( w_S \geq 0 \) also must hold and so can be ignored. In short, the principal minimizes \( w_S \) subject to
\[ w_S \geq w_F + \frac{c}{1 - p} \]
\[ w_F \geq 0. \]

Clearly, the solution is \( w_F = 0 \) and \( w_S = c/(1-p) \).

Just as in the model discussed in class, we see that the cost of inducing the low effort is the same with or without the limited liability constraint, while the cost of inducing the high effort is higher with limited liability since it goes from \( c \) to \( c/(1-p) > c \). Thus if the induced effort changes when we add the constraint, it changes from \( e_H \) to \( e_L \).

4. Clearly, a constant wage of 0 is the cheapest way to induce the agent to choose low effort on both tasks. So the profit to this is still \( C \). Also, because effort on task 2 is effectively observable, the cheapest way to induce high effort on this task still costs \( g^2 \): the principal tells the agent he will receive 0 (or worse) if \( \pi_2 \neq B \) and \( g^2 \) otherwise. This again makes the payoff to this option equal to \( B - g^2 \). Since \( B - C > g^2 \) by assumption, this is better than low effort on both tasks.

Inducing high effort on task 1 is more complex. Here we have to minimize costs subject to the individual rationality constraint

\[
\frac{1}{2} \sqrt{w_A} + \frac{1}{2} \sqrt{w_0} - g \geq 0
\]

and the incentive constraint

\[
\frac{1}{2} \sqrt{w_A} + \frac{1}{2} \sqrt{w_0} - g \geq \sqrt{w_0}.
\]

Explanation: The principal should insist on \( \pi_2 = C \) (otherwise, he knows that the agent put low effort into task 1). Conditional on this, he has to decide how much to pay when \( \pi_1 = A \) and how much to pay when \( \pi_1 = 0 \). I’m using \( w_A \) to denote the wage in the first case and \( w_0 \) for the second.

We know that both constraints are binding at the optimum. Hence

\[
\frac{1}{2} \sqrt{w_A} + \frac{1}{2} \sqrt{w_0} - g = \sqrt{w_0}.
\]

The second equality tells us that \( w_0 = 0 \). Plugging this into the first equality, we get \( \sqrt{w_A} = 2g \) or \( w_A = 4g^2 \). The principal’s expected payoff from this is

\[
\frac{1}{2} [A - 4g^2] + C.
\]

For his decision in (b) to be different from that in (a), the payoff to inducing high effort on task 2 must be larger than this or

\[
\frac{1}{2} A - (B - C) < g^2.
\]

5. (a) This change gives the principal more options but under the assumptions given, these extra options have no value. To see this, first note that the principal can now offer
only one contract, say \((w_H, e_H)\), chosen for only for the high type and “shut down” if it’s the low type. That is, he can set this so that

\[
w_H - g(e_H, \theta_H) \geq 0 \geq w_H - g(e_H, \theta_L).
\]

Note that he cannot set the contract to induce the low type to work but the high type to quit since the high type earns more from any given contract than the low type. The best contract of this form would maximize

\[
\lambda[\pi(e_H) - w_H] + (1 - \lambda)(0)
\]

subject to the constraint above. But note that this gives the principal the same payoff as in the original formulation of the model but with \(e_L = w_L = 0\). Since \(\bar{u} = 0\) and \(v(0) = 0\), this contract is actually available to the principal in the original model. Thus even though we seemed to constrain the principal to deal with both types, he could deal with the low type only in the trivial sense of offering him no money in exchange for no work.

(b) Now the argument of (a) breaks down. If the principal does try to get the low type to take the contract but not take effort, he still has to pay that agent \(g(0, \theta_L) > 0\) which is worse than what happens if he induces him to turn down the contract instead. Essentially the same thing happens if \(g(0, \theta_L) = 0\) but \(v^{-1}(\bar{u}) > 0\). Again, the principal has to pay the low type a strictly positive amount which is worse than what happens if the principal instead induces the low type to reject the contract.

I’ll focus my comments on the fixed cost case, setting \(v^{-1}(\bar{u}) = 0\), but it’s easy to rewrite for the case where \(g(0, \theta_L) = 0\) but \(v^{-1}(\bar{u}) > 0\).

It is easy to see that the best contract which excludes the low type is to set \(e_H = e_H^*\) and \(w_H = g(e_H^*, \theta_H)\). Hence this is optimal whenever \(\lambda[\pi(e_H^*) - g(e_H^*, \theta_H)]\) exceeds the profits from the usual second-best. The latter is

\[
\lambda[\pi(e_H^*) - g(e_H^*, \theta_H)] - \lambda[g(e_L^{**}, \theta_L) - g(e_L^{**}, \theta_H)] + (1 - \lambda)[\pi(e_L^{**}) - g(e_L^{**}, \theta_L)].
\]

Hence the new approach is better iff

\[
\lambda[g(e_L^{**}, \theta_L) - g(e_L^{**}, \theta_H)] \geq (1 - \lambda)[\pi(e_L^{**}) - g(e_L^{**}, \theta_L)].
\]

In other words, it’s better to “get rid of” the low type when the expected rents he forces the owner to pay the high type exceeds the profits the owner earns from him. Intuitively, this is likely to hold as the low type becomes more and more inefficient. The more inefficient is the low type, the higher the rents the high type earns from imitating him and the lower the surplus the owner gets from dealing with him. For example, if the low type is so inefficient that \(\max_e \pi(e) - g(e, \theta_L)\) is negative (which is possible with the fixed cost), then certainly this new option would be better.
6. (a) If the owner can observe $\theta$, he will pay the worker $e/\theta$ and choose $e$ to maximize $2\sqrt{e} - (e/\theta)$. The first–order condition is $e^{-1/2} = \theta^{-1}$ so $e = \theta^2$. So the optimal contract is $e_j = \theta_j^2$ and $w_j = \theta_j$ for $j = L, H$. The owner’s expected profits are $p\theta_H + (1 - p)\theta_L$.

(b) If the owner cannot observe $\theta$, he will choose $e_H$, $w_H$, $e_L$, and $w_L$ to maximize his expected profits subject to the individual rationality and incentive compatibility constraints:

$$w_j - \frac{e_j}{\theta_j} \geq 0, \quad j = L, H$$

$$w_L - \frac{e_L}{\theta_L} \geq w_H - \frac{e_H}{\theta_L}$$

$$w_H - \frac{e_H}{\theta_H} \geq w_L - \frac{e_L}{\theta_H}.$$

As usual, the only binding constraints are the individual rationality constraint for type $\theta_L$ and the incentive compatibility constraint for type $\theta_H$. Hence

$$w_L = \frac{e_L}{\theta_L}$$

$$w_H = \frac{e_H}{\theta_H} + \frac{e_L}{\theta_L} - \frac{e_L}{\theta_H}.$$

Substituting these results into the objective function, the owner will choose $e_L$ and $e_H$ to maximize

$$p \left[ 2\sqrt{e_H} - \frac{e_H}{\theta_H} - \frac{e_L}{\theta_L} + \frac{e_L}{\theta_H} \right] + (1 - p) \left[ 2\sqrt{e_L} - \frac{e_L}{\theta_L} \right]$$

The first–order condition for $e_H$ is the same as in the first–best (as usual), so $e_H = \theta_H^2$. The first–order condition for $e_L$ is

$$(1 - p) \left[ \frac{1}{\sqrt{e_L}} - \frac{1}{\theta_L} \right] = p \left[ \frac{1}{\theta_L} - \frac{1}{\theta_H} \right].$$

Rearranging gives $e_L = [(1 - p)\theta_H \theta_L / (\theta_H - p \theta_L)]^2$. The wages are obtained by substituting for $e_L$ and $e_H$ into the equations above.

(c) When there is a probability that the owner observes the manager’s type, the problem changes dramatically. Recall that a contract can depend on anything mutually observed. Hence a contract can depend on the observation made by the owner, if any, as well as the reported type by the manager. Suppose the contract says that if the owner observes that the manager’s type is not what he claimed, then the manager is paid $-\infty$. Then for any strictly positive probability that his type is observed, the manager will not lie. Hence we are back into the first–best world and the contracts are as in (a) with the provision that the manager receives a bad payoff if he is caught lying.