1. The expected quality conditional on the wage is
\[ E[\theta \mid r(\theta) \leq w] = E[\theta \mid \theta \leq w/a]. \]

Let \( H(w) \) denote the function on the right-hand side. One has to be careful to distinguish cases where \( w/a \) is in the interval \([\theta, \bar{\theta}]\) from those cases where it is outside this range. First, suppose \( w/a \leq \bar{\theta} \). When this inequality is strict, the expectation is not well defined since we’re conditioning on a zero probability event. When \( w/a = \bar{\theta} \), the expectation is \( \theta \). That is, \( H(a\theta) = \theta \). Since \( a < 1 \), \( a\theta < \theta \). So \( w < H(w) \) at \( w = a\theta \). Hence at this \( w \), demand is infinite and supply is zero, so we don’t have an equilibrium. Note that the same conclusion would follow if we defined \( H(w) = \theta \) for \( w < a\theta \).

Next consider the possibility that \( w/a \geq \bar{\theta} \). For any such \( w \), \( H(w) \) is the unconditional expectation of \( \theta \) which is \( \frac{1}{2}(\theta + \bar{\theta}) \). So is \( H(w) \) above or below \( w \) at \( w = a\theta \)? It is above the diagonal if and only if
\[ a\bar{\theta} \leq \frac{1}{2}(\theta + \bar{\theta}). \]

When this holds, \( H(w) > w \) for all \( w \in [a\theta, a\bar{\theta}] \). \( H \) then stays at the unconditional expectation for all higher \( w \), so the unique equilibrium in this case is where \( w = \frac{1}{2}(\theta + \bar{\theta}) \), the unconditional expectation. Here all workers are hired.

So suppose that (1) does not hold. In this case, the \( H \) function crosses the diagonal somewhere between \( w = a\theta \) and \( w = a\bar{\theta} \). In this range, we have
\[ H(w) = \frac{1}{2} \left[ \frac{w}{a} + \theta \right]. \]
So \( H(w) = w \) only at the point where
\[ 2aw = w + a\theta \]
or \( w = a\bar{\theta}/(2a - 1) \). How do we know this is well-defined — that is, that \( a \neq 1/2 \)? By hypothesis, (1) does not hold, so we must have
\[ a\bar{\theta} > \frac{1}{2}(\theta + \bar{\theta}) \]
or $(2a - 1)\bar{\theta} > \theta$. Because both $\bar{\theta}$ and $\theta$ are strictly positive, this requires $2a > 1$.

In short, we have two possibilities. If (1) holds, the unique equilibrium wage is the unconditional expectation. If (1) does not hold, it is $a\bar{\theta}/(2a - 1)$. It is not hard to show that the equilibrium wage is weakly decreasing in $a$. It’s strictly decreasing in $a$ in the region where (1) does not hold. But once $a$ is small enough that (1) holds, the wage no longer depends on $a$, so it’s only weakly decreasing in $a$ in general. Similarly, the wage is weakly increasing in $\bar{\theta}$.

If we set $\theta = 0$, our first step changes. Above, we said that $a\theta < \theta$, but this is not true when $\theta = 0$. Hence we see that $w = 0$ is an equilibrium since $H(0) = \bar{\theta} = 0$. Also, in this case, (1) holds iff $a \leq 1/2$. If $a < 1/2$, $H$ is strictly above the diagonal for all $w \in (0, a\bar{\theta})$. As a result, we have two equilibria: $w = 0$ and $w$ equal to the unconditional expectation. If $a > 1/2$, $H$ is strictly below the diagonal for all $w > 0$, so the unique equilibrium is at $w = 0$.

Finally, suppose $\theta = 0$, $\bar{\theta} = 1$, and $r(\theta) = a\theta - b$. Now the conditional expectation we are interested in is

$$H(w) = E \left[ \theta \mid \theta \leq \frac{w + b}{a} \right].$$

For all $w \geq 0$, $(w + b)/a \geq 0$. For all $w \leq 1/2$, $(w + b)/a < 1$ by the assumption that $a > b + .5$. Since $1/2$ is the unconditional expectation, this tells us that $(w + b)/a \leq 1$ for the “relevant range” of values of $w$. Since $H(0) > 0$ and $H(1/2) < 1/2$, there is a unique $w \in (0, 1)$ satisfying the equilibrium condition. This $w$ is defined by $w = (1/2)(w + b)/a$, so $w = b/(2a - 1)$. Now the wage is increasing in $b$ and decreasing in $a$.

2. (a) In the complete information case, the interest rate will depend on the type of the entrepreneur. Let $r_i$ be the interest rate for type $i$, where $i$ is either $h$ or $\ell$. By assumption, this interest rate is determined by $p_i(1 + r_i)L - L = 0$ or $1 + r_i = 1/p_i$. The entrepreneur takes the loan if

$$p_i[S - (1 + r_i)L] + (1 - p_i)D \geq 0.$$ 

Substituting for $1 + r_i$ and rearranging gives

$$p_iS + (1 - p_i)D - L \geq 0.$$ 

By assumption, $p_hS + (1 - p_h)D - L \geq D > 0$, so bad entrepreneurs take the loan. By assumption, $p_h > p_\ell$ and $S > D$, so good entrepreneurs take the loan also.

(b) There are three possibilities for equilibrium. First, it could be that only the $h$ types take the loan. In this case, we know from the above that $1 + r = 1/p_h$. Second, it could be true that both types take the loan. In this case, we have $1 + r = 1/\bar{p}$ where $\bar{p} = \lambda p_h + (1 - \lambda)p_\ell$. Finally, it could be true that only the $\ell$ types take the loan. In this case, $1 + r = 1/p_\ell$. 

2
Is the first an equilibrium? From the above, we know that the $h$ types would take this loan. What about the $\ell$ types? They would take the loan if

$$p_\ell S + (1 - p_\ell) D - \frac{p_\ell}{p_h} L \geq 0.$$  

Because $p_\ell < p_h$, the left–hand side is strictly larger than $p_\ell S + (1 - p_\ell) D - L$ which we know is positive. Hence the $\ell$ types take the loan also, contradicting the hypothesis that only the $h$ types do. Hence this is not an equilibrium.

When is the second an equilibrium? Low types take the loan if

$$p_\ell S + (1 - p_\ell) D \geq \frac{p_\ell}{\bar{p}} L.$$  

Because $p_\ell < p_h$ implies $p_\ell < \bar{p}$, again, low types take the loan. High types take the loan if

$$p_h S + (1 - p_h) D \geq \frac{p_h}{\bar{p}} L.$$  

Rearranging, this is

$$\lambda p_h + (1 - \lambda)p_\ell \geq \frac{p_h L}{p_h S + (1 - p_h) D}.$$  

By assumption, this does not hold. Hence only low types take the loan, contradicting the hypothesis that both types do. Hence this is not an equilibrium either.

Finally, is the last an equilibrium? From (a), we know that the low types would take the loan. The high types would not take the loan in part (b) and the interest rate here is higher. Hence they will not take the loan. So this is the unique equilibrium.

3. (a) The proof is the same as the one we gave in class. Simply replace $L(s)$ in the proof from class with $L^2(s)$ everywhere.

(b) It is quite possible to have the event that $(E_1 L(s))^2 > \alpha > (E_2 L(s))^2$ be common knowledge. For example, suppose $S = \{a, b\}$, $L(a) = -1$, and $L(b) = 1$. Suppose the priors put probability 1/2 on each state. Suppose the partitions are $\Pi_1 = \{\{a\}, \{b\}\}$ and $\Pi_2 = \{\{a, b\}\}$. Then

$$(E_1 L(a))^2 = (E_1 L(b))^2 = 1$$  

and

$$(E_2 L(a))^2 = (E_2 L(b))^2 = 0.$$  

So the square of the value 1 puts on $L$ is always strictly larger than the square of the value 2 puts on $L$. Since it’s true in every state, it is common knowledge. Note that this relies on the fact that 1’s valuation for $L$ is negative in state $a$. So while it is not common knowledge that 1’s expected value is higher than 2’s, it is common knowledge that the square of 1’s expected value is higher than the square of 2’s.

4. While there are surely a variety of ways one could formalize this, here’s my choice. As before, let $S$ be the set of states. A state will specify who committed the murder as well.
as how, what kind of evidence they left behind, etc. Let $\Pi_1$ and $\Pi_2$ be the information partitions for the two detectives. The interpretation would be that these describe the information the two detectives end up with after tracking down evidence as best they can. Assume they have a common prior $\mu$. Let $E_A$ be the set of states where $A$ committed the murder (so $B$ committed the crime in all states not in $E_A$). $A$ would be a detective’s prime suspect if the probability he gives to $E_A$ is (strictly) greater than $1/2$. If the probability is strictly below $1/2$, then $B$ would be the prime suspect. Otherwise, both are prime suspects.

So could they have different prime suspects? Let $p_i(A \mid s)$ be the probability detective $i$ assigns to $A$ having committed the murder when the true state is $s$. That is,

$$p_i(A \mid s) = \frac{\sum_{s' \in \pi_i(s) \cap E_A} \mu(s')} {\sum_{s' \in \pi_i(s)} \mu(s')} .$$

Let

$$F_1 = \{ s \in S \mid p_1(A \mid s) > \frac{1}{2} > p_2(A \mid s) \} .$$

So $F_1$ is the set of states where $A$ is 1’s prime suspect but $B$ is 2’s prime suspect. If you take the proof we did in class, set $\alpha = 1/2$, and set $L(s) = 1$ for all $s$, you get a proof that $F_1$ can never be common knowledge. Similarly, if we let

$$F_2 = \{ s \in S \mid p_1(A \mid s) = \frac{1}{2} > p_2(A \mid s) \} ,$$

then $F_2$ can never be common knowledge. This is the set of states where 1 sees both $A$ and $B$ as equally likely to have committed the murder, while 2 sees $B$ as the prime suspect. The argument here is essentially the one we did in class replacing $>$ with $=$ when looking at person 1. One can go through the other combinations of possibilities to see that if it’s common knowledge between them who their prime suspects are, then the prime suspects must be the same.