1. Consider the adverse selection model we looked at in class. Suppose that $\theta$ is distributed uniformly on $[\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} < \bar{\theta}$. Assume that $r(\theta) = a\theta$ where $0 < a < 1$. What is the unique equilibrium price and average quality as a function of $a$, $\underline{\theta}$, and $\bar{\theta}$? How do you know the equilibrium is unique? How does the price depend on $a$, $\underline{\theta}$, and $\bar{\theta}$ (is it increasing, decreasing, or ambiguous as a function of the parameter)? What happens if $\underline{\theta} = 0$? What if we assume $\theta$ is uniformly distributed between 0 and 1 and change $r$ to $r(\theta) = a\theta - b$ where $0.5 + b < a < 1$ and $b > 0$?

2. Suppose there are two types of entrepreneurs in the world who differ only in terms of how likely their project is to succeed. More specifically, each entrepreneur has a project which pays off $S > 0$ if it succeeds and 0 if it fails. Good entrepreneurs have a project that has a probability of $p_h$ of success, while the projects of bad entrepreneurs have probability $p_\ell$ of success where $0 < p_\ell < p_h < 1$. A fraction $\lambda \in (0, 1)$ of the entrepreneurs are good. To do the project, the entrepreneur must get a loan of $L > 0$. If an entrepreneur’s project succeeds, he gets $S$ minus $L$ and the interest on the loan. He cannot default on the loan in this case. If his project fails, he defaults and gets $D > 0$, while the bank gets nothing. If he doesn’t get a loan at all, his payoff is 0. Assume that $S - L > D$. Also, assume

$$p_\ell S + (1 - p_\ell)D - L > 0$$

and

$$\lambda p_h + (1 - \lambda)p_\ell < \frac{p_h L}{p_h S + (1 - p_h)D}.$$ 

Banks are risk neutral and compete so that their expected profits are zero. If the borrower borrows $L$ and repays with interest, the bank gets $L(1 + r)$. If the borrower defaults, the bank gets 0. If the bank doesn’t make the loan, its payoff is $L$.

(a) Suppose the bank sees the type of the entrepreneur before setting the interest rate. What interest rate will be offered to an entrepreneur? Under what conditions does he take the loan?
(b) Now suppose the bank cannot see the type of the entrepreneur. What is the equilibrium outcome and how does it depend on $\lambda$?

3. Consider the Glosten–Milgrom model with noise traders. Suppose there are $N$ buy orders and $M$ sell orders (in any particular sequence). Show that the buying and selling prices depend only on $N - M$, not $N$ or $M$ separately. Let $b_T$ be the buying price after seeing $N$ buys and $M$ sells where $T = N - M$. Show that $b_T \rightarrow 1$ as $T \rightarrow \infty$.

4. Suppose we have a finite state space $S$ and two agents with information partitions $\Pi_1$ and $\Pi_2$. Let $\mu$ be the common prior and $\pi_i(s)$ the event in $\Pi_i$ containing $s$. Suppose we have a random variable $L : S \rightarrow \mathbb{R}$. Let $E_iL(s)$ be the expected value of $L$ by person $i$ when the state is $s$ — that is,

$$ E_iL(s) = \sum_{s' \in \pi_i(s)} \frac{\mu(s) L(s)}{\mu(\pi_i(s))}. $$

Similarly, let $E_iL^2(s)$ be the expected value of $L$ squared by person $i$ when the state is $s$. In other words,

$$ E_iL^2(s) = \sum_{s' \in \pi_i(s)} \frac{\mu(s) L^2(s)}{\mu(\pi_i(s))}. $$

For any event $E \subseteq S$, let $CK(E)$ be the set of states where it is common knowledge that the true state is in $E$.

(a) Show that for all $\alpha$,

$$ CK \left( \{ s \in S \mid E_1L^2(s) > \alpha > E_2L^2(s) \} \right) = \emptyset. $$

(b) Is it necessarily true that for all $\alpha$,

$$ CK \left( \{ s \in S \mid (E_1L(s))^2 > \alpha > (E_2L(s))^2 \} \right) = \emptyset? $$

If so, prove this. If not, give a counterexample.