Answers to Final Examination

Economics 703

Spring 2015

1. (a) The consumer’s expected utility if he is cautious is \((1/2)\sqrt{100} + (1/2)\sqrt{36} - 1 = 7\). His expected utility if he is not cautious is \(\sqrt{36} = 6\). So the consumer will be cautious and have expected utility of 7.

(b) First, note that if the contract \((t_s, t_L)\) induces the consumer to not be cautious, then the consumer would accept it iff \(\sqrt{36} + t_L \geq 7\) or \(t_L \geq 13\). Clearly, the insurance company’s profit from this contract is \(-13\), so they’d rather not offer any contract than offer this one.

Hence the optimal contract must satisfy

\[
\frac{1}{2} \sqrt{100 + t_s} + \frac{1}{2} \sqrt{36 + t_L} - 1 \geq \sqrt{36 + t_L},
\]

so that the consumer will choose to be cautious. It must also satisfy the individual rationality constraint

\[
\frac{1}{2} \sqrt{100 + t_s} + \frac{1}{2} \sqrt{36 + t_L} - 1 \geq 7.
\]

So the consumer chooses \((t_s, t_L)\) to maximize \((1/2)(-t_s) + (1/2)(-t_L)\) subject to these constraints. It is not hard to see that both are binding. To see that the first one must bind, note that if it didn’t, the contract would have to have the insurance company bearing all the risk, in which case the consumer would never incur a cost to be cautious. If the first is binding but the second isn’t, then the company could lower \(t_L\) and be better off.

Since both constraints are binding, we have

\[
\sqrt{100 + t_s} = \sqrt{36 + t_L} + 2 \]
\[
\sqrt{100 + t_s^2 + 36 + t_L} = 16.
\]

Substituting from the first into the second gives

\[2\sqrt{36 + t_L} + 2 = 16\]
so \( \sqrt{36 + t_L} = 7 \) or \( t_L = 13 \). Hence
\[
\sqrt{100 + t_s} = 7 + 2 = 9,
\]
so \( t_s = -19 \). The insurance company’s expected profit is \((1/2)(19 - 13) = 3\). The consumer’s expected utility is 7 since the individual rationality constraint is binding.

2. We know that we must have

\[
U_i(\theta_i) = U_i(0) + \int_{0}^{\theta_i} \bar{y}_i(s) \, ds.
\]

Since the good is always allocated to the bidder with the highest type, \( \bar{y}_i(s) \) is the probability that the other \( I - 1 \) bidders have types below \( s \). Hence \( \bar{y}_i(s) = s^{I-1} \). So
\[
U_i(\theta_i) = U_i(0) + \int_{0}^{\theta_i} s^{I-1} \, ds = \frac{\theta^I_i}{I},
\]
where I also use the assumption that \( U_i(0) = 0 \). We also know that
\[
-\bar{e}_i(\theta_i) = \theta_i \bar{y}_i(\theta_i) - U_i(\theta_i) = \theta^I_i - \frac{\theta^I_i}{I},
\]
so the seller’s expected revenue is
\[
I \int_{0}^{1} \left( 1 - \frac{1}{I} \right) \theta^I \, d\theta = \frac{I - 1}{I + 1}.
\]
This is exactly what we computed for the first and second price auctions in class using the equilibrium strategies in these auctions.

3. As usual, the individual rationality constraint is binding for the low type and the incentive constraint is binding for the high type. Thus
\[
w_L = \alpha e_L
\]
\[
w_H = e_H + \alpha e_L - e_L.
\]
So the principal chooses the efforts, \( e_H \) and \( e_L \), to maximize
\[
\frac{1}{2} \left[ 2\sqrt{e_H} - e_H + \alpha e_L - e_L \right] + \frac{1}{2} \left[ 2\sqrt{e_L} - \alpha e_L \right].
\]
The first–order condition for \( e_H \) is
\[
\frac{1}{\sqrt{e_H}} = 1,
\]
so \( e_H = 1 \). The first–order condition for \( e_L \) is
\[
\frac{1}{\sqrt{e_L}} - \alpha = \alpha - 1.
\]
So
\[ e_L = \frac{1}{(2\alpha - 1)^2}. \]

Hence
\[ w_L = \frac{\alpha}{(2\alpha - 1)^2} \]
\[ w_H = 1 + \frac{\alpha - 1}{(2\alpha - 1)^2}. \]

(b) The low type gets his outside option regardless of \( \alpha \), so he doesn’t care what \( \alpha \) is. The high type’s payoff is
\[ w_H - e_H = \frac{\alpha - 1}{(2\alpha - 1)^2}. \]

The derivative of his payoff with respect to \( \alpha \) is
\[ \frac{1}{(2\alpha - 1)^2} - \frac{4(\alpha - 1)}{(2\alpha - 1)^3}. \]

This is greater than or equal to zero iff
\[ (2\alpha - 1) - 4(\alpha - 1) \geq 0 \]
or \( 2\alpha \leq 3 \) or \( \alpha \leq 3/2 \). Hence the derivative is positive for \( \alpha \leq 3/2 \) and negative for \( \alpha \geq 3/2 \). Hence high type’s preferred \( \alpha \) is \( \alpha = 3/2 \).

Finally, the principal’s expected payoff is
\[ \frac{1}{2} [2\sqrt{1 - w_H} + \frac{1}{2} \left( 2\sqrt{\frac{1}{(2\alpha - 1)^2} - w_L} \right)] \]
\[ = \frac{1}{2} \left[ 2 - 1 - \frac{\alpha - 1}{(2\alpha - 1)^2} + \frac{2}{2\alpha - 1} - \frac{\alpha}{(2\alpha - 1)^2} \right] \]
\[ = \frac{1}{2} \left[ 1 - \frac{2\alpha - 1}{(2\alpha - 1)^2} + \frac{2}{2\alpha - 1} \right] \]
\[ = \frac{1}{2} \left[ 1 + \frac{1}{2\alpha - 1} \right] \]

This is obviously decreasing in \( \alpha \), so the best \( \alpha \) for the principal, given the constraint \( \alpha \geq 1 \), is \( \alpha = 1 \).