1. (a) The cheapest way to induce the low effort is to pay a flat wage of 0. This gives the principal profits of $\pi_L$. So consider the minimum cost way to induce the high effort. Let $w_H$ denote the wage when profits are $\pi_H$ and $w_L$ the wage when profits are $\pi_L$. The principal’s problem is to minimize $pw_H + (1-p)w_L$ subject to individual rationality

$$p\sqrt{w_H} + (1-p)\sqrt{w_L} - c \geq 0$$

and the incentive constraint

$$p\sqrt{w_H} + (1-p)\sqrt{w_L} - c \geq \sqrt{w_L}.$$ 

Both constraints must bind. (If the second doesn’t bind, we’ll have $w_H = w_L$, but this will lead the agent to choose low effort, a contradiction. If the first doesn’t bind, we can lower $w_L$, making the second easier to satisfy and raising the principal’s profits.) Hence

$$p\sqrt{w_H} + (1-p)\sqrt{w_L} - c = 0 = \sqrt{w_L}.$$ 

So $w_L = 0$. Therefore, $p\sqrt{w_H} = c$, so $w_H = (c/p)^2$. The principal’s expected payoff from inducing high effort is then

$$p\pi_H + (1-p)\pi_L - p(c/p)^2.$$ 

This exceeds $\pi_L$ if $p\pi_H - p\pi_L - p(c/p)^2 > 0$ or

$$\pi_H - \pi_L > \frac{c^2}{p^2},$$

which holds by assumption.

(b) Let $w_S$ denote the wage when the principal sees high effort and $w_N$ the wage when he does not. Just as before, if the principal wants to induce low effort, he can do
so by setting \( w_S = w_N = 0 \), yielding him profits of \( \pi_L \). The principal’s cost minimization problem for high effort is to minimize \( qw_S + (1 - q)w_N \) subject to

\[
q \sqrt{w_S} + (1 - q) \sqrt{w_N} - c \geq 0 \\
q \sqrt{w_S} + (1 - q) \sqrt{w_N} - c \geq \sqrt{w_N}.
\]

This is the same as the problem from (b) with \( q \) replacing \( p \), \( w_S \) replacing \( w_H \), and \( w_N \) replacing \( w_L \). Hence we know that the cost minimizing contract is \( w_N = 0 \) and \( w_S = c^2/q^2 \). The principal’s payoff to inducing high effort this way is

\[
p\pi_H + (1 - p)\pi_L - qw_S - (1 - q)w_N = p\pi_H + (1 - p)\pi_L - q(c^2/q^2).
\]

This exceeds the payoff to inducing low effort iff

\[
\pi_H - \pi_L > \frac{c^2}{pq}
\]

Hence if this inequality holds, the optimal contract is \( w_S = c^2/q^2 \) and \( w_N = 0 \). Otherwise, it is \( w_S = w_N = 0 \).

Comparing the principal’s payoff in (a) to his payoff in (b), we have two cases. First, suppose it is optimal for the principal to induce low effort in (b). Since this was an option in (a) and he chose high effort, in this case, the principal is better off in (a). So suppose it is optimal for the principal to induce high effort in (b). Then his profits in (b) are higher if his expected wage cost is lower — that is, if

\[
\frac{c^2}{q} < \frac{c^2}{p}
\]

or \( q > p \). If \( q > p \), the principal will induce high effort in (b). In short, the principal is better off in (b) if and only if \( q > p \). Intuitively, this says that the principal wants to base the contract on the best possible signal of high effort.

2. Consider the payoff to bidder \( i \) with value \( \theta_i \) from bidding \( b \) when for all \( j \neq i \), \( j \) follows the strategy of bidding \( \alpha \theta_j^\beta \). The payoff is

\[
\theta_i \Pr[\alpha \theta_j^\beta \leq b, \forall j \neq i] - b
\]

since \( i \) wins iff his bid is larger than everyone else’s (and ties are probability zero) but pays his bid with probability 1. We can rewrite this as

\[
\theta_i \left( \Pr \left[ \theta_j \leq \left( \frac{b}{\alpha} \right)^{1/\beta} \right] \right)^{1-1} - b = \theta_i \left( \frac{b}{\alpha} \right)^{\frac{1}{\beta}} - b.
\]
The first–order condition for the optimal $b$ is

$$\frac{I - 1}{\beta} \frac{\theta_i}{\alpha^{(t-1)/\beta}} b^{\frac{t-1}{\beta}} - 1 = 0.$$ 

Rearranging:

$$b = \theta^{\frac{\beta}{\beta - I + 1}} \left( \frac{I - 1}{\beta} \right)^{\frac{\beta}{\beta - I + 1}} \alpha^{-\frac{t-1}{\beta - I + 1}}.$$ 

To be consistent with our hypothesis, we must have

$$\beta = \frac{\beta}{\beta - I + 1}$$

or $\beta = I$. Substituting, this gives

$$b = \theta^I \left( \frac{I - 1}{I} \right)^I \alpha^{-(I-1)}.$$ 

So

$$\alpha = \left( \frac{I - 1}{I} \right)^I \alpha^{-(I-1)}$$

implying $\alpha^I = [(I - 1)/I]^I$ or $\alpha = (I - 1)/I$.

Hence we have an equilibrium where $\sigma_i(\theta_i) = (I - 1)\theta_i^I/I$.

The seller’s expected revenue is

$$\sum_i E \left( \frac{(I - 1)\theta_i^I}{I} \right) = (I - 1)E(\theta_i^I) = (I - 1) \int_0^1 \theta^I d\theta = \frac{I - 1}{I + 1}.$$ 

We know that this must be the same as the expected revenue in the first price auction, at least if we focus on the equilibrium we computed earlier for that auction. Reason: Note that the bidder with the highest type always gets the object in this auction, just as in the first price auction. Also, the lowest type of a bidder gets a payoff of 0 since he bids 0 and gets the object with probability 0. Again, this is the same as the payoff to the lowest type of a bidder in the first price auction. Hence the revenue equivalence theorem tells us that the revenue in the first price auction must also be $(I - 1)/(I + 1)$.

3. (a) If the principal can observe $\theta$, he will pay the worker $e/\theta$ and choose $e$ to maximize $2\sqrt{e} - (e/\theta)$. The first–order condition is $e^{-1/2} = \theta^{-1}$ so $e = \theta^2$. So the optimal contract is $e_j = \theta_j^2$ and $w_j = \theta_j$ for $j = L, H$. So $e_L = w_L = 1$, $e_H = 4$, and $w_H = 2$. The principal’s expected profits are $(1/2)((\theta_H + \theta_L) = 3/2$. 

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(b) If the principal cannot observe $\theta$, he will choose $e_H$, $w_H$, $e_L$, and $w_L$ to maximize his expected profits subject to the individual rationality and incentive compatibility constraints:

$$w_j - \frac{e_j}{\theta_j} \geq 0, \quad j = L, H$$

$$w_L - \frac{e_L}{\theta_L} \geq w_H - \frac{e_H}{\theta_L}$$

$$w_H - \frac{e_H}{\theta_H} \geq w_L - \frac{e_L}{\theta_H}.$$  

As usual, the only binding constraints are the individual rationality constraint for type $\theta_L$ and the incentive compatibility constraint for type $\theta_H$. Hence

$$w_L = \frac{e_L}{\theta_L}$$

$$w_H = \frac{e_H}{\theta_H} + \frac{e_L}{\theta_L} - \frac{e_L}{\theta_H}.$$  

Substituting these results into the objective function, the principal will choose $e_L$ and $e_H$ to maximize

$$\frac{1}{2} \left[ 2\sqrt{e_H} - \frac{e_H}{\theta_H} - \frac{e_L}{\theta_L} + \frac{e_L}{\theta_H} \right] + \frac{1}{2} \left[ 2\sqrt{e_L} - \frac{e_L}{\theta_L} \right]$$

The first–order condition for $e_H$ is the same as in the first–best (as usual), so $e_H = \theta_H^2 = 4$. The first–order condition for $e_L$ is

$$\frac{1}{2} \left[ \frac{1}{\sqrt{e_L}} - \frac{1}{\theta_L} \right] = \frac{1}{2} \left[ \frac{1}{\theta_L} - \frac{1}{\theta_H} \right]$$

or

$$\frac{1}{\sqrt{e_L}} - 1 = 1 - \frac{1}{2}.$$  

Rearranging gives $e_L = 4/9$. The wages are $w_L = 4/9$ and

$$w_H = 2 + \frac{4}{9} - \frac{2}{9} = \frac{20}{9}.$$  

(c) Suppose the principal offers only one wage–effort pair. The problem says the principal must hire the agent, so this must satisfy the individual rationality constraint for both types. Since the low type has higher costs, if $(w, e)$ satisfies individual rationality for the low type, it must satisfy it for the high type. Incentive compatibility is no longer relevant since there’s no “lies” anyone can tell. So the best version of this option for the principal is to offer the $w$ and $e$ which maximize $2\sqrt{e} - w$ subject to $w - (e/\theta_L) \geq 0$. This is the same problem as the first–best for the low type, so we know that $e = 1$ and $w = 1$. If the principal follows this approach, his payoff is $2\sqrt{e} - w = 2 - 1 = 1$.  

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If the principal offers two wage–effort pairs, then he has to choose the first–best efforts in each case. Otherwise, the contract cannot be ex post efficient since the total payoff to the principal and agent could be made larger. Hence we must have $e_L = 1$ and $e_H = 4$. We also have to satisfy incentive compatibility and individual rationality. So our constraints are

$$
\begin{align*}
w_L - 1 &\geq 0 \\
w_H - 2 &\geq 0 \\
w_H - 2 &\geq w_L - \frac{e_L}{\theta_H} = w_L - \frac{1}{2} \\
w_L - 1 &\geq w_H - \frac{e_H}{\theta_L} = w_H - 4.
\end{align*}
$$

Rewriting:

$$
\begin{align*}
w_L &\geq 1 \\
w_H &\geq 2 \\
w_H &\geq w_L + \frac{3}{2} \\
w_L &\geq w_H - 3.
\end{align*}
$$

It’s easy to see that the first and third constraints imply the second. Also, if we ignore the fourth constraint, we get $w_L = 1$ and $w_H = 5/2$. Since this implies the fourth constraint, it can’t bind and these must be the wages. The principal’s profit is $(1/2)(6 - 1 - 2.5) = 5/4 > 1$. Hence this is the optimal contract.