Econ 701: Answers to Extra Problems

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1. (a) To minimize $\alpha w \cdot z$ over a certain set of z, it is obviously optimal to pick z in this set to minimize $w \cdot z$. Hence $z(\alpha w, q) = z(w, q)$, so $c(\alpha w, q) = (\alpha w) \cdot z(\alpha w, q) = \alpha w \cdot z(w, q) = \alpha c(w, q)$. Hence c is homogeneous of degree 1 in w.

(b) Fix any w^* and q^*. Let $z^* = z(w^*, q^*)$. Define the function $g(w) = c(w, q^*) - w \cdot z^*$. By definition, $g(w) \leq 0$ and $g(w^*) = 0$. Hence g is maximized over w at $w = w^*$. Thus the derivatives of g with respect to each input price must equal 0 at $w = w^*$. That is, $D_w c(w^*, q^*) = z^*$. Since w^* and q^* are arbitrary, this says $D_w c(w, q) = z(w, q)$.

(c) Fix input price vectors w and w' and let $z = z(w, q)$ and $z' = z(w', q)$. Fix any $\lambda \in (0, 1)$ and let $z^\lambda = z(\lambda w + (1-\lambda) w', q)$. By definition, $w \cdot z \leq w \cdot z^\lambda$ and $w' \cdot z' \leq w' \cdot z^\lambda$. Multiply both sides of the first inequality by λ, the second by $1 - \lambda$, and add to get

$$\lambda c(w, q) + (1-\lambda) c(w', q) \leq (\lambda w + (1-\lambda) w') \cdot z^\lambda = c(\lambda w + (1-\lambda) w', q).$$

Hence c is concave in w.

(d) See (a).

(e) Same argument as (a).

(f) This is essentially the same argument as (c) but I'll give it anyway to show exactly how it differs. Fix (p, w) and (p', w'). Let $(q, -z) = (q(p, w), -z(p, w))$ and $(q', -z') = (q(p', w'), -z(p', w'))$. Fix any $\lambda \in (0, 1)$ and let $(q^\lambda, -z^\lambda) = (q(\lambda(p, w) + (1 - \lambda)(p', w')), -z(\lambda(p, w) + (1-\lambda)(p', w')))$. By definition, $(p, w) \cdot (q, -z) \geq (p, w) \cdot ((q^\lambda, -z^\lambda)$ and $(p', w') \cdot (q, -z) \geq (p', w') \cdot ((q^\lambda, -z^\lambda)$. Multiply both sides of the first inequality by λ, both sides of the second by $1 - \lambda$, and add to get

$$\lambda \pi(p, w) + (1-\lambda) \pi(p', w') \geq [\lambda(p, w) + (1-\lambda)(p', w')] \cdot (q^\lambda, -z^\lambda) = \pi(\lambda(p, w) + (1-\lambda)(p', w')),$$

so $\pi(p, w)$ is convex.

(g) From Roy's Identity,

$$x_\ell^i(p, w_i) = \frac{1}{b(p)} \left[\frac{\partial a_i}{\partial p_\ell} + \frac{\partial b}{\partial p_\ell} w_i \right].$$

Note that the terms multiplying w_i are a function of p which is independent of i. Hence we can write this as

$$x_\ell^i(p, w_i) = A_i(p) + B(p)w_i.$$

So aggregate demand is

$$\sum_i x_\ell^i(p, w_i) = B(p) \sum_i w_i + \sum_i A_i(p)$$

which depends only on aggregate wealth $\sum_i w_i$.

2. We know that if this cost function is the result of cost minimization, then it must be concave in w and homogeneous of degree 1. If these conditions hold, then this estimate will be a good one in the sense that if it were true, then cost minimization would generate the same cost function.

To state this more precisely, let

$$\hat{c}(w, q) = \min_{z \in V_q} w \cdot z.$$

Then I claim that if $c(w, q)$ satisfies the properties stated above, we'll have $\hat{c}(w, q) = c(w, q)$. Proof: Suppose $z \in V_q$. By definition, $w \cdot z \geq c(w, q)$ for any $w \gg 0$. Hence

$$\hat{c}(w, q) = \min_{z \in V_q} w \cdot z \geq c(w, q).$$

Now we just need to show the reverse inequality.

Concavity of c in w implies

$$c(w', q) \leq c(w, q) + \nabla_w c(w, q) \cdot (w' - w).$$

Using the fact that c is homogeneous of degree 1 in w, Euler's theorem implies that $\nabla_w c(w, q) \cdot w = c(w, q)$. Substituting:

$$c(w', q) \leq \nabla_w c(w, q) \cdot w'.$$

In other words, $\nabla_w c(w, q)$ costs more than $c(w', q)$ at prices w'. Since this must be true for every w', this says $\nabla_w c(w, q) \in V_q$.

Then

$$\hat{c}(w, q) = \min_{z \in V_q} w \cdot z \leq w \cdot \nabla_w c(w, q) = c(w, q),$$

where the last equality again uses Euler's theorem and the fact that c is homogeneous of degree 1 in w.

3. Basically, this says that the most important thing to the consumer is good 1, but only up to 10 units of it. So the demands are as follows. If $10p_1 \geq w$, then $x_1(p, w) = w/p_1$

and $x_2(p,w) = 0$. If $10p_1 < w$, then $x_1(p,w) = 10$ and $x_2(p,w) = (w - 10p_1)/p_2$. To see this, suppose $10p_1 \geq w$ but the consumer spends some strictly positive amount of money on good 2. Then the bundle with only good 1 is strictly preferred by condition (1) in the statement of the preference. So the demand for this case is clear. So suppose $10p_1 < w$. The same argument as above shows that buying fewer than 10 units of good 1 cannot be optimal. So consider any feasible bundle with strictly more than 10 units of good 1. Then this bundle must include strictly less of good 2 than given above. By (3) in our statement of the preference, the demand given above is strictly preferred.

To verify symmetry, note that the demands are not differentiable when $10p_1 = w$. So let's just consider (p, w) which do not satisfy this. First, suppose $10p_1 > w$. In this case, $(\partial x_1/\partial p_2) = x_2 = (\partial x_2/\partial p_1) = (\partial x_2/\partial w) = 0$, so

$$\frac{\partial x_1}{\partial p_2} + x_2 \frac{\partial x_1}{\partial w} = 0 = \frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial w}.$$

Similarly, when $10p_1 < w$, we have $(\partial x_1/\partial p_2) = (\partial x_1/\partial w) = 0$ and

$$\frac{\partial x_2}{\partial p_1} + x_1 \frac{\partial x_2}{\partial w} = -\frac{10}{p_2} + 10\frac{1}{p_2} = 0,$$

so symmetry holds.

4. Letting x be the number of units of insurance, the consumer picks x to maximize

$$\frac{1}{2}\log(W - L + x - px) + \frac{1}{2}\log(W - px).$$

The first–order condition is

$$\frac{1}{2}\frac{1}{W - L + x - px}(1-p) - \frac{1}{2}\frac{1}{W - px}p = 0.$$

Rearranging:

$$(1-p)(W - px) = p(W - L + x(1-p))$$
$$2p(1-p)x = (1-p)W - p(W - L) = (1-2p)W + pL$$

so

$$x = \frac{(1-2p)W + pL}{2p(1-p)}.$$

Note: It isn't clear whether this x is positive or not. In principle one could allow people to purchase "negative" amounts of insurance. If one wants to explicitly rule that out, then this is the optimal x only when it is nonnegative and the optimal x is zero otherwise. Of course, we have to assume $p \neq 1$ for this to be well–defined.

At $p = 1/2$, the right–hand side is just L, so the agent fully insures when $p = 1/2$. Note that

$$\frac{\partial x}{\partial L} = \frac{p}{2p(1-p)}.$$

If $p > 1$, this is negative, while if $p < 1$, it is positive. Also,

$$\frac{\partial x}{\partial W} = \frac{1-2p}{2p(1-p)}.$$

If $p > 1$ or $p < 1/2$, this is positive. If $1/2 < p < 1$, it is negative.

5. We can differentiate with respect to p_ℓ to get the Hicksian demand for good ℓ. So

$$h_1(p,u) = \frac{1}{2}u\sqrt{\frac{p_2}{p_1}}$$

$$h_2(p,u) = \frac{1}{2}u\sqrt{\frac{p_1}{p_2}}.$$

To get the indirect utility function, use the fact that $e(p, v(p,w)) = w$, so

$$v(p,w)\sqrt{p_1 p_2} = w$$

or

$$v(p,w) = \frac{w}{\sqrt{p_1 p_2}}.$$

Now we can use Roy's Identity or we could use $h(p, v(p,w)) = x(p,w)$ to get the Walrasian demands. I'll go with the latter:

$$x_1(p,w) = \frac{1}{2}\frac{w}{\sqrt{p_1 p_2}}\sqrt{\frac{p_2}{p_1}} = \frac{w}{2p_1}$$

$$x_2(p,w) = \frac{w}{2p_2}.$$

To verify symmetry, note that

$$\frac{\partial x_1}{\partial p_2} + x_2\frac{\partial x_1}{\partial w} = 0 + \frac{w}{2p_2}\frac{1}{2p_1} = \frac{w}{4p_1p_2},$$

and

$$\frac{\partial x_2}{\partial p_1} + x_1\frac{\partial x_2}{\partial w} = 0 + \frac{w}{2p_1}\frac{1}{2p_2} = \frac{w}{4p_1p_2}.$$

So symmetry holds.

6. This preference violates the independence axiom. To see this, note that $(1/2)L_1 + (1/2)L_2 = L_3$. The independence axiom says that $L_1 \succ L_2$ implies $(1/2)L_1 + (1/2)L_1 \succ (1/2)L_2 + (1/2)L_1$ or $L_1 \succ L_3$. (To be picky: We stated independence in terms of weak preferences, so we really can't use our version of independence to conclude that $L_1 \succ L_3$. The strict version of independence follows from the weak plus the other axioms. So a more precise statement would be that our version of independence would require $L_1 \succeq L_3$, a conclusion which is contradicted in this example.)