# Econ 701: Answers to Problem Set 5

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1. For convenience, define

$$U(p, q, r) = \{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)r \succeq q\}$$

and

$$L(p, q, r) = \{\alpha \in [0, 1] \mid \alpha p + (1 - \alpha)r \preceq q\}.$$

Let's first show that Axiom 2 implies Axiom 1. So suppose Axiom 2 holds and fix $p$, $q$, and $r$ with $p \succ q \succ r$. Obviously, then, $0 \notin U(p, q, r)$. Since this set is closed, it cannot be true that there is a sequence of $\alpha$'s converging to 0 in the set. Thus there must be some $\bar{\alpha} > 0$ such that $(0, \bar{\alpha}) \cap U(p, q, r) = \emptyset$. But then for any $\beta \in (0, \bar{\alpha})$, completeness of $\succeq$ implies $\beta p + (1 - \beta)r \preceq q$. A similar argument shows the existence of the needed $\alpha$.

Now let's show that Axiom 1 implies Axiom 2. Since $p$ and $r$ enter the statement of Axiom 2 completely symmetrically, we can focus without loss of generality on cases where $p \succeq r$. If $p \sim r$, then independence gives us the result directly. If $p \sim q$, then $U(p, q, r) = L(p, q, r) = [0, 1]$ which is closed. If $p \succ q$, then $U(p, q, r) = [0, 1]$ and $L(p, q, r) = \emptyset$, both of which are closed. A symmetric argument covers the case where $q \succ p$. So we may as well assume $p \succ r$.

Again, if $q \succ p \succ r$ or if $p \succ r \succ q$, the analysis is trivial since one set is empty and the other is $[0, 1]$. If $q \sim p \succ r$, then independence implies $U(p, q, r) = \{1\}$ and $L(p, q, r) = [0, 1]$ and, again, both sets are closed. A symmetric argument applies to the case where $p \succ r \sim q$.

Hence without loss of generality, we can focus on the case where $p \succ q \succ r$. In this case, we know that $U(p, q, r)$ and $L(p, q, r)$ are convex sets with $1 \in U(p, q, r)$ and $0 \in L(p, q, r)$. To see why these sets are convex, suppose $\alpha, \alpha' \in U(p, q, r)$ and fix any $\beta \in (0, 1)$. By definition, $\alpha p + (1 - \alpha)r \succeq q$ and $\alpha' p + (1 - \alpha')r \succeq q$. Using the fact shown in Problem 2(d) (where the $p$ there is $\alpha p + (1 - \alpha)r$, the $r$ there is $\alpha' p + (1 - \alpha')r$, the $\alpha$ there is $\beta$, and both the $q$ and $s$ there are $q$), we have

$$\beta(\alpha p + (1 - \alpha)r) + (1 - \beta)(\alpha' p + (1 - \alpha')r) \succeq q,$$

or

$$(\beta\alpha + (1-\beta)\alpha')p + [1 - (\beta\alpha + (1-\beta)\alpha')]r \succeq q,$$

so $\beta\alpha + (1-\beta)\alpha' \in U(p,q,r)$. Hence this set is convex. A similar argument holds for $L(p,q,r)$.

Since $U(p,q,r)$ is convex and contains 1, it must take either the form $[a,1]$ or $(a,1]$ for some $a \in [0,1)$. If it's the former, we're done, so suppose it's the latter.

Since $a \notin U(p,q,r)$, we have $p \succ q \succ ap+(1-a)r$. But then Axiom 1 implies there is an $\beta \in (0,1)$ with $q \succ \beta p + (1-\beta)(ap+(1-a)r)$. That is, $q \succ (\beta+(1-\beta)a)p + (1-\beta)(1-a)r$. Note, though, that we must have $\beta + (1-\beta)a > a$. Hence $\beta + (1-\beta)a \in U(p,q,r)$, implying $(\beta + (1-\beta)a)p + (1-\beta)(1-a)r \succeq q$, a contradiction.

A similar argument applies to $L(p,q,r)$.

2. (a) Suppose $p \sim q$. By definition, $p \succeq q$ and $q \succeq p$. Applying independence to each of these statements, we see that for any $\alpha \in [0,1]$, we get $\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$ and $\alpha p + (1-\alpha)r \preceq \alpha q + (1-\alpha)r$, so $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$ by definition. For the converse, suppose we have $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$ for all $r$ and all $\alpha \in [0,1]$. Since this holds at $\alpha = 1$, this says $p \sim q$.

(b) The first part of the proof is the same as for (a). The converse is not as trivial, though, since the statement we're working with only applies for $\alpha \in (0,1)$. So suppose we have that for every $r$ and every $\alpha \in (0,1)$, $\alpha p + (1-\alpha)r \sim \alpha q + (1-\alpha)r$ but $p \not\sim q$. For concreteness, suppose $p \succ q$. Since our hypothesis holds for all $r$, it holds for $r = q$. That is, we must have $\alpha p + (1-\alpha)q \sim q$ for all $\alpha \in (0,1)$. Since $p \succ q$, though, this does not hold for $\alpha = 1$. Obviously, it does hold at $\alpha = 0$. So consider the set

$$\{\alpha \in [0,1] \mid \alpha p + (1-\alpha)q \preceq q\}.$$

From our hypotheses, we see that this set must be $[0,1)$. But MWG continuity implies that this set must be closed, a contradiction.

(c) One direction is trivial since it's just a restatement of the version of independence we used. So let's show the converse. The proof is very similar to that for the converse of (b). So suppose that for every for every $r$ and every $\alpha \in (0,1)$, $\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$ but $p \not\succeq q$. So we must have $q \succ p$. Since our hypothesis holds for all $r$, it holds for $r = q$. That is, we must have $\alpha p + (1-\alpha)q \succeq q$ for all $\alpha \in (0,1)$. Since $q \succ p$, though, this does not hold for $\alpha = 1$. Obviously, it does hold at $\alpha = 0$. So consider the set

$$\{\alpha \in [0,1] \mid \alpha p + (1-\alpha)q \succeq q\}.$$

From our hypotheses, we see that this set must be $[0,1)$. But MWG continuity implies that this set must be closed, a contradiction.

(d) By independence, $p \succeq q$ implies $\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$ for any $\alpha \in [0,1]$. But independence applied to $r \succeq s$ says $\beta r + (1-\beta)q \succeq \beta s + (1-\beta)q$ for any $\beta \in [0,1]$. Taking $\beta = 1-\alpha$, we see that

$$\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r \succeq \alpha q + (1-\alpha)s,$$

so transitivity gives the desired conclusion.