Econ 701: Answers to Problem Set 4

Bart Lipman
Fall 2009

1. Verifying Slutsky for problem 5: note that:

$$\frac{\partial h_i}{\partial p_j} = \frac{-1}{\alpha} u p_i^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{(-1/\alpha)-1} \gamma p_j^{1/(\alpha-1)}$$

(for $j \neq i$) which can be rewritten as

$$\frac{\partial h_i}{\partial p_j} = \frac{-1}{\alpha - 1} u (p_i p_j)^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-(1/\alpha)-1}$$

Also,

$$\frac{\partial x_i}{\partial p_j} = -w p_i^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-2} \gamma p_j^{1/(\alpha-1)}$$

$$\frac{\partial x_i}{\partial y} = p_i^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-1}$$

so that

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial w} = -w p_i^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-2} \gamma p_j^{1/(\alpha-1)} + w (p_i p_j)^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-2}$$

or

$$w (p_i p_j)^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-2} [-\gamma + 1] = \frac{-1}{\alpha - 1} w (p_i p_j)^{1/(\alpha-1)} \left(\sum_k p_k^\gamma \right)^{-2}$$

If you substitute $v(p, w)$ for u in the expression above for $\partial h_i / \partial p_j$ and rearrange, you should get this.

Completing problem 6: To get the expenditure function, use the identity $v(p, e(p, u)) = u$ to obtain

$$(e(p, u)/p_1) + (e(p, u)/p_2) = u$$

or

$$e(p,u) = \frac{p_1 p_2 u}{p_1 + p_2}.$$

Finally, to get the Hicksian demands, we can differentiate the expenditure function:

$$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1} = \frac{p_2 u}{p_1 + p_2} - \frac{p_1 p_2 u}{(p_1+p_2)^2}$$

or

$$h_1(p,u) = \frac{p_2^2 u}{(p_1+p_2)^2}$$

and, analogously,

$$h_2(p,u) = \frac{p_1^2 u}{(p_1+p_2)^2}.$$

2. If preferences are homothetic, then (as seen in Problem 1 on the last problem set)

$$\frac{\partial x_i}{\partial w} = \frac{x_i}{w}$$

But then

$$\frac{\partial x_i}{\partial p_j} + x_j \frac{\partial x_i}{\partial w} = \frac{\partial x_i}{\partial p_j} + \frac{x_j x_i}{w}.$$

So, using the symmetry restriction implied by the Slutsky equation:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial x_j}{\partial p_i}$$

so that we see that $D_p x(p,y)$ is indeed a symmetric matrix. Showing that this matrix is negative semidefinite is a bit trickier. We can use the Slutsky equation and the fact above to write:

$$D_p x(p,y) = D_p h(p,u) - \frac{1}{y}\, x x^T.$$

By definition, then, $D_p x$ is negative semidefinite if for any vector z,

$$z D_p h(p,u) z' - \frac{1}{y} z x x' z' \leq 0.$$

We already know that $D_p h$ is negative semidefinite so the first term is nonpositive. If you write out the second term, you will see that it is just $-(1/w)(\sum_i z_i x_i)^2$. Therefore, this equation must hold for any z.

3. a) The Walrasian demand would be that x such that x maximizes $u(x)$ subject to $p \cdot x \leq w$ and $q \cdot x \leq I$ (and $x \geq 0$). Denote this by $x(p,q,w,I)$. The indirect utility function would be $v(p,q,w,I) = u(x(p,q,w,I))$.

b) The Walrasian demand and hence the indirect utility function is homogeneous of degree 0 in (p, q, w, I) but not in just (p, w) or (q, I) separately. The proof of homogeneity is clear: If we multiply p, q, w, and I by a scalar, we don't change the feasible set and hence cannot change the optimum. It should also be clear why we cannot have homogeneity with respect to only (p, w). If we multiply p and w by a scalar but don't change q and I, the constraints do change and so there is no reason the optimum wouldn't change.

c) Just as in class, indirect utility must be nonincreasing in (p, q). If p and/or q increase, then the constraints become tighter, so the solution cannot improve. Unlike the situation in class, it could be the case that an increase in p_ℓ has no effect even though the consumer does purchase good ℓ. Reason: It's entirely possible that the budget line defined by $p \cdot x \leq w$ is completely outside the budget line defined by $q \cdot x \leq I$. That is, it may be the case that the latter constraint is so severe that if you satisfy it, you automatically satisfy the other, making the other irrelevant.

As this intuition suggests, it is no longer true that local nonsatiation ensures that v is strictly increasing in w. If $p \cdot x < w$ for every point satisfying $q \cdot x \leq I$, then the consumer simply can't spend all his dollars, so getting more dollars has no effect. On the other hand, our earlier argument does imply that if we increase *both* w and I, then v must increase strictly.

d) The indirect utility function is not quasiconvex in general here. To see this, recall our proof that v is quasiconvex in the usual model. We let x and x' be the demands at (p, w) and (p', w') respectively where $v(p, w) \leq \bar{v}$ and $v(p', w') \leq \bar{v}$. Let $(p_\alpha, w_\alpha) = \alpha(p, w) + (1 - \alpha)(p', w')$ and suppose $v(p_\alpha, w_\alpha)) > \bar{v}$. Let x_α be the demand at (p_α, w_α). Then we knew x_α must be unaffordable at (p, w) and unaffordable at (p', w'), giving us a contradiction to it being affordable at (p_α, w_α).

The problem is that now we have two constraints. To see how this messes up the argument, let's try to follow the line above. So fix (p, q, w, I) and (p', q', w', I') where v at each of these vectors is below $\bar{v}$. Let x be the demand at the first vector and x' the demand at the second. Suppose v is not quasiconvex — specifically, that there is an $\alpha \in (0, 1)$ such that $v(p_\alpha, q_\alpha, w_\alpha, I_\alpha) > \bar{v}$ where $(p_\alpha, q_\alpha, w_\alpha, I_\alpha) = \alpha(p, q, w, I) + (1 - \alpha)(p', q', w', I')$. Let x_α be the demand at $(p_\alpha, q_\alpha, w_\alpha, I_\alpha)$. Then we know that x_α is not feasible at (p, q, w, I), but we don't know which constraint is violated and similarly for (p', q', w', I'). The same argument from before will yield a contradiction if we assume x_α is not feasible in dollars under either vector or if we assume it is not feasible in points under either. But what if it's feasible under points but not dollars in one case and feasible under dollars but not points in the other? Here we can't get a contradiction.

To see this concretely, suppose there are two goods and all prices are 1 under each vector. Suppose $w = 2$, $w' = 10$, $I = 10$, and $I' = 2$. Suppose $\alpha = 1/2$. Then the budget constraints at (p, q, w, I) are $x_1 + x_2 \leq 2$ and $x_1 + x_2 \leq 10$. Only the former matters.

Similarly, the budget constraints at (p', q', w', I') are $x_1 + x_2 \leq 10$ and $x_1 + x_2 \leq 2$, so again only $x_1 + x_2 \leq 2$ matters. But the budget constraints at $(p_\alpha, q_\alpha, w_\alpha, I_\alpha)$ are both $x_1 + x_2 \leq 6$. Hence we certainly see no contradiction to the agent being better off in this situation.

e) Since the expenditure function is rather complex here (as we'll see shortly), we can't use the simpler approach to Roy's Identity to get anywhere here. Instead, we have to use the direct approach. First, suppose $\partial v/\partial w > 0$ and $\partial v/\partial I = 0$. Then we know that the dollars budget constraint is binding and the points budget constraint isn't. In this case, the extra constraint is irrelevant and we get the same Roy's Identity as before. Similarly, if $\partial v/\partial w = 0$ but $\partial v/\partial I > 0$, we just replace p with q and w with I and get a version of Roy's Identity that holds. We know that at least one budget constraint must bind, so the last case to consider is where $\partial v/\partial w > 0$ and $\partial v/\partial I > 0$. In this case, both budget constraints must bind.

So let's try to reconstruct our "direct" proof of Roy's Identity from class and see what we get. Note that

$$\frac{\partial v(p, q, w, I)}{\partial p_k} = \sum_\ell \frac{\partial u}{\partial x_\ell} \frac{\partial x_\ell}{\partial p_k}.$$

What do the first order conditions look like now? We have two Lagrange multipliers now since we have two constraints. Let λ be the multiplier for the $p \cdot x \leq w$ constraint and μ the multiplier for the $q \cdot x \leq I$ constraint. Then the first order condition says

$$\frac{\partial u}{\partial x_\ell} = \lambda p_\ell + \mu q_\ell.$$

So

$$\frac{\partial v(p, q, w, I)}{\partial p_k} = \sum_\ell [\lambda p_\ell + \mu q_\ell] \frac{\partial x_\ell}{\partial p_k}. \tag{1}$$

We said we're at a point where both constraints bind, so we can differentiate both sides of either budget constraint. Differentiating $p \cdot x = w$ with respect to p_k yields

$$\sum_\ell p_\ell \frac{\partial x_\ell}{\partial p_k} + x_k = 0$$

so $\sum_\ell p_\ell (\partial x_\ell / \partial p_k) = -x_k$. We can also differentiate $q \cdot x = I$ with respect to p_k to get

$$\sum_\ell q_\ell \frac{\partial x_{\ell|}}{\partial p_k} = 0.$$

Substituting both back into equation (1), we get

$$\frac{\partial v(p, q, w, I)}{\partial p_k} = -\lambda x_k.$$

Our proof that $\partial v/\partial w = \lambda$ still holds, so, again, we get Roy's Identity. Note that we could apply the same argument to the version of Roy's Identity with q in place of p and I in place of w. In other words, we get *both* versions of Roy's Identity if both constraints bind.

f) The tricky part about defining the expenditure function is that we have more than one kind of expenditure. In principle, you could take a number of different approaches to this. One approach would be to think about minimizing some weighted sum of the expenditures. That is, we could fix some $\alpha \in [0,1]$ and define the expenditure function to be

$$e(p,q,u) = \min_x \alpha p \cdot x + (1-\alpha) q \cdot x \text{ subject to } u(x) \geq u.$$

Alternatively, we could do this as a function of α. That is, instead of doing the above for a *fixed* α, we could define

$$e(p,q,u,\alpha) = \min_x \alpha p \cdot x + (1-\alpha) q \cdot x \text{ subject to } u(x) \geq u.$$

Another approach which is, in a way, almost identical to this last one, would be to minimize *both* expenditures. That is, we could think about all the pairs of expenditures which give utility at least u and take the "undominated" expenditures. In other words,

$$e(p,q,u) = \{(e_1,e_2) \in \mathbf{R}^2_+ \mid v(p,q,e_1,e_2) \geq u \text{ and } \nexists (e'_1,e'_2) < (e_1,e_2) \text{ with } v(p,q,e'_1,e'_2) \geq u\}.$$

Recall that we defined $<$ for vectors to mean less than or equal in every component with at least one of these inequalities strict. Thus this says that (e_1,e_2) is income enough to get utility at least u and we couldn't get u with less of one kind of income unless we had more of the other.

The "right" definition for the expenditure function in this setting depends on what you want to do with the function. There are some nice properties that can be shown for this last definition and I expect for the other definitions as well.

4. The Walrasian demands are $x_i(p,w) = w/2p_i$, $i = 1,2$. So the indirect utility function is $v(p,w) = w^2/4p_1p_2$. We can invert this to get the expenditure function. Plugging into $v(p,e(p,u)) = u$ gives

$$\frac{[e(p,u)]^2]}{4p_1p_2} = u$$

or $e(p,u) = 2\sqrt{up_1p_2}$. The initial utility level is $v(1,1,40) = 400$. The utility level after the change is $v(4,1,40) = 100$.

So the compensating variation is $e(4,1,400) - e(1,1,400) = 80 - 40 = 40$. The equivalent variation is $e(4,1,100) - e(1,1,100) = 40 - 20 = 20$.

The consumer surplus is what we get from integrating the Walrasian demand from the old price to the new one. (Note that the demand for good 2 is unaffected, so we can focus on good 1.) So it is

$$\int_1^4 \frac{20}{p_1}\, dp_1 = 20 \ln(4) \approx 27.7.$$