Econ 701: Answers to Problem Set 3

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1. (a) The easiest way to show this is to calculate the slope of the indifference curve. The slope (or MRS, marginal rate of substitution) is given by the ratio of the "marginal utilities" or

$$MRS = \frac{\partial u(x_1, x_2)/\partial x_1}{\partial u(x_1, x_2)/\partial x_2}.$$

If u is homothetic, we can write $u(x) = h(g(x))$. Substituting, we get

$$MRS = \frac{h'(g(x))\partial g(x_1, x_2)/\partial x_1}{h'(g(x))\partial g(x_1, x_2)/\partial x_2}.$$

Notice that the h' terms cancel. Since g is homogeneous of degree one, the partials in both numerator and denominator are homogeneous of degree zero. This means that

$$\frac{\partial g(x_1, x_2)}{\partial x_i} = \frac{\partial g(\lambda x_1, \lambda x_2)}{\partial x_i}$$

for any scalar λ. Take λ to be $1/x_2$ and we see that the MRS depends only on x_1/x_2. But this means that along any ray from the origin (where this ratio must be constant), the MRS must be constant. (One can generalize this to being two out of n arguments to establish the same thing for the n dimensional case.)

(b) This isn't hard to see graphically. As we increase w, we shift out the budget constraint so that the new constraint is parallel to the old one. If we move out along a ray from the origin from the old optimum, we see that where this ray intersects the new budget constraint, the slope of the indifference curve is the same as the slope of the indifference curve at the old optimum. Since the slope of the new constraint is the same as the slope of the old, this means that this point must be a tangency of the new budget constraint and an indifference curve and (given convexity at least) must be the new optimum. Hence the income expansion paths are just rays from the origin.

To prove it more precisely, first note two facts. Since u is homothetic, it takes the form $h(g(x))$ where $h' > 0$ and g is homogeneous of degree 1. Since maximizing u is the same as maximizing $h^{-1}(u)$ (since this is just a monotonic transformation), we can simply take u to be homogeneous of degree one. (It's a little messier if you don't take advantage of this, but you can still do the proof without this.) Second, given this, it is easy to see that preferences satisfy local nonsatiation since for any $\alpha > 1$, we have $u(\alpha x) = \alpha u(x) > u(x)$. Hence Walras' Law must hold.

So let x' be the Walrasian demands at prices p and income $w' = 1$. That is, x' maximizes $u(x)$ subject to $p \cdot x = 1$ (using equality here since we know it has to hold at the optimum). Now consider the problem of maximizing $u(x)$ subject to $p \cdot x = w''$ for an arbitrary $w'' > 0$.

The key point to notice is that $p \cdot x = w''$ if and only if $p \cdot (1/w'')x = 1$. That is, x is on the budget line for (p, w'') if and only if it is w'' times some point on the budget line for $(p, 1)$. Hence instead of picking an x on the budget line for (p, w''), we can pick one on the budget line for $(p, 1)$ and then scale it up by w''. That is, $x(p, w'')$ must be w'' times that x which solves

$$\max_x u(w''x) \ \text{ subject to } p \cdot x = 1.$$

Since u is homogeneous of degree 1, $u(w''x) = w''u(x)$, so the maximizing x here must be x'. Hence $x(p, w'') = w''x'$. Thus demand is linear in w.

(c) From (b), we know that

$$\frac{\partial \frac{x_1}{x_2}}{\partial w} = 0$$

or

$$\frac{1}{x_2}\frac{\partial x_1}{\partial w} - \frac{x_1}{x_2^2}\frac{\partial x_2}{\partial y} = 0.$$

Rearranging:

$$\frac{w}{x_1}\frac{\partial x_1}{\partial w} = \frac{y}{x_2}\frac{\partial x_2}{\partial w}$$

so that the income elasticities are equal to one another. But differentiation of the budget constraint with respect to w yields

$$p_1\frac{\partial x_1}{\partial w} + p_2\frac{\partial x_2}{\partial w} = 1.$$

This is equivalent to

$$\frac{p_1 x_1}{w}\eta_{1w} + \frac{p_2 x_2}{w}\eta_{2w} = 1$$

where η_{iw} is the income elasticity for good i. Since the elasticities are equal and since $p_1x_1 + p_2x_2 = w$, we see that the elasticities must equal 1.

2. This problem is a generalization of Problem 5b on the last problem set. The solution is obtained by generalizing its answer: the first–order condition for x_1 will be give us the value of the Lagrange multiplier immediately. We can then use this to get each of the

demands for the other goods as a function only of p, not y, since we get $n-1$ equations of the form $\partial U/\partial x_i = p_i/p_1$ with the same number of unknowns. Finally, this works only if the answer we get this way leaves enough money left over to buy a nonnegative amount of good 1.

3. The fact that w enters the utility function has no effect on the demands. To see the point, note that for a fixed w, the (x_1, x_2) which maximizes $x_1^{\alpha_1} x_2^{\alpha_2} w^{1-\alpha_1-\alpha_2}$ over the feasible set must also maximize $x_1^{\alpha_1} x_2^{\alpha_2}$ over the feasible set. Hence the demands are just like the ones we computed for Problem 5a on the last problem set. More specifically, we have

$$x_i(p, w) = \frac{\alpha_i w}{(1-\alpha_1-\alpha_2)p_i}, \quad i = 1, 2.$$

On the other hand, because w enters the utility function, the indirect utility function is not the same as in Problem 5a. Instead, we have

$$v(p, w) = \left[\frac{\alpha_1}{(1-\alpha_1-\alpha_2)p_1}\right]^{\alpha_1} \left[\frac{\alpha_2}{(1-\alpha_1-\alpha_2)p_2}\right]^{\alpha_2} w.$$

So

$$\frac{\partial v(p, w)}{\partial w} = \left[\frac{\alpha_1}{(1-\alpha_1-\alpha_2)p_1}\right]^{\alpha_1} \left[\frac{\alpha_2}{(1-\alpha_1-\alpha_2)p_2}\right]^{\alpha_2}.$$

Is this the same as the Lagrange multiplier? Recall that the Lagrange multiplier is the ratio of "marginal utility" to the price. Hence

$$\lambda^*(p, w) = \frac{\alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} w^{1-\alpha_1-\alpha_2}}{p_1}$$

or

$$= \frac{\alpha_1}{p_1} \left[\frac{\alpha_1 w}{p_1(1-\alpha_1-\alpha_2)}\right]^{\alpha_1 - 1} \left[\frac{\alpha_2 w}{(1-\alpha_1-\alpha_2)p_2}\right]^{\alpha_2} w^{1-\alpha_1-\alpha_2}$$

$$= (1-\alpha_1-\alpha_2) \left[\frac{\alpha_1}{(1-\alpha_1-\alpha_2)p_1}\right]^{\alpha_1} \left[\frac{\alpha_2}{(1-\alpha_1-\alpha_2)p_2}\right]^{\alpha_2}.$$

Note that this is not $\partial v/\partial w$. Why? Recall our envelope theorem argument for why λ normally equals $\partial v/\partial w$. The argument was that λ should equal the derivative of the Lagrangian with respect to w since the first order effect of the change in w on the Lagrangian through the effect of w on x is zero. Now, though, we have a separate first order effect of a change in w on the Lagrangian through the fact that w enters the objective function.

4. The answer is the same for both (a) and (b): the Walrasian demands are $x_1(p, w) = w/p_1$ and $x_2(p, w) = 0$. Thus even though the preferences are discontinuous, the demands are not only continuous but are very simple. Indeed, the preferences are the same as one another and the same as those arising from a utility function of $u(x_1, x_2) = x_1$.

5. a) Strong monotonicity requires

$$\frac{\partial u}{\partial x_i} > 0$$

or

$$\frac{1}{\alpha}\left[\sum_j x_j^\alpha\right]^{1/\alpha - 1} \alpha x_i^{\alpha-1} > 0$$

which holds whenever the x_i's are positive. Convexity is a bit messier. Preferences are convex if

$$\{x \mid (\sum_i x_i^\alpha)^{1/\alpha} \geq a\}$$

is convex for each a. (That is, preferences are convex if u is quasi–concave.) Clearly, this holds iff the set

$$\{x \mid \sum_i x_i^\alpha \geq a\}$$

is convex. Put differently, if the preferences are convex according to the utility function given, they must be convex when we use the utility function obtained by taking the old function to the α power. This function, though, is concave whenever $\alpha < 1$. Hence this must hold.

b) The first–order conditions can be written as

$$\frac{\alpha x_i^{\alpha-1}}{p_i} = \frac{\alpha x_1^{\alpha-1}}{p_1} \tag{1}$$

for each i (plus the budget constraint. Rearranging this yields:

$$x_i = x_1 \left(\frac{p_i}{p_1}\right)^{1/(\alpha-1)}$$

Substituting into the budget constraint for all the x_i's except x_1 yields:

$$p_1 x_1 + \sum_{i \neq 1} p_i x_1 \left(\frac{p_i}{p_1}\right)^{1/(\alpha-1)} = w$$

or

$$x_1(p,w) = \frac{w}{p_1 + \sum_{i\neq 1}\left(\frac{p_i^{\alpha/(\alpha-1)}}{p_1^{1/(\alpha-1)}}\right)} = \frac{y p_1^{1/(\alpha-1)}}{\sum_i p_i^\gamma}$$

where $\gamma = \alpha/(\alpha-1)$. Substituting for x_1 into (1) yields the desired result. To get the indirect utility function, simply substitute for the x_i's into the utility function as follows:

$$v(p,w) = \left(\sum_i \frac{w^\alpha p_i^\gamma}{(\sum_i p_i^\gamma)^\alpha}\right)^{1/\alpha} = \frac{w}{\sum_i p_i^\gamma}\left(\sum_i p_i^\gamma\right)^{1/\alpha}$$

or

$$v(p,w) = w\left(\sum_i p_i^\gamma\right)^{(1-\alpha)/\alpha} = w\left(\sum_i p_i^\gamma\right)^{-1/\gamma}$$

To get the Hicksian demands and expenditure function, you can go either of two ways. First, you can directly solve the expenditure minimization problem. Second, you can take advantage of the identities we talked about on Tuesday 9/29. Since you only saw these identities the day the problem set is due, I'm guessing most of you took the first approach. I'll be lazy and take the second since it's quicker.

Recall that $v(p, e(p,u)) = u$ so that

$$e(p,u)\left(\sum_i p_i^\gamma\right)^{-1/\gamma} = u$$

or

$$e(p,u) = u\left(\sum_i p_i^\gamma\right)^{1/\gamma}$$

We can then get the Hicksian demands by differentiating the expenditure function. So

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i} = \frac{1}{\gamma}u\left(\sum_j p_j^\gamma\right)^{(1/\gamma)-1} \gamma p_i^{1/(\alpha-1)}$$

or

$$h_i(p,u) = u p_i^{1/(\alpha-1)}\left(\sum_j p_j^\gamma\right)^{-1/\alpha}$$

c) Roy's Identity holds as

$$\frac{\partial v(p,w)}{\partial w} = \left(\sum_i p_i^\gamma\right)^{-1/\gamma}$$

and

$$\frac{\partial v(p,w)}{\partial p_i} = \frac{-1}{\gamma}\; w \left(\sum_j p_j^\gamma\right)^{(-1/\gamma)-1} \gamma p_i^{1/(\alpha-1)}$$

so that minus the ratio of the partials is

$$\frac{w p_i^{1/(\alpha-1)}}{\sum_j p_j^\gamma.}$$

6. Using Roy's Identity,

$$x_1(p,w) = \frac{w/(p_1)^2}{(1/p_1) + (1/p_2)} = \frac{wp_2}{p_1(p_1+p_2)}$$

and

$$x_2(p,w) = \frac{wp_1}{p_2(p_1+p_2)}.$$

So this gives the Walrasian demands.

7. As we discussed in class, we know that the matrix $D_p x + D_w x \; x^T$ is symmetric and negative semidefinite and that x is homogeneous of degree zero in (p, w). Monotonicity implies that the budget constraint holds with equality. We can write these in elasticities as follows. Let θ_i be the budget share of good i — that is, $\theta_i = p_i x_i/w$. Let η_{ij} denote $(p_j/x_i)(\partial x_i/\partial p_j)$ and $\eta_{iw} = (w/x_i)(\partial x_i/\partial w)$. Then it's not hard to show that

$$\theta_1[\eta_{12} + \theta_2\eta_{1w}] = \theta_2[\eta_{21} + \theta_1\eta_{2w}] \tag{2}$$

$$\eta_{11} + \eta_{12} + \eta_{1w} = 0 \tag{3}$$

$$\eta_{21} + \eta_{22} + \eta_{2w} = 0 \tag{4}$$

$$\theta_1\eta_{11} + \theta_2\eta_{21} + \theta_1 = 0 \tag{5}$$

$$\theta_1\eta_{12} + \theta_2\eta_{22} + \theta_2 = 0 \tag{6}$$

and

$$\theta_1\eta_{1w} + \theta_2\eta_{2w} = 1 \tag{7}$$

Some of these we showed in class. Briefly, you can get (2) from the symmetry restriction of the Slutsky substitution matrix. Multiply both sides by $p_i p_j$ and regroup to get equation (2). Equations (3) and (4) come from Euler's theorem and the fact that x_i is homogeneous of degree 0 in (p, w). Equations (5), (6), and (7) come from differentiating the budget constraint with respect to p_1, p_2, and w respectively.

Monotonicity of preferences implies that each consumer spends his entire income. Hence $w = 10 + 5 = 15$. Thus we can calculate the θ_i's — specifically, $\theta_1 = 2/3$ and $\theta_2 = 1/3$. The fact that the monopolist producing the first good has zero costs implies that he maximizes $p_1 x_1(p_1, p_2, w)$. This yields first order conditions of $x_1 + p_1 \frac{\partial x_1}{\partial p_1} = 0$ or $\eta_{11} = -1$. Thus, you have six equations relating the six elasticities plus the extra fact that $\eta_{11} = -1$. However, one can show that the six equations are not independent. In fact, there are actually only four equations that you can use. So far, then, we have five independent pieces of information.

This is where homotheticity comes in. You might think that with this information, you don't need $\eta_{11} = -1$ to calculate the elasticities. However, you still need this elasticity. You might have suspected this was unnecessary because, as seen in Problem 1(c) above, homotheticity implies that $\eta_{1w} = \eta_{2w} = 1$. Thus you have two pieces of information. However, this was actually determined using one of the equations above (specifically, the sixth one). You can show that it is not possible to use the six equations above plus $\eta_{1y} = \eta_{2y} = 1$ to solve for the elasticities — you're still one piece of information short. So suppose we have homotheticity and that $\eta_{11} = -1$. Then we can calculate the remaining elasticities as follows. Substituting these two facts into (3) immediately implies that $\eta_{12} = 0$. Substituting these facts into (2) implies

$$\theta_1\theta_2 = \theta_2\eta_{21} + \theta_1\theta_2$$

s so $\eta_{21} = 0$ also. But then (4) implies $\eta_{22} = -1$. Notice that these results do not depend on θ_1 and θ_2.