Econ 701: Answers to Problem Set 2

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1. To make this a little easier to write, I'll call the definitions Definition 1, Definition 2, and Definition 3 (going in the same order as these are given in the problem set). First, let's show that Definition 1 implies Definition 3. Suppose $\succeq$ is continuous according to Definition 1. Fix any $y \in X$ and consider the set $U(y) = \{x \in X \mid x \succeq y\}$. Suppose this set is *not* closed. Then there exists a sequence $\{x_n\}$ converging to x such that $x_n \in U(y)$ for all n but $x \notin U(y)$. That is, $x_n \succeq y$ for all n but $x \not\succeq y$. By definition, then, this implies $y \succ x$. But then we know that there is $\varepsilon_x > 0$ such that for all x' within ε_x of x, we have $y \succ x'$. But if x_n converges to x, we must have some N such that for all $n \geq N$, x_n is within ε_x of x. Hence we have for all such n that $x_n \succeq y$ (from our original statement) and $y \succ x_n$ (from continuity), a contradiction. A similar argument applies for the lower contour set. So if $\succeq$ is continuous according to Definition 1, it is continuous according to Definition 3.

Now let's show that Definition 3 implies Definition 2. Since Rubinstein has shown the equivalence of Definitions 1 and 2, this completes the proof. Suppose $\succeq$ is continuous according to Definition 3. Suppose $(x_n, y_n) \to (x, y)$ with $x_n \succeq y_n$ for all n but $x \not\succeq y$, so $y \succ x$. I claim (proof below) that there must be some z with $y \succ z \succ x$. By the Definition 3, the fact that $z \succ x$ and $x_n \to x$ implies that $z \succ x_n$ for all sufficiently large n. (Otherwise, $x_n \succeq z$ so the limit must also be weakly preferred to z, a contradiction.) Similarly, $y \succ z$ and y_n converging to y implies $y_n \succ z$ for all sufficiently large n. But then $y_n \succ z \succ x_n$. Since $\succeq$ (and hence $\succ$) is transitive, this implies $y_n \succ x_n$, a contradiction. Hence $\succeq$ is continuous according to Definition 2.

So for the key step: How do we show the existence of this z? Let $L = \{z \in X \mid y \succ z\}$ and $U = \{z \in X \mid z \succ x\}$. By Definition 3 of continuity, these sets are open (since they are the complements of closed sets). Fix any point $w \in X$ and suppose $w \notin L$. Then $y \not\succ w$, so $w \succeq y$. Hence by transitivity, $w \succeq y \succ x$, so $w \succ x$. Hence $w \in U$. Therefore, $U \cup L = X$. Since X is connected, this means that $U \cap L \neq \emptyset$. So there exists $z \in U \cap L$, just as asserted.

2. Suppose u is continuous and represents $\succeq$, but that $\succeq$ is not continuous. Since $\succeq$ is not continuous, there must be either an upper or lower contour set that is not

closed. I'll give the argument for an upper contour set; an analogous argument applies for lower. So suppose we have an upper contour set which is not closed. That is, we have a sequence y_n converging to y such that $y_n \succeq x$ for all n but $y \not\succeq x$. That is, $x \succ y$. But since u represents $\succeq$, this says that $u(y_n) \geq u(x)$ for all n and $u(x) > u(y)$. Clearly, though, $u(y_n) \geq u(x)$ for all n implies $\lim_{n\to\infty} u(y_n) \geq u(x)$, so we must have $u(y) \neq \lim_{n\to\infty} u(y_n)$. But this contradicts the definition of u being continuous.

3. Fix any pair of nonempty closed sets A and B such that $Z = A \cup B$. Suppose, contrary to what we want to show, that $A \cap B = \emptyset$. Let A^c and B^c denote the complement of A and B respectively — that is, $A^c = Z \setminus A$ and $B^c = Z \setminus B$. Since A and B are closed, A^c and B^c are open. Also, A^c and B^c both have to be nonempty. To see this, note that A is nonempty, $A \cup B = Z$, and $A \cap B = \emptyset$. Hence there must be points which are not in B — that is, $B^c \neq \emptyset$. A similar argument applies to A^c. Finally, note that $A^c \cup B^c = Z$. To see this, note that by definition, $A^c \cup B^c \subseteq Z$ (since we defined A^c to be points in Z that were not in A and similarly for B^c). To show that $Z \subseteq A^c \cup B^c$, fix any point $z \in Z$. We know $A \cup B = Z$, so either $z \in A$ or $z \in B$. Since $A \cap B = \emptyset$ by hypothesis, we know it can't be in both. So suppose $z \in A$. Then $z \notin B$, so $z \in B^c$. Similarly, if $z \in B$, then $z \in A^c$. Hence, either way, $z \in A^c \cup B^c$, so $A^c \cup B^c = Z$.

From connectedness, then, $A^c \cap B^c \neq \emptyset$. But this is impossible: if $z \in A^c \cap B^c$, then it's not in either A or B, contradicting $A \cup B = Z$.

4. (a) A function f is quasiconcave if for every $\bar{f}$, the set $\{x \mid f(x) \geq \bar{f}\}$ is convex. Suppose g is a monotonic function. Then $g(f(x)) \geq g(\bar{f})$ if and only if $f(x) \geq \bar{f}$. But note that $g(f)$ is quasiconcave if and only if for every $\bar{g}$, the set $\{x \mid g(f(x)) \geq \bar{g}\}$ is convex. But

$$\{x \mid g(f(x)) \geq \bar{g}\} = \{x \mid f(x) \geq g^{-1}(\bar{g})\}.$$

So $g(f)$ is quasiconcave if f is.

(b) Pretty much immediate. A function f is quasiconcave if the upper contour sets are convex. A preference $\succeq$ is convex if the upper contour sets are convex.

A slightly less trivial way to (re)state it: A function f is quasiconcave if for every x and y and $\lambda \in [0,1]$,

$$f(x) \geq f(y) \implies f(\lambda x + (1-\lambda)y) \geq f(y)$$

Preferences $\succeq$ are convex if for every x and y and $\lambda \in [0,1]$,

$$x \succeq y \implies \lambda x + (1-\lambda)y \succeq y.$$

Clearly, for u to represent a convex $\succeq$, u must be quasiconcave. Similarly, if a quasiconcave u represents $\succeq$, it must be true that $\succeq$ is convex.

5. (a) Clearly, if either x_1 or x_2 is 0, utility is 0. This cannot be optimal since positive utility is possible. Hence we know that both must be strictly positive at the optimum,

so we can use a Lagrangian. The Lagrangian is

$$\mathcal{L} = x_1^{\alpha} x_2^{\beta} - \lambda[p_1 x_1 + p_2 x_2 - w]$$

The first order conditions, then, are

$$\alpha x_1^{\alpha-1} x_2^{\beta} - \lambda p_1 = 0$$

and

$$\beta x_1^{\alpha} x_2^{\beta-1} - \lambda p_2 = 0$$

and, of course, the budget constraint. We can solve the first equation for λ and substitute into the second to get

$$\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{p_1} = \frac{\beta x_1^{\alpha} x_2^{\beta-1}}{p_2}$$

or

$$\frac{\alpha}{p_1 x_1} = \frac{\beta}{p_2 x_2}$$

or

$$p_1 x_1 = \frac{\alpha}{\beta} p_2 x_2$$

Substituting this into the budget constraint yields

$$p_2 x_2 [1 + \frac{\alpha}{\beta}] = w$$

or

$$x_2 = \frac{\beta y}{p_2[\alpha + \beta]}$$

and

$$x_1 = \frac{\alpha y}{p_1[\alpha + \beta]}$$

These are the Walrasian demands. The indirect utility function is obtained by substituting into the utility function. Thus

$$v(p, w) = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{\beta}{p_2}\right)^{\beta} \left(\frac{w}{\alpha + \beta}\right)^{\alpha+\beta}.$$

(b) Let's assume for a moment that the nonnegativity constraints do not bind. The Lagrangian is

$$\mathcal{L} = \log(x_1) + x_2 - \lambda[p_1 x_1 + p_2 x_2 - w]$$

The first order conditions are

$$\frac{1}{x_1} - \lambda p_1 = 0$$

and

$$1 - \lambda p_2 = 0$$

This immediately implies

$$x_1 = \frac{p_2}{p_1}$$

Substituting into the budget constraint

$$p_2 + p_2 x_2 = w$$

so

$$x_2 = \frac{w - p_2}{p_2}$$

Notice that this solution only makes sense if $w \geq p_2$. If this condition does not hold, then the constraint $x_2 \geq 0$ is binding and we have $x_2 = 0$ and $x_1 = w/p_1$. These are the Walrasian demands. In the case where $w \geq p_2$, the indirect utility function is

$$v(p, w) = \log(\frac{p_2}{p_1}) + \frac{w - p_2}{p_2}.$$

When $w < p_2$, the indirect utility function is just $\log(w/p_1)$.

(c) I think the easiest way to see this is to substitute from the budget constraint into the objective function. That is, from the budget constraint, we see that $x_2 = (w - p_1 x_1)/p_2$. Hence we wish to pick x_1 to maximize $a x_1 - (p_1/p_2) x_1 + (w/p_2)$. This is linear in x_1. If $a > (p_1/p_2)$, then utility is strictly increasing in x_1, so the optimal choice is $x_2 = 0$ and $x_1 = w/p_1$. If $a < (p_1/p_2)$, then utility is strictly decreasing in x_1, so the optimal choice is $x_1 = 0$ and $x_2 = w/p_2$¿ Finally, if $a = (p_1/p_2)$, then any choice on the budget constraint is optimal. It is easy to see that the indirect utility function is

$$v(p, w) = \frac{w}{\min\{p_1/a, p_2\}}.$$

(d) Could the optimal consumption for some p and w satisfy $x_1 \neq \sqrt{x_2}$? No. If, for example, x_1 is larger, then a small reduction in the expenditure on x_1 and a small increase in the expenditure on x_2 must increase utility. Hence we must have $x_1 = \sqrt{x_2}$ at the optimum. So $x_2 = x_1^2$. Substituting into the budget constraint yields $p_1 x_1 + p_2 x_1^2 = w$. Using the quadratic formula, we know that

$$x_1 = \frac{-p_1 \pm \sqrt{p_1^2 + 4 p_2 w}}{2 p_2}.$$

Since $\sqrt{p_1^2 + 4 p_2 w} > \sqrt{p_1^2} = p_1$, we see that we must have $\pm = +$, so

$$x_1(p, w) = \frac{-p_1 + \sqrt{p_1^2 + 4 p_2 w}}{2 p_2}.$$

$$x_2(p,w) = \left(\frac{-p_1 + \sqrt{p_1^2 + 4p_2 w}}{2p_2} \right)^2.$$

Hence

$$v(p,w) = \frac{-p_1 + \sqrt{p_1^2 + 4p_2 w}}{2p_2}.$$