Econ 701: Answers to Problem Set 1

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1. First, let's show that $\succ$ is asymmetric if and only if $\succeq$ is complete. By definition of completeness, $\succeq$ is complete if and only if for every x and y, either $x \succeq y$ or $y \succeq x$. Hence by definition of $\succeq$, $\succeq$ is complete iff for every x and y, either $x \not\succ y$ or $y \not\succ x$. But this is exactly the definition of $\succ$ being asymmetric.

Next, let's show that $\succ$ is negatively transitive if and only if $\succeq$ is transitive. By definition, $\succ$ is negatively transitive iff whenever $x \succ y$, we have either $z \succ y$ or $x \succ z$. Taking the contrapositive, this holds iff whenever $z \not\succ y$ and $x \not\succ z$, we have $x \not\succ y$. By definition of $\succeq$, this is true iff whenever $y \succeq z$ and $z \succeq x$, we have $y \succeq x$. This is exactly the definition of $\succeq$ being transitive.

2. Assume $\succeq$ is transitive. Suppose $x \succeq y$ and $y \succ z$. By definition, the second statement says $y \succeq z$ and $z \not\succeq y$. By transitivity of $\succeq$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$. So we need to show that $z \not\succeq x$. Suppose not. Then $z \succeq x$. Then we have $z \succeq x$ and $x \succeq y$, so transitivity of $\succeq$ implies $z \succeq y$. But we said above that $z \not\succeq y$, so this is impossible. Hence $z \not\succeq x$ so $x \succ z$.

3. This procedure can violate WARP but doesn't necessarily. For example, if δ is smaller than the smallest possible utility difference, then the procedure won't violate WARP. In other words, if whenever $u(x) > u(y)$, we actually have $u(x) > u(y) + \delta$, then the δ is irrelevant and this agent is really just maximizing u. So he will necessarily satisfy WARP.

For an example violating WARP, suppose $u(x_1) = 0$, $u(x_2) = 9$, $u(x_3) = 10$, and $\delta = 1.1$. If $B = \{x_1, x_2, x_3\}$, then $x^* = x_1$, $u(x^*) = 0$, and $\bar{u} = 10$. Since $\bar{u} > u(x^*) + \delta$, the decision maker chooses x_3. Now suppose $\bar{B} = \{x_2, x_3\}$. WARP would require x_3 to still be chosen. However, now we have $x^* = x_2$, so $u(x^*) = 9$. Hence we no longer have $\bar{u} > u(x^*) + \delta$, so the decision maker chooses x_2.

4. This one must violate WARP. Simply note that $C(\{x_1, x_3\}) = \{x_1\}$ and $C(\{x_1, x_2, x_3\}) = \{x_3\}$. Again, x_1 and x_3 are in both sets, only x_1 is chosen from the first, and only x_3 is chosen from the second, a violation of WARP.

5. One point I should have clarified: When I stated the alternative definition of WARP

in class, I stated it as an "iff." It's really only an "iff" in the case where $\mathcal{B}$ is rich. To illustrate this, I'll write out a longer answer than you needed to give which explains this point.

So suppose our first definition of WARP is satisfied. That is, if $x, y \in B \cap \bar{B}$, $x \in C(B)$, and $y \in C(\bar{B})$, then we have $x \in C(\bar{B})$. Now let's show the second definition (using "iff") must hold. So I want to show that $x \not\succeq^*_C y$ iff y is strictly revealed preferred to x. So we need to rule out the possibility that the first holds but not the second or that the second holds but not the first. Let's take these one at a time. Suppose $x \not\succeq^*_C y$ but y is not strictly revealed preferred to x. Here we need richness. y not strictly revealed preferred to x says that $\{y\} \neq C(\{x, y\})$. Note that this uses richness since $\{x, y\}$ might not be in $\mathcal{B}$ otherwise. Since $C(\{x, y\})$ is nonempty, this tells us $x \in C(\{x, y\})$, so $x \succeq^*_C y$, a contradiction. So now rule out the second possibility. Suppose $x \succeq^*_C y$ but y is strictly revealed preferred to x. By definition, the first statement says that there is some set B such that $x, y \in B$ and $x \in C(B)$. The second statement says, by definition, that there is some set $\bar{B}$ with $x, y \in \bar{B}$, $y \in C(\bar{B})$, and $x \notin C(\bar{B})$. But then $x, y \in B \cap \bar{B}$, $x \in C(B)$, and $y \in C(\bar{B})$, so we must have $x \in C(\bar{B})$, a contradiction. Note that this part does *not* use richness, only the definitions.

In short, our first definition of WARP implies the second with the "iff" if richness holds and implies the second phrased as "if $x \succeq^*_C y$, then y is not strictly revealed preferred to x."

For the converse, suppose WARP holds according to the *weaker* version of the second definition — that is, the one that doesn't have an "iff." Suppose $x \succeq^*_C y$. By the second definition, this means that y is not strictly revealed preferred to x. That is, we have that there is some B with $x, y \in B$ and $x \in C(B)$ and that for any $\bar{B}$ such that $x, y \in \bar{B}$ and $y \in C(\bar{B})$, we also have $x \in C(\bar{B})$. But this is our first definition. Note again that this does not use richness.

6. I will show that the function

$$u(x) = \#\{y \in X \mid x \succ y\}$$

represents $\succeq$ where $\#$ denotes the cardinality of a set (in the finite case, the number of items in the set). So suppose that $u(x) \geq u(x')$. I wish to show that this implies that $x \succeq x'$. Suppose not. Then, by definition, $x' \succ x$. For simplicity, let

$$A = \{y \in X \mid x \succ y\}$$

and

$$B = \{y \in X \mid x' \succ y\}.$$

Consider any $z \in A$. By the definition of A, $x \succ z$. By hypothesis, $x' \succ x$ so, by transitivity, $x' \succ z$. This implies that $z \in B$. By asymmetry, we cannot have $x \succ x$, so we must have $x \notin A$. Thus B contains every element of A plus at least one thing — x — which is not in A. But then the definition of the function $u(x)$ implies that we must have $u(x') > u(x)$, a contradiction. Hence we see that $u(x) \geq u(x')$ implies $x \succeq x'$. To complete the proof, we need to show that $x \succeq x'$ implies $u(x) \geq u(x')$. This is easily shown by a slight variation on the argument given above.

7. Suppose we have such a representation. First, show $\succ$ is asymmetric. Suppose, to the contrary, that we have x and y with $x \succ y$ and $y \succ x$. Then we must have $u(x) > u(y)+\delta$ and $u(y) > u(x) + \delta$. Substituting the second into the right–hand side of the first, we get $u(x) > u(x) + 2\delta$ or $\delta < 0$, a contradiction. Hence $\succ$ is asymmetric. To show $\succ$ is transitive, suppose we have $x \succ y$ and $y \succ z$. Then $u(x) > u(y) + \delta$ and $u(y) > u(z) + \delta$. Substituting the second into the right–hand side of the first gives $u(x) > u(z)+2\delta$. Since $\delta > 0$, then, $u(x) > u(z) + 2\delta > u(z) + \delta$. Hence $x \succ z$, so $\succ$ is transitive.

For the next property, suppose $w \succ x$ and $y \succ z$. Then $u(w) > u(x) + \delta$ and $u(y) > u(z) + \delta$. Suppose we do not have $w \succ z$. Then $u(w) \leq u(z) + \delta$. Hence $u(y) > u(z) + \delta \geq u(w) > u(x) + \delta$, so $y \succ x$.

For the final property, suppose $w \succ x$ and $x \succ y$. Then $u(w) > u(x) + \delta$ and $u(x) > u(y) + \delta$. Suppose we do not have $z \succ y$. Then $u(z) \leq u(y) + \delta$. Hence $u(w) > u(x) + \delta > u(y) + 2\delta \geq u(z) + \delta$, so $w \succ z$.