**Microeconomic Theory Qualifying Exam**
June 4, 2007

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**Instructions.** You have five hours to complete this exam. Answer all four questions. Maximum points for each part are in brackets.

Write on **one side** of the provided paper only. Start the answer to each question on a new sheet of paper and be sure to write your exam number on each sheet.

Be concise in your answers, and think before you write. Good luck!

1. [22 points] Consider an economy consisting of $N$ agents who have potentially different wealth levels but whose preferences over $\mathbf{R}_+^L$ are identical and denoted by $\succsim$ .

a) [11] Suppose $\succsim$ satisfies completeness, transitivity, continuity, homotheticity ($x \succsim y$ if and only if $\lambda x \succsim \lambda y$ for any $\lambda > 0$), and strict convexity. Prove that aggregate demand is a function of prices and aggregate wealth.

b) [11] Suppose that $L = 2$ and that $\succsim$ is represented by the function $U(x_1, x_2) = x_1 x_2$. Prove that a representative agent exists for this economy.

Answer:

*a) From strict convexity, we know that individual demands are single-valued (they are well defined by continuity). Let $x^i(p, m^i)$ denote i's demand at price vector $p$ and income $m^i$. By homotheticity, we know that*

$$x^i(p, \lambda m^i) = \lambda x^i(p, m^i).$$

*Proof: Suppose not. Obviously, $p \cdot \lambda x^i(p, m^i) = \lambda p \cdot x^i(p, m^i) \leq \lambda m^i$. Hence $\lambda x^i(p, m^i)$ is feasible at $(p, m^i)$. So there exists $x'$ with $p \cdot x' \leq \lambda m^i$ such that $x' \succ \lambda x^i(p, m^i)$. But then $p(1/\lambda) \cdot x' \leq m^i$, and, by homotheticity, $(1/\lambda)x' \succ x^i(p, m^i)$, a contradiction.*

*Let $\bar{x}^i(p) = x^i(p, 1)$. Since all the agents are identical except for income levels, we can write $\bar{x}^i(p) = \bar{x}(p)$ for all $i$. Then $x^i(p, m^i) = m^i \bar{x}^i(p) = m^i \bar{x}(p)$. So aggregate demand is $\sum_i x^i(p, m^i) = \sum_i m^i \bar{x}(p) = \bar{x}(p) \sum_i m^i$. Thus aggregate demand depends only on $p$ and $\sum_i m^i$.*

*b) The individual demands are*

$$x_1^i(p, m^i) = \frac{m^i}{2p_1}, \quad x_2^i(p, m^i) = \frac{m^i}{2p_2}.$$

*So $\sum_i x_1^i(p, m^i) = \sum_i \frac{m^i}{2p_1} = \frac{\sum_i m^i}{2p_1} = x_1^j(p, \sum_i m^i)$ for any of the agents $j$. The same is true for the other good as well. Hence aggregate demand is identical to the demand of a single agent with utility $U(x_1, x_2) = x_1 x_2$ and income equal to aggregate income $\sum_i m^i$.*

2. [28 points] A large factory owned by Pondskum Inc. emits effluent into a river that encourages the growth of algae in a lake downstream from the factory. The shore of the lake is inhabited by $N$ identical fisherwomen, and the algae negatively affects their businesses.

a) [4] Suppose that Pondskum's profits are given by $s - \frac{1}{2}s^2$, where $s$ is the amount of effluent emitted. Calculate the level of effluent $\hat{s}$ that you expect Pondskum will emit and the profit $\hat{\pi}$ it makes.

b) [4] A typical fisherwoman's profit is $\frac{1}{N} - \frac{1}{N}s$, where $N \geq 2$ is the number of fisherwomen located around the lake. Compute the socially optimal level of effluent $s^*$. How does it compare with $\hat{s}$?

c) [4] Mr. Busybody, the husband of one of the fisherwomen, decides to organize a coalition to persuade Pondskum to change its effluent emissions. He tells his neighbors that anyone who joins the coalition will be asked to contribute a sum $c/I$, where $I \leq N$ is the number of members; the coalition will make a take-it-or-leave-it offer to Pondskum, consisting of an emission level $s$ and a transfer $c$. If Pondskum accepts, it will be legally bound to the keep its emissions at or below $s$; if it rejects, it can continue to behave as before.

What offer $(\tilde{s}, \tilde{c})$ would the Busybody Coalition make to Pondskum if its membership is $I$?

d) [4] Compare $\tilde{s}$ with $s^*$. Is there a socially optimal number of coalition members, and if so what is it?

e) [4] To form his coalition, Mr. Busybody will mail reply forms to all of the fisherwomen at once; the forms are due at post office closing time the day the are received. Each fisherwoman must reply "join" or "decline." Since the mail arrives at 2 in the afternoon while the fisherwomen are on the lake, all fisherwomen return from the lake at 4pm, and the post office closes as 4:05, the fisherwomen will have to make their choices simultaneously. Compute the benefit $J(I)$ to someone who has joined the coalition, assuming it has $I$ members. Compute the benefit $D(I)$ to someone who has not joined the coalition of $I$ members.

f) [4] Write down two necessary conditions for $I$ to be a subgame-perfect Nash equilibrium level of membership of the Busybody coalition. Use these to compute the possible equilibrium levels of $I$.

g) [4] Will Mr. Busybody be effective in bringing about efficient pollution levels? How does the answer depend on $N$? Provide some intuition for your answer.

Answer:
*a) Pondskum maxes* $s - \frac{1}{2}s^2$ *to obtain* $\hat{s} = 1$, $\hat{\pi} = \frac{1}{2}$
*b)* $\max\limits_{s} s - \frac{1}{2}s^2 + N\left(\frac{1}{N} - \frac{s}{N}\right) = 1 - \frac{1}{2}s^2$ *yields* $s^* = 0$.
*c)* $\max\limits_{s,c} I\left([1/N] - [s/N]\right) - c$ *s.t.* $s - [1/2]s^2 + c \geq 1/2$ *(else offer will be rejected); constraint binds, else c can be lowered. Thus* $c = [1/2] + [1/2]s^2 - s$; *substituting gives* $\max\limits_{s} I\left([1/N] - [s/N]\right) - [1/2] - \frac{1}{2}s^2 + s$ *and the optimum is*
$\tilde{s} = 1 - [I/N]$,
$\tilde{c} = [I/N]\text{-}[1/2]\text{+}[1/2]\left(1 - [I/N]\right)^2 = \frac{I^2}{2N^2}$.
*d)* $1 - [I/N] > 0$ *unless* $I = N$, *which is the optimal coalition size.*
*e) member of coalition of size I gets* $[1/N] - \frac{\tilde{s}}{N} - \frac{\tilde{c}}{I} = [1/N] - [1/N] + \frac{I}{N^2} - \frac{I}{2N^2} = \frac{I}{2N^2} = J(I)$
*nonmember benefits from* $\tilde{s}$ *anyway and gets* $\frac{1}{N} - \frac{\tilde{s}}{N} = \frac{I}{N^2} = D(I)$
*f) member doesn't want to deviate (which reduces the coalition size by 1) :* $J(I) \geq D(I-1)$
*nonmember doesn't want to deviate (and increase coalition size):* $D(I) \geq J(I+1)$
*from the first we have* $\frac{I}{2N^2} \geq \frac{I-1}{N^2} \Longrightarrow 2 \geq I$; *from the second,* $\frac{I}{N^2} \geq \frac{I+1}{2N^2} \Longrightarrow I \geq 1$.
*hence the possible equilibria are* $I = 1$ *or* $I = 2$ *for any* $N \geq 2$.
*g) Since I is at most 2, he will fail to bring about efficient pollution levels if* $N > 2$ *(he may if the "good" equilibrium occurs in case* $N = 2$*). As N gets large,* $\tilde{s}$ *approaches* 1; *efficiency declines with population. Fisherwomen free ride off each other, enjoying the benefit of the coalition without paying (joining reduces pollution, but this benefit is shared and the fee reduces the incentive further). Other membership games might improve things (for instance, introducing sequential membership with discounts for early joiners).*

3. [24 points] Consider the following complete information game. There are three players, 1, 2, and 3, bidding on a single item for sale. The strategies are bids. More specifically, the strategy set for player $i$ is $S_i = \{0, .01, .02, \ldots, 100, 100.01, 100.02, \ldots\}$. That is, each player can bid any nonnegative multiple of 1 cent. The value of the object to player $i$ is $v_i$ (specified below).

The auction is a "third price" auction. More specifically, the payoffs are as follows. If $i$ does not at least tie for the highest bid, his payoff is 0. If he bids strictly more than either of the other two bidders, his payoff is $v_i - s^*$ where $s^*$ is the *lowest* of the three bids. If he ties with one or both of the other bidders for the high bid, then each of the high bidders has an equal chance to win the object, paying the lowest of the three bids.

The values for the players are $v_1 = 100$, $v_2 = 90$, and $v_3 = 80$.

a) [6] Show that it is *not* a Nash equilibrium for each player to bid his value.

b) [6] Is there any Nash equilibrium in which player 3 wins? If so, give an example. If not, show why not.

c) [6] Is there any Nash equilibrium in which player 2 wins? If so, give an example. If not, show why not.

d) [6] What is the most player 1 can pay in a Nash equilibrium where he wins? Give an equilibrium in which he pays this amount. Similarly, what is the least bidder 1 can pay in a Nash equilibrium in which he wins? Give an equilibrium in which he pays this amount.

Answer:

*Denote i's bid by $b_i$. A strategy profile will be written as $(b_1, b_2, b_3)$. A deviation by player i is $b'_i$.*

*(a) $(100, 90, 80)$ cannot be a NE as player 2 can do strictly better by setting $b_2 = 100$.*

*(b) WTS: there does not exist a NE with $(b_1, b_2, b_3)$ such that $b_3 \geq b_1, b_2$. It is not possible that $\min\{b_1, b_2\} > 80$ since player 3 deviates to $b'_3 < \max\{b_1, b_2\}$. If $\min\{b_1, b_2\} \leq 80$ and $b_3 < 100$ then player 1 deviates to $b'_1 > b_3$. If $\min\{b_1, b_2\} \leq 80$ and $b_3 \geq 100$ then, assuming $b_2 \leq 80$ (resp $b_1 \leq 80$), player 1 (resp player 2) deviates to $b'_1 > b_3$ (resp $b'_2 > b_3$).*

*(c) Similar to (b). Here $\min\{b_1, b_3\} \leq 90$.*

*(d) A NE with $(b_1, b_2, b_3)$ such that $b_1 \geq b_2, b_3$ must have $\min\{b_2, b_3\} \leq 100$ otherwise player 1 will deviate to $b'_i < \max\{b_2, b_3\}$. Given this, it must be that $b_1 > b_2, b_3$ since player 1 can win with probability 1 by bidding an epsilon amount higher (ie, 1 cent more) that what the others are bidding. Finally, it must also be that $b_1 > 90$ otherwise player 2 will deviate to $b_1 < b'_2 \leq 90$. It is easy to see that a strategy profile with these three properties is a NE where player 1 wins. Thus we can deduce that within the set of NE where player 1 wins, the most player 1 pays is 100 (e.g. NE $(b_1, 100, 100)$ and $b_1 > 100$) and the least is 0 (e.g. NE $(b_1, 0, 0)$ and $b_1 > 90$).*

4. [26 points] A large number of risk-averse traders of varying initial wealth $w$ trade in shares of IBM on a given day. They purchase shares during the day at a price $p$ which is set (endogenously) in such a way that the market clears. At the time of purchase the traders are uncertain about the value of the shares at the end of the day: it will either be $P_H$ or $P_L$, where $P_H > P_L$.

Purchase decisions are made to maximize expected utility. The utility function of a trader equals $-\exp(-aW)$ where $a > 0$ and $W$ is wealth at the end of the day: for a trader with initial wealth $w$ and $\theta$ shares purchased (where $\theta \in \Re$), $W$ will equal $\theta[P_s - p] + w$ if the end-of-day price of the share is $P_s$. Each traders' wealth $w$ is sufficiently large so that they are not constrained in the number of shares $\theta$ they can purchase.

The total supply of shares in the market is exogenously given by $X$ shares per trader.

A fraction $\alpha \in (0, 1)$ of traders have inside information about IBM earnings before they make their purchases; these can help them predict the end-of-day price. IBM earnings are either $e_1$ or $e_2$, where $e_1 > e_2$, and $\Pr[P_H|e_i] = q_i$, where $1 > q_1 > q_2 > 0$. Remaining traders do not have access to this earnings information, but they are all aware of the existence of the informed traders in the market. All agents have a common prior over earnings, represented by $\pi$, the probability that $e = e_1$.

a) [11] Define a fully revealing rational expectation equilibrium (REE) for this stock market. Explain carefully whether or not it exists.

b) [10] Define a non-revealing REE for this stock market. Explain carefully whether or not it exists.

c) [5] Can there be any other kind of REE in this setting?

Answer**:**

*a) A fully revealing REE is a set of state-contingent stock prices $p_i, i = 1, 2$ corresponding to earnings $e_i$, with $p_1 \neq p_2$, and stock purchase plans for all traders: $\theta(p|i)$ which maximizes their expected utility conditional on knowledge of i:*

$$[-q_i \exp\{-a\theta(P_H - p)\} - (1 - q_i) \exp\{-a\theta(P_L - p)\}] \exp\{-aw\} \tag{1}$$

*such that the market clears in each state i:*

$$\theta(p_i|i) = X. \tag{2}$$

*A fully revealing REE exists. $\theta(p|i)$ is described by the solution to the first-order condition (since the objective function (1) is seen to be strictly concave in $\theta$):*

$$q_i(P_H - p)\exp\{-a\theta(P_H - p)\} + (1 - q_i)(P_L - p)\exp\{-a\theta(P_L - p)\} = 0$$

*Notice here that the condition does not involve $a$ or $w$. The lefthand side of (3) is strictly decreasing in $\theta$ owing to the concavity of conditional expected utility. At a price $p$ at or above $\bar{p}_i \equiv q_i P_H + (1 - q_i)P_L$, the posterior expected share return, the demand $\theta(p|i)$ is zero; below that it is positive, strictly decreasing in $p$, and strictly increasing in $q_i$. If we solve the market-clearing condition (2) for a fully revealing REE, note that $p_i$ will be less than $\bar{p}_i$, for each $i$. Hence the solution for state 1: $p_1$ must exceed $p_2$, the solution for state 2, because $q_1 > q_2$. So a fully revealing REE exists, described by the solution to (2).*

*b) A non-revealing REE is a constant price $p_1 = p_2 = p^*$, purchase plans for informed traders the same as above (i.e., $\theta(p|i)$ maximizing (1))) and for uninformed traders given by $\theta_U(p)$ which maximizes their unconditional expected utility (where $Q \equiv q_1\pi + q_2(1 - \pi)$):*

$$[-Q\exp\{-a\theta(P_H - p)\} - \{1 - Q\}\exp\{-a\theta(P_L - p)\}]\exp\{-aw\} \tag{4}$$

*such that the market clears in each state $i$:*

$$\alpha\theta(p^*|i) + (1 - \alpha)\theta_U(p^*) = X. \tag{5}$$

*A non-revealing REE does not exist, because the demand of informed traders is greater in state 1 than 2, while that of uninformed traders is the same in both states. Therefore market demand will be higher in state 1, while supply is fixed, so the market clearing price solving (5) must be higher in state 1.*

*c) No, since there are just two states, so either $p_1$ and $p_2$ are the same (whence uninformed traders stay uninformed), or they are different (in which case the information leaks fully from informed to uninformed traders).*