Econ 701: Answers to Midterm Exam

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1. (a) The independence axiom (as we stated it) says that if $p \succeq q$, then for every r and every $\alpha \in [0,1]$, we have $\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$. To show that this holds, suppose $p \succeq q$. Using the representation, we have

$$\sum_z p(z)u(z) \geq \sum_z q(z)u(z).$$

Fix any lottery r and any $\alpha \in [0,1]$. Multiply both sides of the inequality above by α and add $(1-\alpha)\sum_z r(z)u(z)$ to both sides to obtain

$$\alpha \sum_z p(z)u(z) + (1-\alpha)\sum_z r(z)u(z) \geq \sum_z q(z)u(z) + (1-\alpha)\sum_z r(z)u(z)$$

or

$$\sum_z [\alpha p(z) + (1-\alpha)r(z)]u(z) \geq \sum_z [\alpha q(z) + (1-\alpha)r(z)]u(z).$$

From the representation, this implies $\alpha p + (1-\alpha)r \succeq \alpha q + (1-\alpha)r$.

(b) The risk premium is that number π such that $\mathrm{E}u(w+\tilde{z}) = u(w+\mathrm{E}(z)-\pi)$. That is, it is the amount the agent would pay to receive the mean of the gamble instead of facing the gamble. An agent is strictly risk averse if u is strictly concave. So by Jensen's inequality, for a strictly risk averse agent, we have

$$u(w+\mathrm{E}(\tilde{z})-\pi) = \mathrm{E}u(w+\tilde{z}) > u(w+\mathrm{E}(\tilde{z})).$$

Since u is strictly increasing, comparing the first and third terms, we see $w+\mathrm{E}(\tilde{z})-\pi > w+\mathrm{E}(\tilde{z})$, so $\pi > 0$.

(c) Fix price vectors p and p', a utility level u, and a number $\alpha \in [0,1]$. Let $p^\alpha = \alpha p + (1-\alpha)p'$. Let $x = h(p,u)$, $x' = h(p',u)$, and $x^\alpha = h(p^\alpha, u)$. By definition, $p \cdot x \leq p \cdot x^\alpha$ and $p' \cdot x' \leq p' \cdot x^\alpha$. Multiply the first inequality by α, the second by $1-\alpha$, and add to obtain

$$\alpha p \cdot x + (1-\alpha)p' \cdot x' \leq (\alpha p + (1-\alpha)p') \cdot x^\alpha = p^\alpha \cdot x^\alpha.$$

Hence

$$\alpha e(p,u) + (1-\alpha)e(p',u) \leq e(\alpha p + (1-\alpha)p', u),$$

so $e(p,u)$ is concave in p.

(d) There were a lot of proofs you could use. I'll go with the shortest. Take the identity $v(p, e(p,u)) = u$ and differentiate both sides with respect to p_ℓ. This gives

$$\frac{\partial v(p, e(p,u))}{\partial p_\ell} + \frac{\partial v(p, e(p,u))}{\partial w}\frac{\partial e(p,u)}{\partial p_\ell} = 0.$$

Since $h = D_p e$, this says

$$\frac{\partial v(p, e(p,u))}{\partial p_\ell} + \frac{\partial v(p, e(p,u))}{\partial w} h_\ell(p,u) = 0.$$

This holds for every p and u. Fix w such that $e(p,u) = w$ (so $e(p, v(p,w)) = w$). Substituting,

$$\frac{\partial v(p,w)}{\partial p_\ell} + \frac{\partial v(p,w)}{\partial w} h_\ell(p, v(p,w)) = 0$$

or

$$\frac{\partial v(p,w)}{\partial p_\ell} + \frac{\partial v(p,w)}{\partial w} x_\ell(p,w) = 0.$$

Rearranging gives the desired result.

2. We can use Roy's identity to get the Walrasian demands. Note that

$$\frac{\partial v}{\partial p_1} = -\frac{1}{p_1}$$

$$\frac{\partial v}{\partial p_2} = \frac{1}{p_2} - \frac{w}{p_2^2}$$

$$\frac{\partial v}{\partial w} = \frac{1}{p_2}$$

So

$$x_1(p,w) = -\frac{-1/p_1}{1/p_2} = \frac{p_2}{p_1}$$

$$x_2(p,w) = -\frac{(p_2 - w)/p_2^2}{1/p_2} = \frac{w - p_2}{p_2}.$$

Some of you noted that this is only correct if $w \geq p_2$. It was good that you picked up on this, but not essential.

We can use $v(p, e(p,u)) = u$ to get the expenditure function. So

$$u = \log\left(\frac{p_2}{p_1}\right) + \frac{e(p,u)}{p_2}$$

or

$$e(p,u) = p_2 \left[u - \log\left(\frac{p_2}{p_1}\right) \right].$$

We can differentiate to get the Hicksian demands. So

$$h_1(p,u) = \frac{\partial e(p,u)}{\partial p_1} = \frac{p_2}{p_1}$$

$$h_2(p,u) = \frac{\partial e(p,u)}{\partial p_2} = u - \log\left(\frac{p_2}{p_1}\right) - 1.$$

3. The key to both parts is to recognize that the only thing the agent cares about is the probability that the investment has a return above K. To see the point, fix an investment and suppose the probability it gives a return above K is p. Then the agent's expected utility is $(1-p)u(x) + pu(y)$. The derivative of this with respect to p is $u(y) - u(x) > 0$. Hence whatever investment gives a higher probability of a return above K is the one she'll choose.

(a) She must at least weakly prefer F to G. The point is that the probability she gets the promotion if she picks F is $1 - F(K)$, while the probability if she picks G is $1 - G(K)$. So the probability of a promotion is weakly better with F if $F(K) \leq G(K)$. By definition, the fact that F FOSD G tells us that this holds for every K. So she's either indifferent or picks F.

(b) Now we don't have enough information to say. It's true that she's risk averse, but as noted above, this is irrelevant. If she were getting the money from these investments, her risk aversion would say she prefers F. But she only cares about these investments through whether she gets a promotion or not, so that's irrelevant.

To see that it could go either way, consider two cases. First, suppose F puts probability 1 on $K-1$ while G puts probability 1/2 on $K-2$ and probability 1/2 on K. It's easy to see that these distributions have the same mean. This is exactly the kind of example we discussed in class of a mean–preserving spread. Note that the probability of a promotion under F is 0, while the probability under G is 1/2. So she will choose G in this case.

Second case: Suppose F puts probability 1 on K, while G puts probability 1/2 on $K-1$ and probability 1/2 on $K+1$. Again, G is a mean–preserving spread of F. Now the probability of a promotion under F is 1, while the probability of a promotion under G is 1/2. So she chooses F in this case.

4. Let $x_1 = (1,2,2)$, $x_2 = (2,1,2)$, and $x_3 = (2,2,1)$. It is easy to calculate the cost of each bundle under each price vector to show that $x_1, x_3 \in B_1$, $x_1, x_2 \in B_2$, and $x_2, x_3 \in B_3$. Recall that WARP says that if $x, y \in B \cap \bar{B}$, $x \in C(B)$, $y \in C(\bar{B})$, then

$x \in C(\bar{B})$. But if you take the intersection of any two of these sets, one of the choices is always excluded. For example, $B_1 \cap B_2$ includes x_1 (which is $C(B_1)$) but not x_2 (which is $C(B_2)$). Thus the only way for WARP to apply is if $B = \bar{B}$ and WARP holds trivially in this case.

On the other hand, this choice correspondence cannot be rationalized by a preference which is a weak order. To see this, suppose the order $\succeq$ rationalizes all choices. Then to rationalize the choice from B_1, we require $x_1 \succ x_3$. Rationalizing the choice from B_2 requires $x_2 \succ x_1$. Rationalizing the choice from B_3 requires $x_3 \succ x_2$. But then $\succ$ is not transitive, so $\succeq$ is not a weak order. This is a case where WARP holds but SARP fails.