Microeconomic Theory Qualifying Exam  
June 2012

Instructions. You have a maximum of 4 hours and 15 minutes to complete this exam (suggested time allocation: 15 mins to review the questions and up to 4 hours to answer them). Answer all four questions. The questions are equally weighted.

Write on one side of the provided paper only. Start the answer to each question on a new sheet of paper and be sure to write your candidate number, question number, and page number on each sheet.

Be concise in your answers, and think before you write. Good luck!

Question 1

Consider a household consisting of two agents (a husband and a wife with preferences $\succeq_h, \succeq_w$ over $R^L$, respectively) who jointly make consumption choices out of Walrasian budget sets by the following procedure: the wife goes first and decides on her (possibly nonsingleton) set of favorite consumption bundles and the husband only gets to make a final pick out of this set. Suppose both agents’ preferences are complete, transitive, continuous, locally nonsatiated and convex and suppose that $\succeq_h$ is strictly convex.

(a) Formulate the decision problem that determines the household’s Walrasian demand.

(b) Prove that there exists a solution whenever $p >> 0$.

(c) Prove that the households’ Walrasian demand is a function.

(d) Prove that the Walrasian demand function satisfies Walras’ law.

(e) Does the Walrasian demand function satisfy WARP?

(f) Is the Walrasian demand function continuous?
Question 2

(a) In the context of an exchange economy in which every consumer has complete and transitive preferences, consider the following assertion. “If initial endowments define a Pareto optimal allocation, and if a competitive equilibrium exists, then there exists a competitive equilibrium in which no one trades.”

Is this “no trade” assertion correct? If so, prove it. If it is false, give a counterexample. In the latter case, is the conclusion valid if one assumes also that all preferences are convex? Again, justify your answer fully.

(b) There are two consumers, denoted \( i \) and \( j \). They maximize expected utility using the common vNM index \( u(c) = \log c \). They face uncertainty because there are four possible equally likely events – each corresponding to a weather-endowment pair – as described in the following table:

<table>
<thead>
<tr>
<th>Weather</th>
<th>Endowment ((e_i, e_j))</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>((\beta, 1))</td>
</tr>
<tr>
<td>b</td>
<td>((1, \beta))</td>
</tr>
<tr>
<td>g</td>
<td>((1, \beta))</td>
</tr>
<tr>
<td>b</td>
<td>((\beta, 1))</td>
</tr>
</tbody>
</table>

Here \( e_i \) and \( e_j \) denote (random) endowments. Assume that \( 0 < \beta \neq 1 \).

Before the true event is realized, there is competitive trade in two securities. The first is a good weather security and pays 1 unit of consumption if the weather is \( g \) and 0 units otherwise; the second is a bad weather security and pays similarly with \( b \) replacing \( g \).

(i) Show that in any equilibrium, the two securities must have the same price.

(ii) Prove that there is a no trade equilibrium and find all corresponding equilibrium prices.

(iii) What can be inferred about the arbitrage-free price of a new security, if added to those above, that pays 1 unit if the endowment pair \((e_i, e_j) = (\beta, 1)\) is realized, and 0 otherwise?

(iv) Is the no-trade allocation Pareto optimal? Justify your answer in either case. If it is not Pareto optimal, how can this be reconciled with the First Welfare Theorem?
1. Ginaville, a town with a continuum of inhabitants, has two new restaurants, Eat at Bart’s (B) and Dine at Larry’s (L). The local population know that one of the restaurants has food of high quality $\theta$, and the other has low quality $\theta$, but they are not sure which is which (Bart and Larry have no more clue than anyone else): everyone commonly believes that Bart’s is the high quality restaurant with probability $\beta \in (0,1)$. Assume for simplicity that the price of all restaurant meals are fixed at $\$0$. Normalize the size (measure) of the population to 1.

An individual’s dining experience (utility) at a restaurant depends positively on quality $\theta$ and because of congestion and the resulting bad service and lousy food, negatively on the fraction $n$ of the population in the restaurant. Specifically his utility can be written

$$v(\theta, n) = \theta - c(n),$$

where $c(n)$ is strictly increasing and $c(0) = 0$. People who don’t eat at either restaurant get utility 0. Assume that $v(\theta, 1) > 0$.

a) Suppose that each consumer chooses a restaurant taking as given the choices of others. Show that in equilibrium,

$$(2\beta - 1)\Delta = c(n_B) - c(1 - n_B),$$

where $\Delta \equiv (\bar{\theta} - \theta)$ and $n_B$ is the number of patrons at Eat at Bart’s and $n_L = 1 - n_B$ is the number at Dine at Larry’s. How does the relative number of diners at each restaurant depend on $\beta$? Derive an expression for how $n_B$ depends on $\beta$ (you may assume as much differentiability as you need).

b) Show that if $c(\cdot)$ is strictly convex, a typical consumer’s expected welfare in equilibrium, considered as a function of $\beta$, is “single-peaked,” with a maximum at $\beta = \frac{1}{2}$. Provide some intuition for this result. Show that consumer welfare is symmetric about $\frac{1}{2}$ (that is, it assumes the same values at $\beta$ and $1 - \beta$).

c) Chamley’s Review, the town’s foodie magazine, which everyone reads, sends their restaurant critic to the new restaurants. Chamley’s ratings have reliability $\sigma > \frac{1}{2}$. That is, the probability that the rating will say a restaurant
is the one with high quality, given that it truly is high quality, is equal to \( \sigma \) (the probability it says low quality, given that it is low quality, is also \( \sigma \)).

Assume that \( 1 > \beta > \frac{1}{2} \) (everyone initially believes that Bart’s is better, but is not certain of this). Suppose that the review turns out to be “confirmatory,” that is, says that Bart’s is better. How does everyone fare after the review is published compared to the situation without the review? What if the review is perfectly reliable (\( \sigma = 1 \)) and is a “surprise” (says Larry’s is better)? Give some intuition for these results.

**Question 4**

Consider a risk-neutral principal and an agent who can choose one of two possible effort levels, \( e_L \) or \( e_H \). There are three possible profit levels, \( \pi_H, \pi_M, \) and \( \pi_L \), where \( \pi_H > \pi_M > \pi_L > 0 \). The agent’s utility if he chooses effort \( e_H \) and is paid wage \( w \) is \( \sqrt{w - 4} \), while his payoff if he chooses effort \( e_L \) and is paid wage \( w \) is \( \sqrt{w} \). The agent’s utility if he does not work for the principal is 2. If the agent does work for the principal, there is a minimum wage that the principal can pay of \( m \) where \( 0 \leq m \leq 4 \).

The principal cannot observe the agent’s effort but does observe the profits earned. Profits are a stochastic function of effort with conditional distributions given by:

\[
\Pr[\pi = \pi_H | e_H] = \frac{1}{2} \\
\Pr[\pi = \pi_M | e_H] = \frac{1}{2} \\
\Pr[\pi = \pi_L | e_H] = 0.
\]

and

\[
\Pr[\pi = \pi_H | e_L] = \frac{1}{4} \\
\Pr[\pi = \pi_M | e_L] = 0 \\
\Pr[\pi = \pi_L | e_L] = \frac{3}{4}.
\]

Assume \( \pi_H - \pi_M \) and \( \pi_M - \pi_L \) are “large.” More precisely, if some conclusion holds if \( \pi_H - \pi_M \) or \( \pi_M - \pi_L \) is large enough, assume this holds. Find an optimal contract for the principal. (Hint: Be careful about identifying binding constraints.)
Q1(a). Let the continuous utility functions $u$ and $v$ represent the husband and wife's preferences respectively (the existence of representations is guaranteed by completeness, transitivity and continuity due to a proposition in MGW). The household's choice solves

$$\max_{x \in \text{arg max}_{y \in B(p,w)} v(y)} u(x).$$

That is, the wife maximizes $v$ over the budget set $B(p,w)$ and the husband then maximizes $u$ over her maximizers.

(b) The set $\text{arg max}_{y \in B(p,w)} v(y)$ is nonempty by Wierstrass' theorem. By the maximum theorem, it is also closed. Since $\text{arg max}_{y \in B(p,w)} v(y)$ is a closed subset of the compact set $B(p,w)$, it is compact as well. Another application of Wierstrass' theorem ensures existence of a solution to the decision problem.

(c) Since the wife's set of maximizers is convex (by convexity of her preference) and the husband's preference is strictly convex, the solution must be unique.

(d) Follows from the local nonsatiation of wife's preference.

(e) Yes. Suppose $x(p,w)$ is chosen from $B(p,w)$ which contains a distinct element $x(p',w')$ that is in fact the choice from $B(p',w')$. We need to show that $x(p,w)$ cannot lie in $B(p',w')$. The fact that $x(p,w)$ is chosen over $x(p',w')$ in $B(p,w)$ implies that either it is strictly preferred by the wife, or that the wife is indifferent and it is strictly preferred by the husband. If $x(p,w)$ was available in $B(p',w')$ it would dominate $x(p',w')$, contradicting the fact that $x(p',w')$ was chosen from it. Therefore $x(p,w)$ cannot lie in $B(p',w')$.

An alternative proof would be to show that the induced preference over alternatives of the household is in fact complete and transitive (the induced preference is defined by: $x$ is strictly better than $y$ if either the wife strictly prefers $x$ or if she is indifferent but the husband strictly prefers $x$, and $x$ is indifferent to $y$ if both wife and husband are indifferent), and that the decision problem in (a) is equivalent to maximizing the induced preference.

(f) The husband is breaking the wife's indifference points so one can write examples where the husband strictly prefers $x$ to $y$, the wife is indifferent between the two, but there is a sequence $x_n, y_n$ where the wife strictly prefers $y_n$ to $x_n$. For suitable prices and wealth levels, it is possible that the household
chooses $y_n$ for each $n$ but demands $x$ where continuity would require $y$ be demanded.

Q2. (a) Correct as stated. Let $e = (e_i)_{i \in I}$ be the initial PO allocation and let $(x^*, p)$ be a competitive equilibrium. Then, for each $i$, because $e_i$ lies in $i$’s budget set given $p$, we must have $x^*_i \succeq_i e_i$. But market clearing implies that the allocation $x^*$ is feasible for the economy. Therefore, PO implies that $x^*_i \sim_i e_i$ for all $i$. It follows (using completeness and transitivity) that $(e, p)$ is an equilibrium.

(b)(i) Consumer $i$ solves
\[
\max_{a_g, a_b} \log (\beta + a_g) + \log (1 + a_g) + \log (1 + a_b) + \log (\beta + a_b)
\]
\[
\text{s.t. } 0 = q_g a_g + q_b a_b
\]
If $q_b > q_g$, then utility increases as one goes long in security $g$ and short in security $b$. There is no maximizer. Similarly for $j$.

(b)(ii) From (i), security prices must be equal in equilibrium. But then concavity of log implies that $a_g = a_b = 0$ is optimal for each consumer. Conclude that there is a no trade equilibrium where (normalized) security prices are $q = (1, 1)$.

(b)(iii) Denote by $\pi_1$ the (shadow) price of the new security and by $\pi_2$ the (shadow) price of the security that pays 1 only if $(e_i, e_j) = (1, \beta)$, which is the only other possibility for endowments. Then $q = (1, 1)$ and no-arbitrage imply only that $\pi_k > 0$ and $\pi_1 + \pi_2 = 1$. Thus $\pi_1$ can be any price in the open unit interval.

(b)(iv) It is not PO. Given common beliefs and no aggregate risk (total endowment is $1 + \beta$ in every event), PO calls for full insurance.

The Welfare Theorem assumes complete markets. The two securities provided here do not give completeness.

Q3(a). a) Each consumer must be indifferent between the two restaurants given the number at each, else he would switch to the other. Thus $\beta \Delta + \theta - c(n) = (1 - \beta) \Delta + \theta - c(1 - n)$; rearranging gives the result. Then $n_B > n_L = 1 - n_B$ iff $\beta > \frac{1}{2}$ since $\Delta > 0$ and $c(\cdot)$ is increasing. Implicitly differentiating (1) gives $n_B'(\beta) = \frac{2 \Delta}{\frac{c(n_B)}{c(n_B^*)} + c(1 - n_B)}$.

b) Since everyone obtains the same utility in either restaurant in equilibrium, write the equilibrium welfare as $V(\beta) = \beta \Delta + \theta - c(n)$. Differentiating
w.r.t. $\beta$ gives $V'(\beta) = \Delta - c'(n_B(\beta))n'_B(\beta) = \Delta \left( 1 - \frac{2c(n_B)}{c(n_B) + c(N-n_B)} \right)$ from above. This is positive iff $c'(n_B) + c'(1-n_B) > 2c'(n_B)$, i.e. if $n_L > n_B$ iff $\beta < \frac{1}{2}$. The inverted-U follows. Because of the convex congestion costs, welfare is maximized when people spread themselves equally between the two restaurants, which happens when they believe equally in the two restaurants.

For symmetry, note that $(1-\beta)\Delta + \theta - c(n(1-\beta)) = \beta \Delta + \theta - c(1-n(1-\beta))$; thus $c(n(\beta) - c(1 - n(\beta)) = c(1- n(1-\beta)) - c(n(1 - \beta))$, from which it follows that $n(\beta) = 1 - n(1 - \beta)$. Then $V(\beta) = \beta \Delta + \theta - c(n(\beta)) = \beta \Delta + \theta - c(1-n(1-\beta)) = V(1-\beta)$.

c) Using Bayes’s rule, a confirmatory review must raise the posterior belief in state $B$: $P(B|b) = \frac{P(b|B)P(B)}{P(b|B)P(B) + P(b|L)P(L)} = \frac{\sigma \beta}{\sigma \beta + (1-\sigma)(1-\beta)} > \beta$ iff $\sigma > \frac{1}{2}$. Since the welfare is declining in the belief for $\beta > \frac{1}{2}$, everyone is worse off after the review. Similarly, after a perfectly reliable review favoring Larry’s, by symmetry everyone’s payoff is the same as it would be if $\beta = 1$, which is the worst possible equilibrium payoff. Both scenarios lead to beliefs and therefore restaurant patronage that are too extreme. The only hope for a welfare gain is from a moderately reliable surprise review, which may generate an approximately uniform posterior and therefore balanced patronage. Note that more information is making things worse in both scenarios here, and that ratings/rankings can therefore be welfare worsening.

Q4. First, note that it must be optimal for the principal to induce effort $e_H$. This is true because it is certainly possible to induce this effort, albeit at a higher cost than the cost of inducing $e_L$. Given that we’re assuming $\pi_H - \pi_M$ and $\pi_M - \pi_L$ are “large,” the extra profit from inducing $e_H$ must be larger than the extra cost. Also, this option must dominate not hiring the agent at all as $\pi_L > 0$ and $\pi_H$ and $\pi_M$ “much larger” than $\pi_L$ implies that the profits must exceed the cost.

So the real question is how the principal induces $e_H$. Let $w_H$ denote the wage paid when $\pi = \pi_H$, let $w_M$ denote the wage paid when $\pi = \pi_M$, and let $w_L$ denote the wage paid when $\pi = \pi_L$. So the problem is to minimize $(1/2)w_H + (1/2)w_M$ subject to

$$\frac{1}{2} \sqrt{w_H} + \frac{1}{2} \sqrt{w_M} - 4 \geq 2$$

$\frac{1}{2} \sqrt{w_H} + \frac{1}{2} \sqrt{w_M} - 4 \geq \frac{1}{4} \sqrt{w_H} + \frac{3}{4} \sqrt{w_L}$
and \( w_H \geq m, w_M \geq m, \) and \( w_L \geq m. \) It is easy to see that lowering \( w_L \) helps with the incentive constraint and doesn’t directly affect the objective function, so, without loss of generality, we can set \( w_L = m. \) (As will be clear, this is not always necessary, which is why the question asks for “an” optimal contract.)

Suppose the incentive constraint does not bind. In this case, the risk aversion of the agent implies that \( w_H = w_M. \) Let \( w^* \) be the common value of these wages. Individual rationality requires

\[
\sqrt{w^*} - 4 \geq 2
\]

so \( w^* \geq 36. \) Obviously, this satisfies the constraint that \( w^* \geq m \) since \( m \leq 4. \) Clearly, the best choice is \( w^* = 36. \) This is the solution if the incentive constraint doesn’t bind at this contract. That is, it is optimal as long as

\[
\sqrt{w^*} - 4 \geq \frac{1}{4} \sqrt{w^*} + \frac{3}{4} \sqrt{m}.
\]

Substituting 36 for \( w^* \) and rearranging gives \( \sqrt{m} \leq 2/3 \) or \( m \leq 4/9. \) (In this range, \( w_L \) could be set to be larger than \( m \) as long as it is still small enough that the incentive constraint doesn’t bind.)

If \( m > 4/9, \) then the incentive constraint must bind. The individual rationality constraint might or might not bind. Suppose it does not bind. In this case, there are (at least) two ways to proceed. First, we can solve the IR constraint for one of the wages, substitute into the objective function, and maximize over the other wage. This is a bit messy. A simpler approach is to use Lagrangians. Letting \( \lambda \) denote the multiplier on the IR constraint, we get first–order conditions of

\[
\frac{1}{2} - \lambda \frac{1}{8} \sqrt{w_H} = 0
\]

\[
\frac{1}{2} - \lambda \frac{1}{4} \sqrt{w_M} = 0
\]

Rewriting:

\[
\sqrt{w_H} = \frac{\lambda}{4}
\]

\[
\sqrt{w_M} = \frac{\lambda}{2}
\]
so $\sqrt{w_M} = 2\sqrt{w_H}$. Substituting into the IC, we get

$$\frac{1}{4} \sqrt{w_H} + \sqrt{w_H} = 4 + \frac{3}{4} \sqrt{m}$$

or $5 \sqrt{w_H} = 16 + 3 \sqrt{m}$. Hence

$$w_H = \frac{(16 + 3 \sqrt{m})^2}{25}$$

and

$$w_L = \frac{4(16 + 3 \sqrt{m})^2}{25}.$$  

Let’s check if the IR is binding. Substituting, we have that IC holds iff

$$\frac{1}{2} \left( \frac{16 + 3 \sqrt{m}}{5} \right) + \frac{1}{2} \left( \frac{2(16 + 3 \sqrt{m})}{5} \right) \geq 6$$

or

$$16 + 3 \sqrt{m} + 32 + 6 \sqrt{m} \geq 60$$

$$9 \sqrt{m} \geq 12$$

or $m \geq 16/9$. So if $m \geq 16/9$, the IR doesn’t bind and we have the solution above. (I’ve omitted it, but it’s easy to verify that these wages must be above $m$ in the relevant range.)

For $m \in (4/9, 16/9)$, both IR and IC must bind. So we have

$$\frac{1}{2} \sqrt{w_H} + \frac{1}{2} \sqrt{w_M} - 4 = 2$$

$$\frac{1}{2} \sqrt{w_H} + \frac{1}{2} \sqrt{w_M} - 4 = \frac{1}{4} \sqrt{w_H} + \frac{3}{4} \sqrt{m}.$$

Hence

$$2 = \frac{1}{4} \sqrt{w_H} + \frac{3}{4} \sqrt{m}$$

or

$$\sqrt{w_H} = 8 - 3 \sqrt{m}$$

For brevity, let $K = \sqrt{m}$. Then we have $w_H = (8 - 3K)^2$. Substituting into the IR, we get

$$\frac{1}{2} (8 - 3K) + \frac{1}{2} \sqrt{w_M} = 6$$
or

\[ \sqrt{w_M} = 4 + 3K \]

so \( w_M = (4 + 3K)^2 \). Are these wages feasible — i.e., are they larger than \( m \)?

Obviously,

\[ w_M = (3 + 3K)^2 > K^2 = m, \]

so \( w_M \) is feasible. We have

\[ w_H = (8 - 3K)^2 = [(8 - 2K) + K]^2 \geq K^2 \]

if \( 8 - 2K \geq 0 \) or \( K \leq 4 \). Since \( K = \sqrt{m} \), this holds iff \( m \leq 16 \) which is certainly true.

Summarizing: If \( m \leq 4/9 \), IC does not bind and the optimal contract has \( w_H = w_M = 36 \) and \( w_L = m \). If \( m \geq 16/9 \), then IR doesn’t bind and the optimal contract is \( w_H = (16 + 3\sqrt{m})^2/25, w_M = 4(16 + 3\sqrt{m})^2/25, \) and \( w_L = m \). In between, both constraints bind and the optimal contract is \( w_H = (8 - 3\sqrt{m})^2, w_M = (4 + 3\sqrt{m})^2, \) and \( w_L = m \).