

Department of Economics
Microeconomics PhD Qualifying Examination
June 5, 2006
SOLUTIONS

Instructions:

- *Do not open the exam booklet until you are told to do so.*
- *Answer all four of the following problems. Each problem has equal weight.*
- *Think before you write.*
- *Do not spend too much time on any one problem.*
- *If you have to leave the room for any reason, please give the instructor your examination on the way out.*
- *You will have 4 hours and 30 minutes to complete the exam.*
- *Answer each question in a SEPARATE answer booklet.*
- *Use a dark pencil or a pen (the exam will be photocopied).*
- *DO NOT WRITE OUTSIDE OF THE BOXES PROVIDED FOR THE ANSWERS. The boxes probably provide much more space than you need for your answers, so do not use the size of the box as a guide to how long your answer should be.*
- *You may use the backs of the answer sheets as scratch paper, but do not write your answers there.*
- *WRITE YOUR EXAM CODE and Question Number at the top of each page.*

Problem 1. [60 minutes, total] Ma Ching-to consumes two goods, rice (r) and whiskey (x). Let p denote the price of rice; whiskey is the numeraire with a price of 1. Ma's utility function is

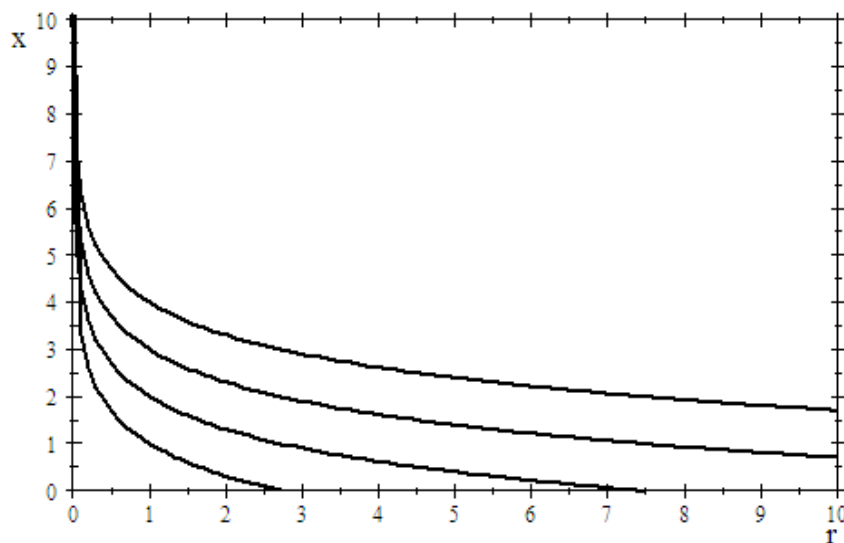
$$u(r, x) = x + \log r.$$

- a) [15 minutes] What is the slope of Ma's indifference curves for general values of u , x and r ? Make a rough drawing of Ma's indifference map (show the shapes of the curves, but do not waste time on a precise drawing). Label your axes. Do Ma's indifference curves intersect the axes?

From the implicit function theorem we know that with u held constant

$$\frac{dx}{dr} = -\frac{\frac{\partial u}{\partial r}}{\frac{\partial u}{\partial x}} = -\frac{1}{r}.$$

This slope is independent of u and x , which means that all the indifference curves are parallel, simply shifting upwards as u and x increase.



From the form of the utility function we see that $r = e^{u-x}$, which means that r cannot be 0 for any finite values of u and x . It follows that the indifference curves do not intersect the whiskey axis. We also have that $x = u - \log r$, so that intersections on the rice axis are the solutions of $u - \log r = 0$, or at $r = e^u$.

- b) [15 minutes] Find Ma's demand for rice $r(p, y)$ and for whiskey $x(p, y)$, where y denotes income.

We have

$$\frac{MU_r}{p} \equiv \frac{1}{rp} \text{ and } \frac{MU_x}{1} \equiv 1.$$

Ma consumes only rice so long as

$$\frac{MU_r}{p} > \frac{MU_x}{1},$$

but if he consumes only rice then from his budget constraint we know that

$$r = y/p.$$

Therefore, Ma consumes only rice if

$$\frac{1}{rp} \equiv \frac{1}{y} > 1$$

or, equivalently, if

$$y < 1.$$

In that case, $r = y/p$ and $x = 0$. Otherwise, Ma sets

$$\frac{MU_r}{p} = \frac{MU_x}{1},$$

so that

$$\begin{aligned} r &= 1/p \\ x &= y - 1. \end{aligned}$$

In sum, demand is given by

$$r(p, y) = \begin{cases} \frac{y}{p} & \text{for } y < 1 \\ 1/p & \text{for } y \geq 1 \end{cases}$$

and

$$x(p) = \begin{cases} 0 & \text{for } y < 1 \\ y - 1 & \text{for } y \geq 1 \end{cases}$$

- c) [5 minutes] Find the indirect utility function.

Substituting demand into the utility function we get

$$v(p, y) = \begin{cases} \log \frac{y}{p} & \text{for } y < 1 \\ y - 1 - \log p & \text{for } y \geq 1 \end{cases}$$

- d) [15 minutes] Find the expenditure function and Hicksian compensated demand functions for rice and for whiskey.

The expenditure function $y = e(p, u)$ is the solution of $u = v(p, y)$ for y . For $y < 1$ we have $u = \log \frac{y}{p}$ so that $y = pe^u$. But $pe^u < 1$ if and only if $p < e^{-u}$. For $y \geq 1$, we have $u = y - 1 - \log p$ or $y = u + 1 + \log p$. But $y \geq 1$ if and only if $\log p \geq -u$ or, equivalently, if and only if $p \geq e^{-u}$. So we have

$$e(p, u) = \begin{cases} pe^u & \text{for } p < e^{-u} \\ u + 1 + \log p & \text{for } p \geq e^{-u} \end{cases} .$$

The Hicksian compensated demand for rice is

$$h_r(p, u) \equiv \frac{\partial e}{\partial p} = \begin{cases} e^u & \text{for } p < e^{-u} \\ 1/p & \text{for } p \geq e^{-u} \end{cases} .$$

We cannot use this method to find the compensated demand for whiskey, because the price of whiskey is the numeraire and it is held constant. However, because at any price the utility derived from the compensated demand for both goods must add up to the specified utility, it must be true that

$$u \equiv h_x(p, u) + \log h_r(p, u)$$

so that

$$h_x(p, u) \equiv u - \log h_r(p, u) .$$

Therefore

$$h_x(p, u) = \begin{cases} 0 & \text{for } p < e^{-u} \\ u + \log p & \text{for } p \geq e^{-u} \end{cases} .$$

- e) [10 minutes] Suppose $y = 1$ and the price changes from $p = 1$ to $p = 2$. Find the changes in the quantities that Ma demands. Decompose the changes into the substitution and income effects.

At $y = 1$ all income is spent on rice, so that $r(p, 1) = 1/p$ and $x = 0$. When p goes from 1 to 2, r goes from 1 to 1/2, but x stays at 0. At $p = 1$, we have $u = x + \log r = 0$. When $p = 2$, compensated demand for rice is 1/2 and that for whiskey is $\log 2$. (Slutsky compensation $\Delta y = 1$, because we need $y = 2$ for Ma to be able to buy his original bundle. This means Slutsky compensated demand for rice would be 1/2 and for whiskey, it would be 1.) Using the Hicksian definition, then we have that the substitution effect is $-1/2$ for rice and $+\log 2$ for whiskey. The income effect is zero for rice and $-\log 2$ for whiskey—it is the income effect that holds that actual demand for whiskey at 0.

Problem 2. [60 minutes, total] An exchange economy has a single consumption good, two dates $t = 0, 1$, and agents $i = 1, \dots, n$ who do not own or consume anything at $t = 0$. At $t = 1$ there are states of nature $s = 1, \dots, S$; agent i has endowment ω_{is} in state s . The state of nature is unknown at $t = 0$, all agents assign the same probability $\pi_s > 0$ to state s . Agent i 's Bernoulli utility belongs to the linear risk-tolerance class, taking the form $u_i(c_i) = \frac{1}{b-1}(a_i + bc_i)^{1-\frac{1}{b}}$ where $b > 0, \neq 1$ and $a_i > 0$ for all i .

- a) [30 minutes] Show that *ex ante* Pareto optimal allocations have the property that $c_{is} = f_i \cdot \omega_s + k_i$ where f_i, k_i are independent of s , where ω_s denotes the per capita endowment of the economy in state s .

The Arrow-Borch PO conditions are: $\lambda_i \pi_s u'_i(c_{is}) = \gamma$, for all i, s , or $\frac{u'_i(c_{is})}{u'_i(c_{ir})} = \frac{\pi_s}{\pi_r}$ for any pair of states s, r and any agent i . So the ratio of marginal utilities across any pair of states is equalized across all agents. Since $u'_i(c_i) = (a_i + bc_i)^{-\frac{1}{b}}$, this implies that $\frac{a_i + bc_{is}}{a_j + bc_{js}} = \frac{a_i + bc_{ir}}{a_j + bc_{jr}}$ across any pair of agents i, j and any pair of states r, s . Hence

$$\frac{a_i + bc_{is}}{a_j + bc_{js}} = m_{ij} \quad (1)$$

for all s . Fix state s ; select an arbitrary agent j , and average (1) across all i to obtain (using the budget balance condition which states that average consumption in any state equals the average endowment):

$$\frac{a + b\omega_s}{a_j + bc_{js}} = m_j \quad (2)$$

where a denotes the average of a and m_j the average of m_{ij} across all i . Hence (2) implies

$$a_j + bc_{js} = \frac{1}{m_j} [a + b\omega_s]$$

and so

$$c_{js} = f_j \cdot \omega_s + k_j \quad (3)$$

where $f_j \equiv \frac{1}{m_j}$ and $k_j \equiv \frac{a}{bm_j} - \frac{a_j}{b}$.

- b) [30 minutes] Suppose that a full set of Arrow securities are traded at $t = 0$. Derive an expression for the equilibrium price of a state s security, as a function of the probability and per capita endowment associated with state s .

If z_{is} denotes i 's net purchase of a state- s security, and p_s its price, then i selects a portfolio $\{z_{is}\}$ to maximize $\sum_s \pi_s u_i(\omega_{is} + z_{is})$ subject to $\sum_s p_s z_{is} = 0$. The first-order conditions for an optimal asset portfolio imply

$$\frac{p_s}{p_r} = \frac{u'_i(c_{is})}{u'_i(c_{ir})} = \frac{\pi_s}{\pi_r} \left[\frac{a_i + bc_{is}}{a_i + bc_{ir}} \right]^{-\frac{1}{b}}, \quad (4)$$

or

$$\frac{p_s}{\pi_s (a_i + bc_{is})^{-\frac{1}{b}}} = \frac{p_r}{\pi_r (a_i + bc_{ir})^{-\frac{1}{b}}} = K_i \quad (5)$$

independent of the state. (5) implies that

$$\left[\frac{p_s}{\pi_s K_i} \right]^{-b} = a_i + bc_{is} \quad (6)$$

and averaging across i we obtain

$$\left[\frac{p_s}{\pi_s K} \right]^{-b} = a + b\omega_s \quad (7)$$

for some constant K , i.e.,

$$p_s = K \pi_s [a + b\omega_s]^{-\frac{1}{b}} \quad (8)$$

Problem 3. [60 minutes, total] Two male deer compete for the right to mate with a female deer. Each male deer is strong (S) or weak (W) with an equal probability. Each male deer knows its own type, but not its rival's. The two male deer make a simultaneous decision whether or not to fight with each other. If they don't fight, then the female deer will not mate with any of them; if they do fight, then the female deer will mate with the winner. (Notice that a male deer can mate only if *both* agree to fight and it wins.) Suppose that a strong deer always wins a fight against a weak deer, and that if both deer have the same type, then each wins with probability $\frac{1}{2}$. Finally, suppose that fighting costs a deer 2, and, on top of that, a deer gets 0 if it doesn't mate, and 10 if it does, regardless of whether it fought or not.

a) [5 minutes] Describe the set of pure strategies available to a male deer.

Each male deer has four pure strategies:

G – Give up in either case;

F – Fight in either case;

C – Clever, fight if strong, give up if weak;

D – Dumb, give up if strong, fight if weak.

	G	F	C	D
S :	N	F	F	N
W :	N	F	N	F

b) [15 minutes] Is the situation in which strong deer fight and weak deer don't a Bayesian-Nash equilibrium?

The question is whether (C, C) is a BNE. Suppose that the other deer employs strategy C . In this case, not fighting gives both types of deer a payoff of zero. Fighting gives a strong deer a payoff of

$$\frac{1}{2} \left(\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 - 2 \right) + \frac{1}{2} \left(\begin{matrix} 0 \\ \text{other is weak and therefore doesn't fight} \end{matrix} \right) = \frac{3}{2} > 0,$$

and it gives a weak deer a payoff of

$$\frac{1}{2} \left(\begin{matrix} 0 - 2 \\ \text{other is strong} \end{matrix} \right) + \frac{1}{2} \left(\begin{matrix} 0 \\ \text{other is weak} \end{matrix} \right) = -1 < 0.$$

That is, it is better to fight if and only if a deer is strong. It therefore follows that (C, C) is a BNE.

c) [30 minutes] Find all the pure strategy Bayesian-Nash equilibria. (Hint: there are three.)

The best response correspondence for each deer is given by:

- Against C , as shown above, the best response is C .
- Against G , payoff is zero regardless of what a deer does, so the best response is $\{G, F, C, D\}$.
- Against F , not fighting gives both types of deer a payoff of zero. Fighting gives a strong deer a payoff of

$$\frac{1}{2}(10 - 2) + \frac{1}{2}\left(\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 - 2\right) = \frac{11}{2} > 0,$$

and it gives a weak deer a payoff of

$$\frac{1}{2}(0 - 2) + \frac{1}{2}\left(\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 - 2\right) = \frac{1}{2} > 0.$$

So the best response against F is F .

- Against D , not fighting gives both types of deer a payoff of zero. Fighting gives a strong deer a payoff of

$$\frac{1}{2}(0) + \frac{1}{2}(10 - 2) = 4 > 0,$$

and it gives a weak deer a payoff of

$$\frac{1}{2}(0) + \frac{1}{2}\left(\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 0 - 2\right) = \frac{3}{2} > 0.$$

So the best response against D is F .

- It follows that the BNE of the game are (C, C) , (F, F) , and (G, G) .

d) [10 minutes] Are all the Bayesian-Nash equilibria you found equally plausible?

The BNE (G, G) is less plausible because G is weakly dominated by C (note that a strong deer always weakly benefits from fighting!).

Problem 4. [60 minutes, total] A worker is equally likely to be either a “Low” or a “High” type worker. Neither the worker, nor the “market” can observe the worker’s type. There are two periods: $t = 1$ and $t = 2$. In every period the worker exerts effort $e \in [0, 1]$. The worker’s choice of effort is unobservable by anyone else. The utility cost of the effort in every period is $\frac{1}{2}e^2$ for both types of the worker. There are two possible output levels in every period: 1 and 0. In each period, the output of the “Low” type worker is equal to 1 with probability e , and is equal to 0 with probability $1 - e$, where e is the effort exerted by the worker in that period. The output of the “High” type worker is always equal to 1 regardless of the exerted effort.

Suppose that in every period the worker is paid a market wage *before the worker exerts effort*. Suppose further that the wage that the worker is paid in each period $t \in \{1, 2\}$ is equal to the probability that the worker’s output in period t is equal to 1 given the available information. Since it is assumed that both the the worker and the market can observe the output produced by the worker in the first period, the information that is available in the second period consists of the output produced by the worker in the first period. Finally, suppose that the worker is a risk-neutral expected utility maximizer with the same utility function $u(w, e) = w - \frac{1}{2}e^2$ where w denotes wage in both periods. There is no discounting between periods.

Let \hat{e}_1 denote what the market believes is the effort of the low type worker in the first period. Let w_{21} and w_{20} denote the wages paid to the worker in the second period when its output in the first period is equal to 1 and 0, respectively.

- a) [5 minutes] Explain the reason that the worker will not exert any effort in the second period.

Since $t = 2$ is the last period, the wage is paid by the market up-front and the effort is costly, the worker will optimally exert no effort.

- b) [5 minutes] Find the wage $w_1(\hat{e}_1)$ that the market pays to the worker in the first period as a function of its belief \hat{e}_1 .

The "High" type delivers output equal to 1 with probability 1 in $t = 1$. The "Low" type delivers output equal to 1 with probability \hat{e}_1 in $t = 1$. Thus $w_1(\hat{e}_1) = \Pr\{\text{output}=1 \text{ in } t = 1\} = \Pr\{\text{output}=1 \text{ in } t = 1 \mid \text{"High"}\} \Pr\{\text{"High"}\} + \Pr\{\text{output}=1 \text{ in } t = 1 \mid \text{"Low"}\} \Pr\{\text{"Low"}\} = 1 \cdot \frac{1}{2} + \hat{e}_1 \cdot \frac{1}{2} = \frac{1}{2}(1 + \hat{e}_1)$.

- c) [10 minutes] Find the wage $w_{21}(\hat{e}_1)$ that the market pays to the worker in the second period if the output in the first period was equal to 1 and the worker is believed to have exerted the effort \hat{e}_1 in the first period.

The probability of the first period output be equal to 1 is $\frac{1}{2}(1 + \hat{e}_1)$ (see part (2)). Since the worker exerts no effort in $t = 2$ (see part (a)), "High" type delivers output equal to 1 with probability 1 in $t = 2$, "Low" type delivers output equal to 1 with probability 0 in $t = 2$. Thus

$$\begin{aligned} w_{21}(\hat{e}_1) &= \Pr\{\text{output}=1 \text{ in } t = 2 \mid \text{output}=1 \text{ in } t = 1\} \\ &= \frac{\Pr\{\text{output}=1 \text{ in } t = 2 \text{ AND output}=1 \text{ in } t = 1\}}{\Pr\{\text{output}=1 \text{ in } t = 1\}} \\ &= \frac{\Pr\{\text{output}=1 \text{ in } t=2 \text{ AND output}=1 \text{ in } t=1 \mid \text{High}\} \cdot \Pr\{\text{High}\} + \Pr\{\text{output}=1 \text{ in } t=2 \text{ AND output}=1 \text{ in } t=1 \mid \text{Low}\} \cdot \Pr\{\text{Low}\}}{\Pr\{\text{output}=1 \text{ in } t=1\}} \\ &= \frac{1 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}}{\frac{1}{2}(1 + \hat{e}_1)} = \frac{1}{1 + \hat{e}_1}. \end{aligned}$$

- d) [10 minutes] Find the wage $w_{20}(\hat{e}_1)$ that the market pays to the worker in the second period if the output in the first period was equal to 0 and the worker is believed to have exerted the effort \hat{e}_1 in the first period.

The probability of the first period output be equal to 0 is $\frac{1}{2}(1 - \hat{e}_1)$ (see part (2)). Since the worker exerts no effort in $t = 2$ (see part (a)), "Low" type delivers output equal to 1 with probability 0 in $t = 2$. Thus $w_{21}(\hat{e}_1) = \Pr\{\text{output}=1 \text{ in } t = 2 \mid \text{output}=0 \text{ in } t = 1\} = \frac{\Pr\{\text{output}=1 \text{ in } t=2 \text{ AND output}=0 \text{ in } t=1\}}{\Pr\{\text{output}=0 \text{ in } t=1\}} = \frac{\Pr\{\text{output}=1 \text{ in } t=2 \text{ AND output}=0 \text{ in } t=1 \mid \text{High}\} \cdot \Pr\{\text{High}\} + \Pr\{\text{output}=1 \text{ in } t=2 \text{ AND output}=0 \text{ in } t=1 \mid \text{Low}\} \cdot \Pr\{\text{Low}\}}{\Pr\{\text{output}=0 \text{ in } t=1\}} = \frac{0 \cdot \frac{1}{2} + 0 \cdot \frac{1}{2}}{\frac{1}{2}(1 - \hat{e}_1)} = 0.$

- e) [10 minutes] Find the worker's optimal first period effort $e_1(w_{21}, w_{20})$ as a function of the wages paid in the second period w_{21} and w_{20} . Assume that the worker takes the wages as given.

By exerting the effort e_1 the worker gets w_{21} with probability $\frac{1}{2}(1 + e_1)$ and w_{20} with probability $\frac{1}{2}(1 - e_1)$. Hence the worker's problem is $\max_{e_1} w_1 - \frac{1}{2}e_1^2 + \frac{1}{2}(1 + e_1)w_{21} + \frac{1}{2}(1 - e_1)w_{20}$. The first order condition is $-e_1 + \frac{1}{2}(w_{21} - w_{20}) = 0$. Rearranging we get: $e_1 = \frac{1}{2}(w_{21} - w_{20})$.

- f) [10 minutes] Using your previous results, find the equilibrium first period effort e_1 , and the second period wages w_{21} and w_{20} .

From (3)-(5) we have: $e_1 = \frac{1}{2}w_{21}$, $w_{21} = \frac{1}{1+e_1}$, and $w_{20} = 0$. Thus $e_1 = \frac{1}{2} \frac{1}{1+e_1}$, or $e_1^2 + e_1 - \frac{1}{2} = 0$. The roots of this quadratic equation are $-\frac{1}{2} + \frac{1}{2}\sqrt{3}$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{3}$. After ruling out the negative root we get $e_1 = \frac{1}{2}\sqrt{3} - \frac{1}{2}$, $w_{21} = \sqrt{3} - 1$, and $w_{20} = 0$.

- g) [10 minutes] What does the above model tell us about the likely effort and earnings profiles of workers over their careers?

The workers work hard early in their careers in order to be perceived as high productivity types by the market and be paid a higher wage. The workers work less hard later in their careers because: (i) they are not that keen on being perceived as high productivity types by the market (i.e. career concerns play less of a role); (ii) the market has more information about the productivity of the workers from their previous performance.