A Sufficient Statistics Construction of Exponential Family
Lévy Measure Densities for Nonparametric Conjugate
Models: Supplementary Materail

Proof of Lemma 1: The result in Lemma 1 follows from the definition of
the functional derivative and a modification of the proof in the case where the
derivative is a standard partial derivative. For the sake of completeness we
sketch the argument. A standard reference for the definitions that follow is [1].

Given a space $B$ of functions, for example the space of piecewise continuous
functions on $(0, \infty)$, and a functional $F : B \rightarrow \mathbb{R}$, the functional derivative of
$F$ with respect to $a(x) \in B$ is defined as

$$\frac{\partial F[a(x)]}{\partial a(x)} = \lim_{\varepsilon \rightarrow 0} \frac{F[a(x) + \varepsilon \delta(x)] - F[a(x)]}{\varepsilon},$$

where $\delta(x)$ is an element of a class of test functions, usually taken to be a class
of indicator functions or the class of bump functions on the domain of $B$. Now,
to prove the result in 1 from Lemma 1, it suffices to prove the result for the
exponential family corresponding to the first component of $\eta(z)$, i.e. for $\eta_1(z)$.
Thus, define the function $\psi(\eta_1(z))$ by

$$\psi(\eta_1(z)) = \int \exp(\eta_1(z)T_1(x)) \mu(dx).$$

Now assume $\eta_1(z)$ is such that $\psi(\eta_1(z))$ is finite. In addition assume $\exists \zeta > 0$
such that $\forall g(z) \in B$ if $|\eta_1(z) - g(z)| < \zeta$ uniformly, then $\psi(g(z))$ exists and is
finite. Let $0 < \delta(z) \leq 1$ and $\varepsilon_0 > 0$ be a pair such that $|\eta_1(z) - (\eta_1(z) + \varepsilon_0 \delta(z))| < \zeta$ uniformly. Define $\xi(z) = \eta_1(z) + \varepsilon_0 \delta(z)$ and note that

$$\frac{\psi(\xi(z)) - \psi(\eta_1(z))}{\varepsilon_0} = \int \frac{e^{\xi(z)T_1(x)} - e^{\eta_1(z)T_1(x)}}{\varepsilon_0} \mu(dx)$$

$$= \int e^{(\eta_1(z)T_1(x))} \frac{e^{(\xi(z)-\eta_1(z))T_1(x)} - 1}{\varepsilon_0} \mu(dx) = \int e^{(\eta_1(z)T_1(x))} \frac{e^{\varepsilon_0 \delta(z)T_1(x)} - 1}{\varepsilon_0} \mu(dx)$$

$$\leq \int e^{(\eta_1(z)T_1(x))} \frac{e^{\zeta(\delta(z))T_1(x)}}{\zeta} \mu(dx)$$

$$\leq \int \left| e^{(\eta_1(z) + \zeta \delta(z))T_1(x)} + e^{(\eta_1(z) - \zeta \delta(z))T_1(x)} \right| \mu(dx).$$
Since the above integral is finite, it follows from the Lebesgue dominated convergence theorem that
\[
\lim_{\varepsilon \to 0} \frac{\psi(\xi(z)) - \psi(\eta_1(z))}{\varepsilon_0} = \int \frac{\partial}{\partial(\eta_1(z))} \left[ \exp(\eta_1(z)T_1(x)) \right] \mu(dx).
\]

Extension of the above to higher order functional derivatives proceeds by a standard induction argument. Finally, part 2. of Lemma 1 follows as both the usual chain rule and multiplication by a constant rule hold for functional derivatives.

**Construction and Proof of Theorem 1:** For every pair \(n,i\) define \(\eta_{n,i} = \eta(i/n - 1/2, n)\), \(A_{0,n,i} = A_0(i/n - 1/2, n)\), and \(T_{k,n,i} = A_{0,n,i}T_k(s)\), where the \(T_{k,n,i}\) are independent random variables with \(T_k(s)\) distributed according to a sufficient statistic from an exponential family with density \(h(s) \exp \left\{ \langle \eta_{n,i}, T(s) \rangle - A(\eta_{n,i}) \right\} \).

We use the notation \(A_0(i/n - 1/2, n)\) as shorthand for \(A_0(i/n) - A_0(i/n +)\). Next define
\[
T_{k,n}(0) = 0 \text{ and } T_{k,n}(t) = \sum_{\xi \leq t} T_{k,n,i}(t) \text{ for } t \geq 0.
\]

Note that with this definition, \(T_{k,n}\) has independent increments. Next, we consider \(T_k(s)\) from the exponential family \(p_\theta(x|\eta(z))\) and perform a Taylor expansion on \(e^{-\theta T_k(s)}\) yielding
\[
- \int (1 - e^{-\theta T_k(s)})dL_t(s) = - \int \left( 1 - \left( \sum_{m=0}^\infty \frac{(-1)^m \theta^m T_k^m(s)}{m!} \right) \right) dL_t(s)
\]
\[
= \sum_{m=1}^\infty \frac{(-1)^m \theta^m}{m!} \int T_k^m(s)dL_t(s)
\]
\[
= \sum_{m=1}^\infty \frac{(-1)^m \theta^m}{m!} \int T_k^m(s) \left\{ \int_0^t e^{\langle \eta(z), T(s) \rangle - A(\eta(z))} dA_0(\eta(z)) \right\} ds
\]
\[
= \sum_{m=1}^\infty \frac{(-1)^m \theta^m}{m!} \int_0^t \left\{ \int T_k^m(s)e^{\langle \eta(z), T(s) \rangle - A(\eta(z))} ds \right\} dA_0(\eta(z))
\]
\[
= \sum_{m=1}^\infty \frac{(-1)^m \theta^m}{m!} \int_0^t e^{-A(\eta(z))} \left[ \frac{\partial^m e^{A(\eta(z))}}{\partial \eta_k^m} \right] dA_0(\eta(z)).
\]

In the above, the last equality holds by Lemma 1.
Next, in order to compute $E[e^{-\theta T_k,n}]$ we must first compute $E[T^m_{k,n,i}]$, where the expectation is with respect to the density of the random variable $T_{k,n,i}$. To that end we compute the density of $T_{k,n,i}$.

Since $s$ is distributed according to the exponential family

$$h(x)e^{\eta(z)A(\eta(z))}$$

as $T_k(s)$ satisfies the conditions of the theorem, by setting $u = T_k(s)$, and $v = A_{0,n,i}T_k(s)$, a simple calculation shows that $T_{k,n,i}$ has a density of the form

$$A_{0,n,i}^{-1}(dT_k^{-1}(dv))h\left(T_k^{-1}\left(\frac{v}{A_{0,n,i}}\right)\right)\exp\left(\langle \eta_{n,i}, vA_{0,n,i}^{-1} \rangle - A(\eta_{n,i})\right).$$

As this integrates to 1 with respect to $v$, we have

$$\int \left\{A_{0,n,i}^{-1}(dT_k^{-1}(dv))h\left(T_k^{-1}\left(\frac{v}{A_{0,n,i}}\right)\right)\exp\left(\langle \eta_{n,i}, vA_{0,n,i}^{-1} \rangle \right)\right\} dv = e^{A(\eta_{n,i})},$$

from which it follows that

$$\int \left\{A_{0,n,i}^{-1}(dT_k^{-1}(dv))h\left(T_k^{-1}\left(\frac{v}{A_{0,n,i}}\right)\right)\exp\left(\langle \eta_{n,i}, vA_{0,n,i}^{-1} \rangle \right)\right\} dv = e^{A(\eta_{n,i})}.$$

Therefore, taking functional derivatives of both sides with respect to $\eta_{k,n,i}A_{0,n,i}^{-1}$, by Lemma 1 we have

$$\int \left\{T^m_{k,n,i}A_{0,n,i}^{-1}(dT_k^{-1}(dv))h\left(T_k^{-1}\left(\frac{v}{A_{0,n,i}}\right)\right)\exp\left(\langle \eta_{n,i}, vA_{0,n,i}^{-1} \rangle \right)\right\} dv$$

$$= \frac{\partial^m}{\partial(\eta_{k,n,i}A_{0,n,i}^{-1})^m} \left[e^{A(\eta_{n,i})}\right] = A_{0,n,i} \frac{\partial^m}{\partial(\eta_{k,n,i})^m} \left[e^{A(\eta_{n,i})}\right].$$

Multiplying both sides of (1) by $e^{-A(\eta_{n,i})}$, we conclude that

$$E[T^m_{k,n,i}] = A_{0,n,i} \left(e^{-A(\eta_{n,i})} \frac{\partial^m}{\partial(\eta_{k,n,i})^m} \left[e^{A(\eta_{n,i})}\right]\right).$$

Thus, as $n \to +\infty$

$$E[T_{k,n}(t)] = \sum_{\frac{n-1}{2} \leq t} E[T_{k,n,i}(t)] =$$

$$\sum_{\frac{n-1}{2} \leq t} A_{0,n,i} \left(e^{-A(\eta_{n,i})} \frac{\partial}{\partial \eta_{k,n,i}} \left[e^{A(\eta_{n,i})}\right]\right) \to \int_0^t e^{-A(\eta(z))} \frac{\partial}{\partial \eta_{k,n,i}} \left[e^{A(\eta(z))}\right] dA_0(z).$$
Now we consider the quantity $E[e^{-\theta T_{k,n,i}}]$. Denoting the density of $T_{k,n,i}$ by $f_{k,n,i}(v)$ and performing a Taylor expansion on $e^{-\theta T_{k,n,i}}$ we have

$$E[e^{-\theta T_{k,n,i}}] = \int e^{-\theta T_{k,n,i}(s)} f_{k,n,i}(v) dv = \sum_{m=0}^{\infty} \frac{(-1)^m \theta^m}{m!} \int T_{k,n,i}^m(s) f_{k,n,i}(v) dv$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \theta^m}{m!} E[T_{k,n,i}^m]$$

$$= 1 + \sum_{m=1}^{\infty} \frac{(-1)^m \theta^m}{m!} \{ A_{0,n,i} \left( e^{-A(\eta_{n,i})} \frac{\partial}{\partial (\eta_{n,i})^m} \left[ e^{A(\eta_{n,i})} \right] \right) \},$$

where, similar to (2), as $n \to +\infty$ we have

$$\sum_{a \leq i \leq b} A_{0,n,i} \left( e^{-A(\eta_{n,i})} \frac{\partial}{\partial (\eta_{n,i})^m} \left[ e^{A(\eta_{n,i})} \right] \right) \to \int_0^t e^{-A(\eta(z))} \frac{\partial}{\partial (\eta(z))^m} \left[ e^{A(\eta(z))} \right] dA_0(z). \quad (3)$$

Defining

$$z_{n,i} = \sum_{m=1}^{\infty} \frac{(-1)^m \theta^m}{m!} \{ A_{0,n,i} \left( e^{-A(\eta_{n,i})} \frac{\partial}{\partial (\eta_{n,i})^m} \left[ e^{A(\eta_{n,i})} \right] \right) \}$$

from (3) we conclude

$$\sum_{a \leq i \leq b} \left( e^{-A(\eta_{n,i})} \frac{\partial}{\partial (\eta_{n,i})^m} \left[ e^{A(\eta_{n,i})} \right] \right) \to \int_0^t (1 - e^{-\theta T_k(s)}) dL(s).$$

As

$$E[e^{-\theta T_{k,n,i}}] = E \left[ \prod_{a \leq i \leq b} \exp(-\theta T_{k,n,i}) \right] = \prod_{a \leq i \leq b} (1 + z_{n,i}),$$

we may now invoke a Lemma A.1 from Appendix 1 of [2]. We state the result here for the sake of completeness:

**Lemma 0.1** Let $z_{n,i}$ be real numbers, for $n \geq 1$ and $i \geq 1$. Assume that, as $n \to +\infty$, (i) $\sum_{a < \frac{a}{b} \leq b} z_{n,i} \to z$, (ii) $\max_{a < \frac{a}{b} \leq b} |z_{n,i}| \to 0$, (iii) $\limsup \sum_{a < \frac{a}{b} \leq b} z_{n,i} \leq M < +\infty$. Then $\prod_{a < \frac{a}{b} \leq b} (1 + z_{n,i}) \to e^z$.

From the above lemma and the preceding calculations, we conclude that

$$E[e^{-\theta T_{k,n,i}(s)}] \to \exp \left\{ - \int (1 - e^{\theta T_k(s)}) dL_1(s) \right\}.$$
As in [2] an analogous argument shows that the finite dimensional distributions of \( \{T_{k,n}(s)\} \) converge properly as well. The fact that, for all \( R > 0 \), the sequence \( \{T_{k,n}(s)\}_{n=1}^{\infty} \) is tight in the space \( D([0,R]) \) of all functions that are right continuous with left hand limits in the Skorohod topology follows from (2) and the proof of 15.6 in [3]. Hence, as in [2], the sequence \( \{T_{k,n}(s)\} \) converges to a random element of \( D([0,R]) \) for every \( R > 0 \), and the process so defined may be taken to be the \([0, R]\) restriction of a Lévy process \( T_k \) on \([0, \infty)\) whose Lévy representation is given in Theorem 1. This completes the proof of the theorem.

References

