Linking Time-reversal to Krylov Methods in Acoustic Focusing and Imaging

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Outline

1. Introduction: Time-reversal focusing
2. Lanczos Iterated Time-reversal focusing
   - Numerical tests
3. MUSIC Imaging via Lanczos Time-reversal
   - Numerical tests

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3. MUSIC Imaging via Lanczos Time-reversal
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Motivation

Time-reversal focusing & imaging

- NDE: Locating scatterers, flaws & voids (Johnson, et al. 2008(R))
- Medical applications: lithotripsy, HIFU radiation (Fink, et al. 2003(R))
- Underwater communication (1960's; 1990's - present)
- Decomposition of Time Reversal Operator (DORT) (Prada & Fink, 1990's - present)
- TR Imaging (2000's - present)

Motivation

Time-reversal focusing & imaging

- Underwater communication:
  - Parvulescu & Clay (1965 - matched-signal processing)
  - Dowling (phase conjugation) (early 1990's).
  - TR Demonstration in ocean (Edelmann, et al. 2002)
- Decomposition of Time Reversal Operator (DORT):
  - Iterative focusing on isolated scatterer: Prada & Fink (1994)
  - Multiple scatterers (2004)
  - Demonstration of TRM in ocean waveguide (Gaumond et al. 2006, Prada et al. 2007)
- TR Imaging:
  - MUltiple SIgnal Classification - MUSIC (Devaney, mid 2000).
  - Extended to multiple scattering (Devany 2005).
  - TR & sampling methods (Cheney 2004, Colton 2000(R), Kirsch 2007).
  - TR = adjoint fields.

Motivation

Time-reversal focusing & imaging

- Drawbacks of iterative time-reversal:
  - Localization of multiple scatterers is tedious.
  - Poor convergence with scatterers of similar strengths.
  - Slow convergence limits application environments and frequency ranges.
  - Alternative: Measure entire multi-static response matrix, is worse.
- Goal: Efficiently identify & image multiple targets with few transmissions.
  - Quasi-stationary medium.
  - Large numbers of channels.
  - Signal strength issues when using small elements.

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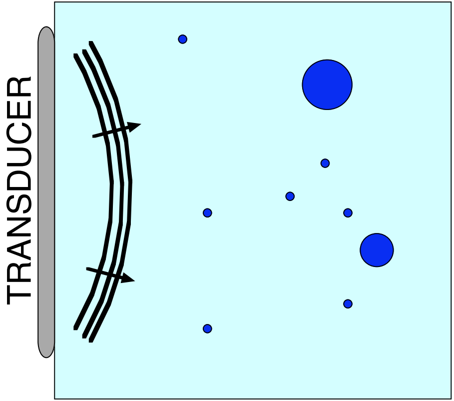
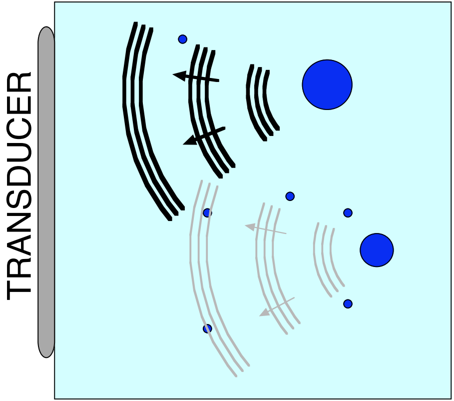
Iterated Time-Reversal



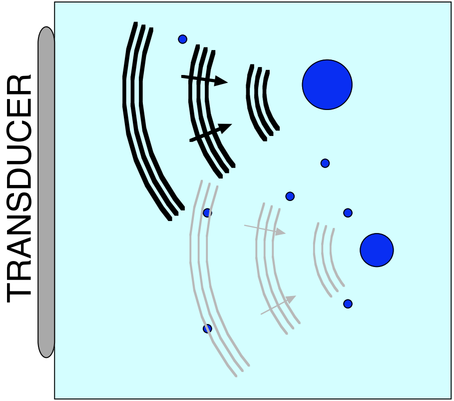
Transmit Pulse

Iterated Time-Reversal



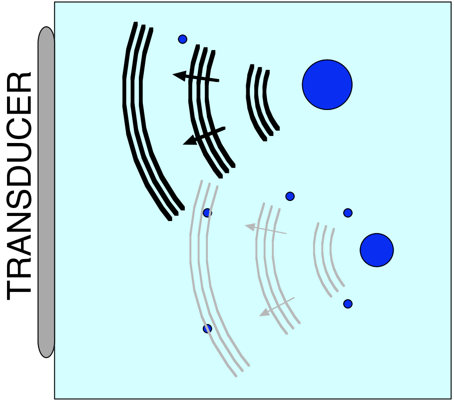
Received Pulse

Iterated Time-Reversal



Re-Transmit Time Reversed Pulse

Iterated Time-Reversal



Receive Pulse

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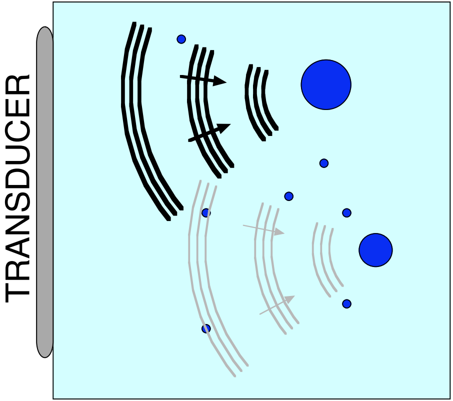

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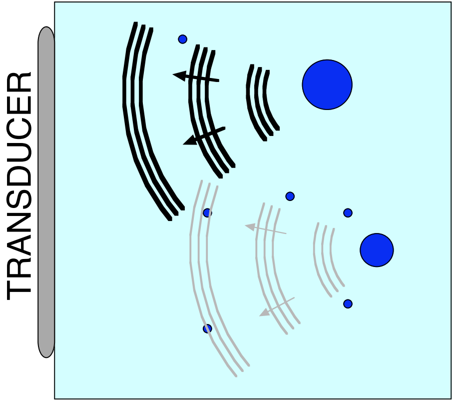

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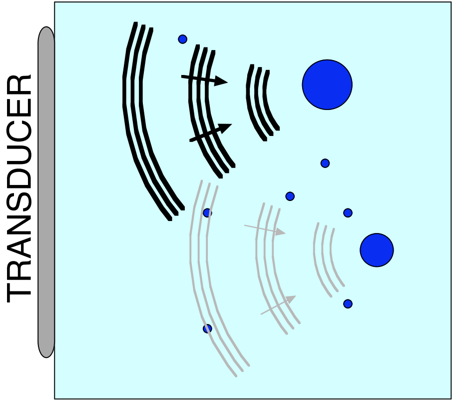

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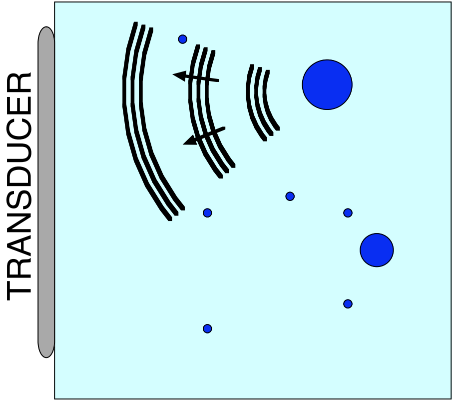

“Final" Received Pulse

Iterated Time-Reversal: Mathematics

Multistatic response matrix: G

G_{ij} = measured signal on xdcr j due to unit excitation on xdcr i.

Properties:

- $G_{ij} = G_{ji}$ symmetric due to reciprocity.
- $G \neq G^{\dagger}$ Not Hermitian.
- $G^{*} = G^{\dagger}$ Conjugate = Hermitian transpose.
- $H = GG^{*}$ = Hermitian.

Iterated Time-Reversal: Mathematics

A single time-reversal iteration

1. Transmit $v^{(0)}(\boldsymbol{x}, t)$
2. Receive $v^{(1/2)}(\boldsymbol{x}, t) = G[v^{(0)}(\boldsymbol{x}, t)]$
3. Transmit $v^{(1/2)}(\boldsymbol{x}, T - t)$
4. Receive $v^{(1)}(\boldsymbol{x}, t) = G[v^{(1/2)}(\boldsymbol{x}, T - t)]$

Can write as $v^{(1)}(\boldsymbol{x}, t) = H[v^{(0)}(\boldsymbol{x}, t)]$, here $H = G^T G$ is the time-reversal operator

Time reversal iterations: $v^{(n)} = \overbrace{H \cdot H \cdots H}^{n}[v^{(0)}]$

As n increases $v^{(n)}$ selectively focuses on the strongest scatterer. Why?

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1. $v^{(n)} = H^n[v^{(0)}]$ are power iterations. As n increases $v^n \approx \phi^{(1)}$, eigenvector of the largest eigenvalue of H
2. Let $\{s^{(i)}, \phi^{(i)}(\boldsymbol{x}, t)\}$ be the ith eigenpair for G; then eigenpairs for H are $\{\lambda^{(i)}, \phi^{(i)}(\boldsymbol{x}, t)\}$, where $\lambda^{(i)} = |s^{(i)}|^2$
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Criticism of Power iterations:

1. Tedious for multiple eigenvalues
2. Poor convergence for similar eigenvalues
3. Non-optimal convergence in general

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Time-reversal using Lanczos iterations

Can we do better? Yes, Lanczos iterations

Let $\mathcal{K}^{n+1}(H, v^{(0)}) = \{v^{(0)}, Hv^{(0)}, \cdots, H^n v^{(0)}\}$ be the Krylov subspace

- For Power iterations $\lambda^{(1)}$ is approximated by the Rayleigh quotient of $H^n v^{(0)}$
- For Lanczos iterations $\lambda^{(1)}$ is approximated by the maximum of the Rayleigh quotient of all vectors in $\mathcal{K}^{n+1}(H, v^{(0)})$
- That is, Lanczos is always better than power iterations.

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Time-reversal using Lanczos iterations

1. Begin with arbitrary signal $v^{(1)}$ such that $(v^{(1)}, v^{(1)}) = 1$
2. Let $w^{(1)} = H[v^{(1)}]$. (standard time-reversal operation.)
3. Iterate:
   1. $\alpha_1 = (w^{(1)}, v^{(1)})$
   2. $w^{(1)} = w^{(1)} - \alpha_1 v^{(1)}$ (orthogonalize)
   3. $\beta_2 = \sqrt{(w^{(1)}, w^{(1)})}$
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   5. $w^{(2)} = H[v^{(2)}] - \beta_2 v^{(1)}$

After n iterations:

- Vectors $v^{(j)}$ represent orthonormal basis for $\mathcal{K}^n(H, v^{(1)})$.
- Create $n \times n$ matrix $T = \mathrm{tridiag}(\alpha_i, \beta_i, \alpha_i)$
- Let $\{\omega^{(i)}, \psi^{(i)}\}$ be the ith eigenpair for $\boldsymbol{T}$ . . . . .
- . . . . .then $\lambda^{(i)} \approx \omega^{(i)}$ and $\phi^{(i)} \approx \sum_{j=1}^{n} \psi_j^{(i)} v^{(j)}$

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- Vectors $v^{(i)}$ represent orthonormal basis for $\mathcal{K}^n(H, v^{(1)})$.
- Create $n \times n$ matrix $T = \mathrm{tridiag}(\alpha_i, \beta_i, \alpha_i)$
- Let $\{\omega^{(i)}, \psi^{(i)}\}$ be the ith eigenpair for $\boldsymbol{T}$ . . . . .
- . . . . .then $\lambda^{(i)} \approx \omega^{(i)}$ and $\phi^{(i)} \approx \sum_{j=1}^{n} \psi_j^{(i)} v^{(j)}$

Time-reversal using Lanczos iterations

1. Begin with arbitrary signal $v^{(1)}$ such that $(v^{(1)}, v^{(1)}) = 1$
2. Let $w^{(1)} = H[v^{(1)}]$. (standard time-reversal operation.)
3. Iterate:
   1. $\alpha_1 = (w^{(1)}, v^{(1)})$
   2. $w^{(1)} = w^{(1)} - \alpha_1 v^{(1)}$ (orthogonalize)
   3. $\beta_2 = \sqrt{(w^{(1)}, w^{(1)})}$
   4. $v^{(2)} = w^{(1)}/\beta_2$ (normalize)
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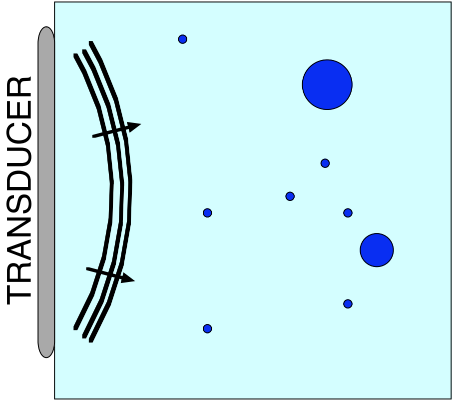
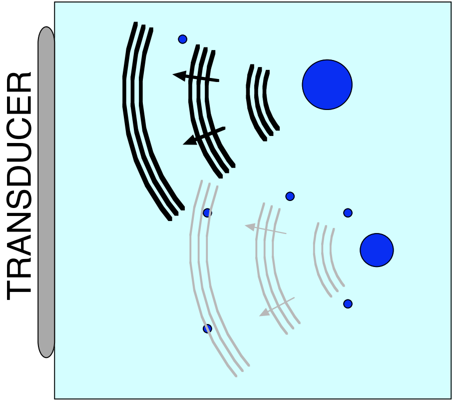
Iterated Lanczos Time-Reversal



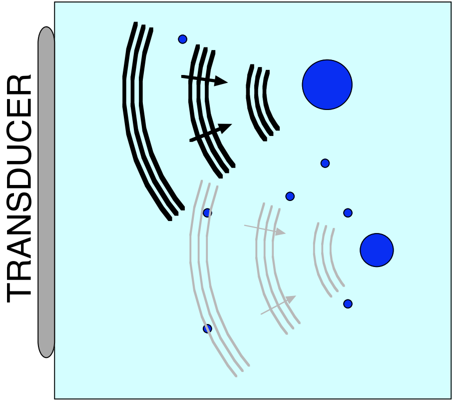
Transmit Pulse

Iterated Lanczos Time-Reversal



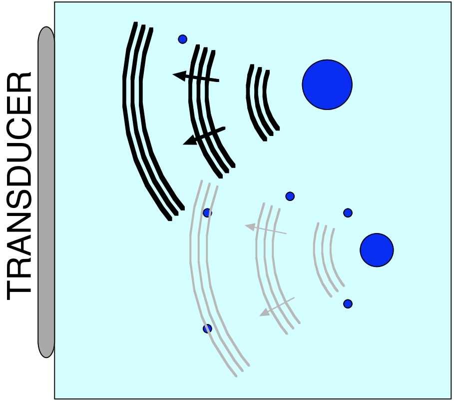
Receive Pulse

Iterated Lanczos Time-Reversal



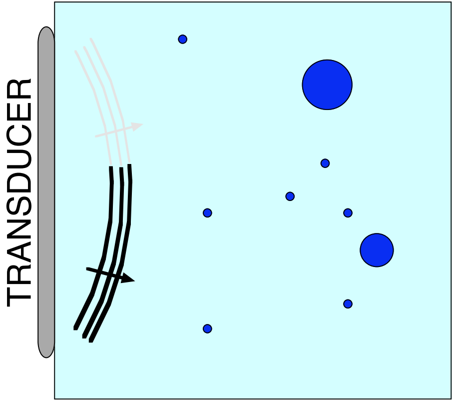
Re-Transmit Time Reversed Pulse

Iterated Lanczos Time-Reversal



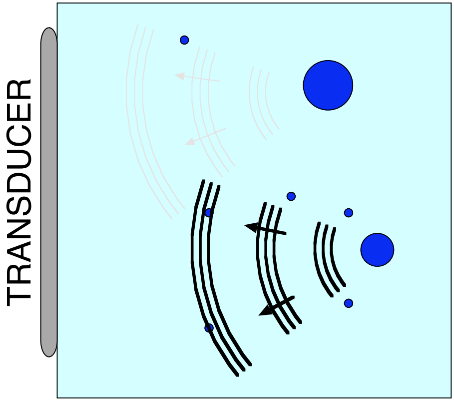
Receive Pulse

Iterated Lanczos Time-Reversal



Transmit Orthogonalized Pulse

Iterated Lanczos Time-Reversal



Receive New Pulse

Time-reversal using Lanczos iterations

Changes to (Power-iterations based) time-reversal experiment

1. Number of time reversal operations remain the same
2. Each time reversal step followed by $\approx 10N$ flops ($N = N_s \times N_t$)
3. Keep 3 vectors of length N in memory
4. Store n Lanczos vectors on hard drive
5. $O(Nn)$ flops to evaluate $\{\lambda^{(i)}, \phi^{(i)}\}$

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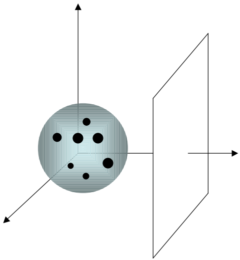
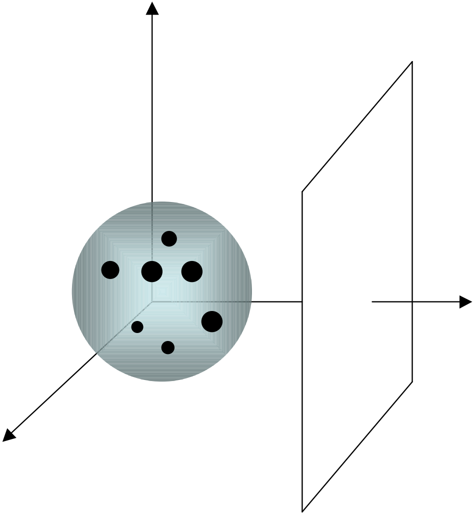
Numerical Tests



Preliminaries

1. Domain Ω with inhomogeneities (scatterers)
2. Surface Γ embedded with transmitters/receivers
3. Transmitted or received signal is a function of $\boldsymbol{x} \in \Gamma$ and $t \in (0, T)$
4. G is the scattering operator; s is transmitted signal; $r = G[s]$ is received signal
5. Sometimes convenient to discretize space $\boldsymbol{x}$ or time t or both

Numerical Test 1



- Time-harmonic case
- Transmitters & receivers at $x = D$
- $D \gg \delta$
- Neglect multiple scattering
- Ideally separated

Numerical Test 1

For the scattering operator G:

- Eigenvalues , $s^{(i)}$, proportional to scatterer strength
- Eigenvectors given by

$$\phi^{(i)}(y,z) = g^{(i)}(D,y,z)/\|g^{(i)}(D,y,z)\|$$

where

$$g^{(i)}(\boldsymbol{x}) = \frac{e^{ik|\boldsymbol{x}-\boldsymbol{x}_i|}}{4\pi|\boldsymbol{x}-\boldsymbol{x}_i|}.$$

For the time reversal operator H:

- Eigenvalues, $\lambda^{(i)} = |s^{(i)}|^2$
- Eigenvectors equal to $\phi^{(i)}(y,z)$

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Numerical Test 1

Case A: Two scatterers ($N = 2$)

- For Power iterations, error in the first eigenvector $\epsilon_1 \propto |\frac{s^{(2)}}{s^{(1)}}|^{2n}$
  - For different strengths, $s^{(2)} = 0.70s^{(1)}$, $\epsilon_1 = 0.08\%$ after 10 iterations
  - For similar strengths, $s^{(2)} = 0.98s^{(1)}$, $\epsilon_1 = 66\%$ after 10 iterations
- Lanczos iterations converge to the correct answer in two iterations

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Numerical Test 1

Case B: Multiple scatterers ($N = 200$)

- Scattering strength selected randomly between 0 and 1
- Initial transmitted signal: point source
- Power iterations: 10 iterations to determine $\phi^{(1)}$ and then 10 to determine $\phi^{(2)}$
- Lanczos iterations: 20 iterations
- 10,000 realizations to quantify performance
- For each realization compute error in the 1st and 2nd eigenfunction
- Plot histogram of error

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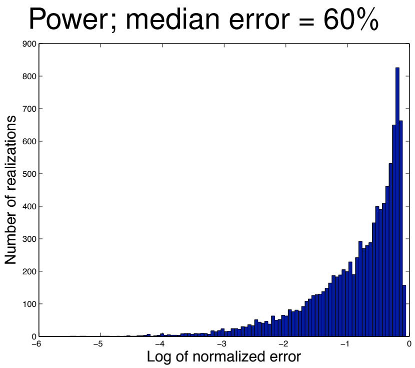
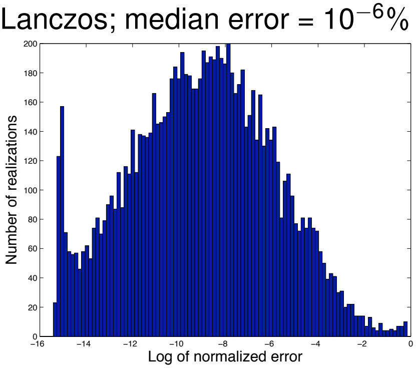
Numerical Test 1

Error in $\phi^{(1)}$

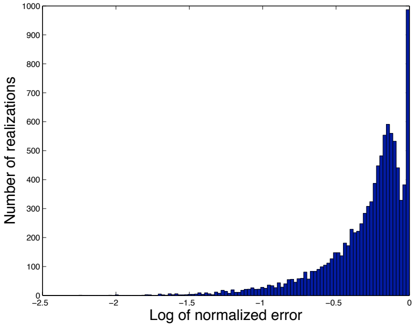




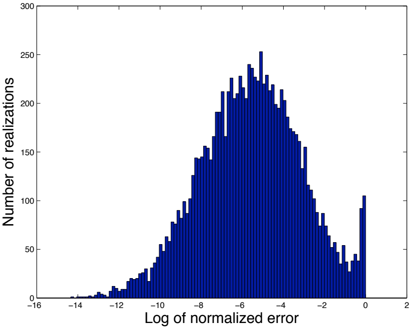
Numerical Test 1

Error in $\phi^{(2)}$

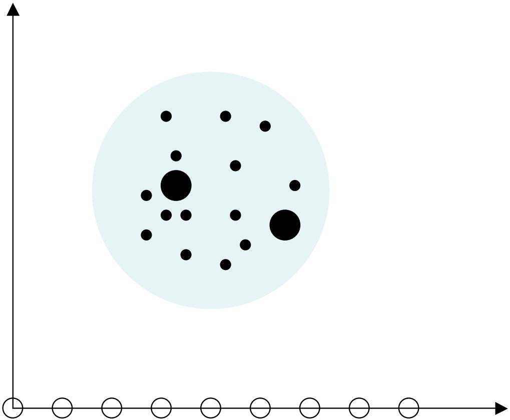
Power; median error = 100%



Lanczos; median error = 10^{-3}%



Numerical Test 2: Focusing

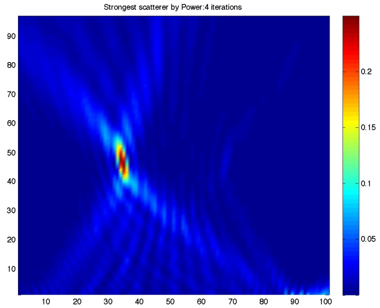


- Time-harmonic case; $\lambda \approx 0.25$
- 40 Transmitters & receivers at $y = 0$; $L_x = 6$
- 30 point scatterers; $D = 6$; 2 strong scatterers
- Solve Helmholtz equation
- Test ability to target

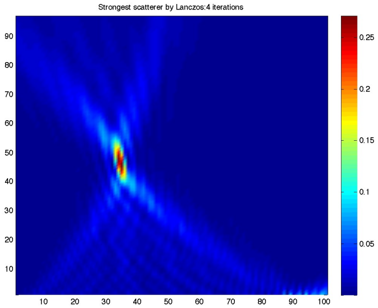
Numerical Test 2: Focusing

Case A: Strong contrast $= 3/2$; number of iterations $n = 4$

Power: strongest



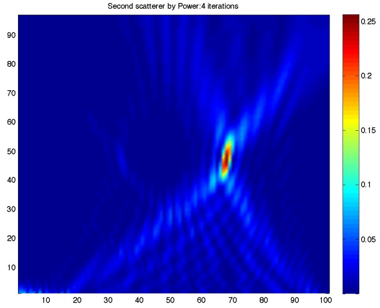
Lanczos: strongest



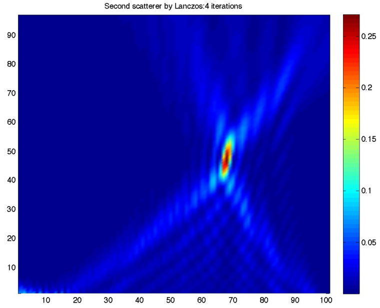
Numerical Test 2: Focusing

Case A: Strong contrast $= 3/2$; number of iterations $n = 4$

Power: next



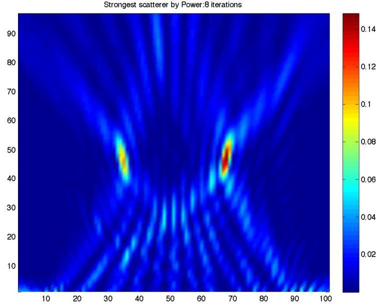
Lanczos: next



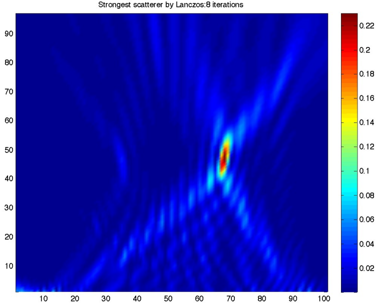
Numerical Test 2: Focusing

Case B: Weak contrast $= 2.1/2$; number of iterations $n = 8$

Power: strongest



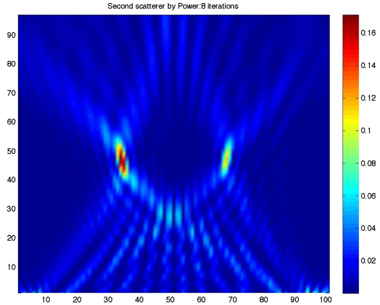
Lanczos: strongest



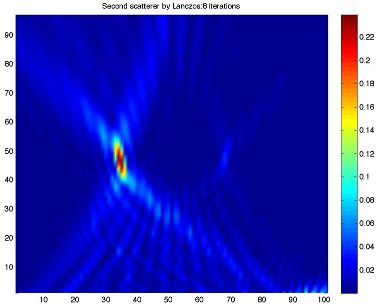
Numerical Test 2: Focusing

Case C: Weak contrast $= 2.1/2$; number of iterations $n = 8$

Power: next



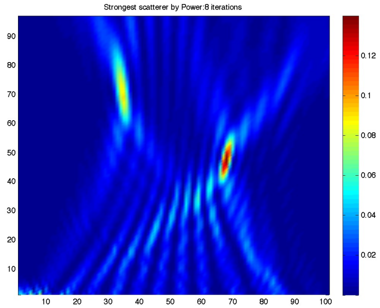
Lanczos: next



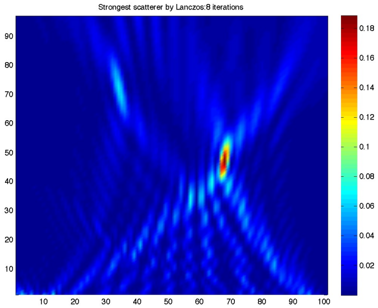
Numerical Test 2: Focusing

Case B: Strong contrast $= 3/2$; ratio of $d_y = 2/3$; $n = 8$

Power: strongest



Lanczos: strongest



Numerical Test 2: Focusing

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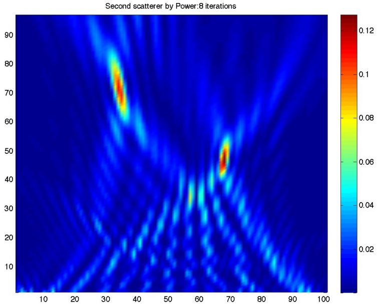
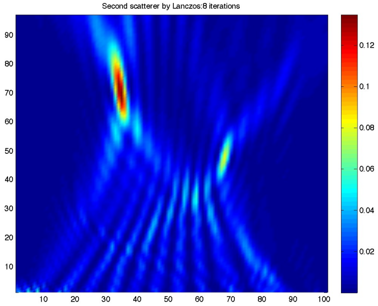


Lanczos: next



Outline

MUSIC Imaging

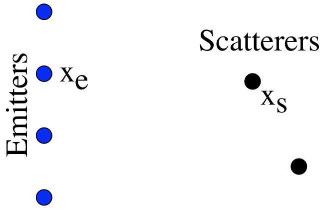
Imaging point scatterers by MUSIC:

- Goal: Identify locations of several point-like targets
- Method based on computing range of multi-static response matrix, G.
- MUSIC = MUltiple SIgnal Classification, designed to pick out dominant frequency content of a signal.
- Adapted to imaging by Devaney 2000.
- Recently recognized as related to Colton's “Linear Sampling Method," and Kirsch's “Factorization Method."

MUSIC Imaging

Consider a scatterer at location $\boldsymbol{x}_s$.
Characterize scattered field as:

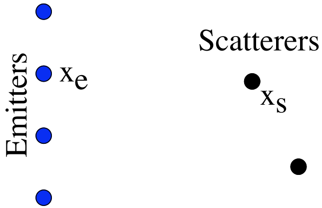
$$u_{scat}(\boldsymbol{x}) = u_{inc}(\boldsymbol{x}_s)\tau_s\, g(\boldsymbol{x}, \boldsymbol{x}_s) \quad (1)$$



Here,

- $u_{inc}(\boldsymbol{x}_s) =$ is the total field incident on the scatterer (including multiply scattered contributions).
- $\tau_s =$ is the scattering strength of the scatterer.
- $g(\boldsymbol{x}, \boldsymbol{x}_s) =$ Green's function = field at $\boldsymbol{x}$ due to a point source at $\boldsymbol{x}_s$.

MUSIC Imaging



Measured field at locations $\boldsymbol{x}_e^j$, $j = 1, \ldots N_e$ is:

$$u(\boldsymbol{x}_e^j) = \sum_{s=1}^{N_s} A_s\, g(\boldsymbol{x}_e^j, \boldsymbol{x}_s) \qquad (2)$$

Remarks:

- Different incident fields give rise to different A_s.
- For any incident field, measured field is in Span $\{g(\boldsymbol{x}_e, \boldsymbol{x}_s^1), g(\boldsymbol{x}_e, \boldsymbol{x}_s^2), \ldots, g(\boldsymbol{x}_e, \boldsymbol{x}_s^{N_s})\}$.

MUSIC Imaging

Let $G =$ multistatic response matrix. Then for $N_e \gg N_s$:

$$\begin{aligned} \text{Range}G &= Span\{g(\boldsymbol{x}_e, \boldsymbol{x}_s^1), g(\boldsymbol{x}_e, \boldsymbol{x}_s^2), \ldots, g(\boldsymbol{x}_e, \boldsymbol{x}_s^{N_s})\} \qquad (3) \\ &= \text{“signal space"} \qquad (4) \end{aligned}$$

Test for scatterer location:

$$\boldsymbol{x} \in \{\boldsymbol{x}_s^1, \boldsymbol{x}_s^2, \ldots \boldsymbol{x}_s^{N_s}\} \Longleftrightarrow g(\boldsymbol{x}_e, \boldsymbol{x}) \in \text{Range}G \qquad (5)$$

MUSIC Imaging

MUSIC Imaging function.

- Let $V = N_e \times N_s$ be orthonormal projector onto signal space.
- Let $V_{noise} = I - V$ is orthonormal projector onto “noise" space.

$$\text{indicator}(\boldsymbol{x}) = \frac{1}{g'(\boldsymbol{x}_e, \boldsymbol{x})V_{noise}g(\boldsymbol{x}_e, \boldsymbol{x})} \tag{6}$$

Remark:

- $V =$non-null eigenvectors of $G G^* = H$.
- $V_{noise} = \text{null}(G G^*) \equiv \text{null}(H)$.

Standard MUSIC Imaging

1. Excite emitters $i = 1, \ldots, N_e \gg N_s$ in turn.
2. For each excitation, i, measure G_{ij} = signal on N_e emitters, j.
3. Compute $H = G\,G^*$.
4. Compute eigenvectors of H.
5. Compute $V = [\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \ldots]$ = non-null eigenvectors of $G\,G^* = H$.
6. Compute $V_{noise} = I - V$.
7. For each point $\boldsymbol{x}$ in region of interest, compute:

$$\text{indicator}(\boldsymbol{x}) = \frac{1}{g'(\boldsymbol{x}_e, \boldsymbol{x})\, V_{noise}\, g(\boldsymbol{x}_e, \boldsymbol{x})} \tag{7}$$

Lanczos MUSIC Imaging

1. Initialization: $v^{(1)} = H[v^{(0)}]$ (Eliminate null component of $v^{(0)}$.)
2. Perform $N_s \ll N_e$ Lanczos iterations to construct $V = [v^{(1)}, v^{(2)}, v^{(3)}, \ldots]$.
3. Compute $V_{noise} = I - V$.
4. For each point $\boldsymbol{x}$ in region of interest, compute:

$$\text{indicator}(\boldsymbol{x}) = \frac{1}{g'(\boldsymbol{x}_e, \boldsymbol{x}) V_{noise} g(\boldsymbol{x}_e, \boldsymbol{x})} \tag{8}$$

Remark:

- Lanczos procedure automatically stops at N_s iterations, when $\beta = 0$.

Lanczos vs. Standard MUSIC Imaging

- Lanczos requires *many* fewer acoustic excitations.
- Lanczos requires no eigenvalue computations.
- Lanczos procedure provides new image with each new measurement.
- But does it work?

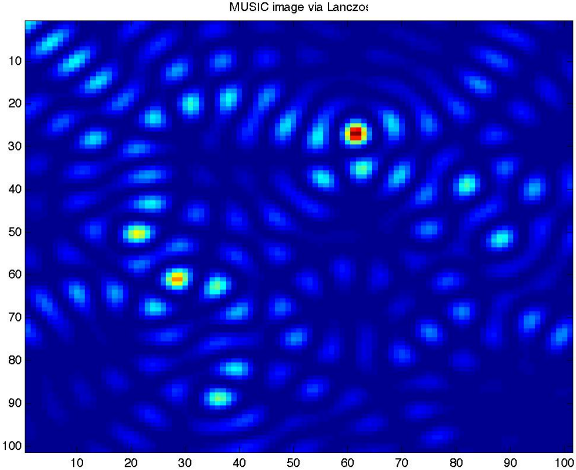
Lanczos vs. Standard MUSIC Imaging

- Lanczos requires *many* fewer acoustic excitations.
- Lanczos requires no eigenvalue computations.
- Lanczos procedure provides new image with each new measurement.
- But does it work?

Numerical Tests: Imaging point scatterers

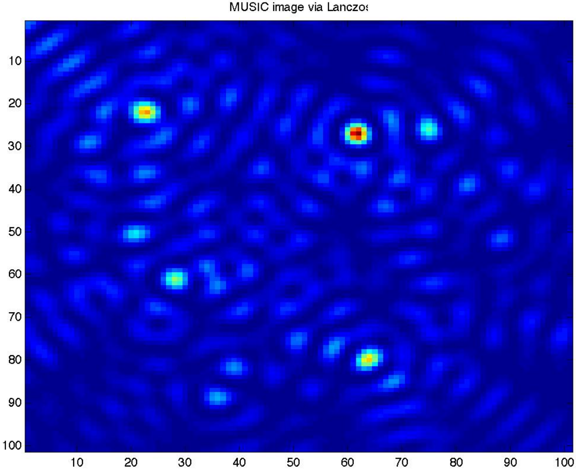
- Far-field array of 240 transducers (emitters).
- Time-harmonic excitation.
- Scattered field computed including multiple scattering.
- For each excitation, update V and compute indicator($\boldsymbol{x}$).
- Stop iterating when $\beta <$ tolerance.
- Field of view = 6 $\times$ 6, so $k =$ number of wavelengths across FOV.

Test 1: 16 scatterers clover pattern; $k = 9$



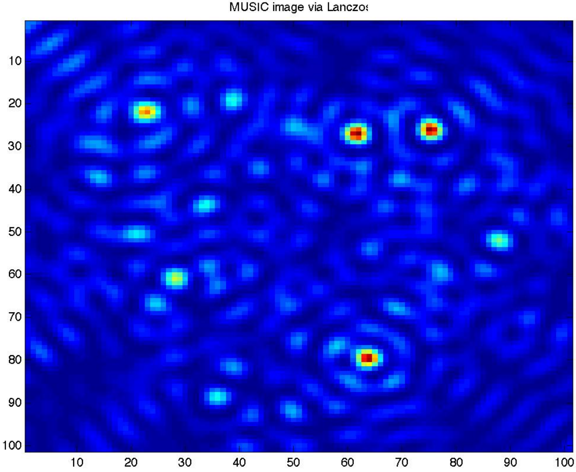
Iteration 1

Test 1: 16 scatterers clover pattern; $k = 9$



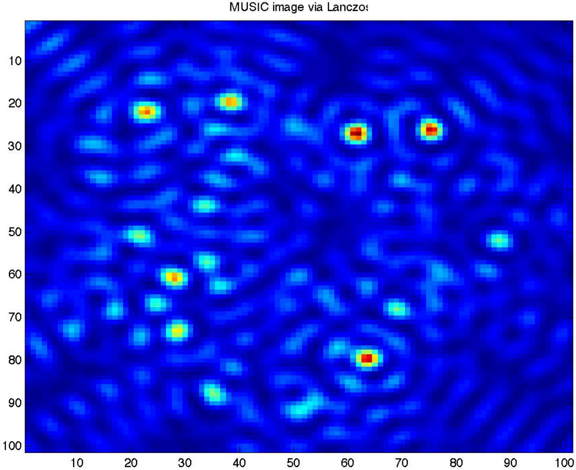
Iteration 2

Test 1: 16 scatterers clover pattern; $k = 9$



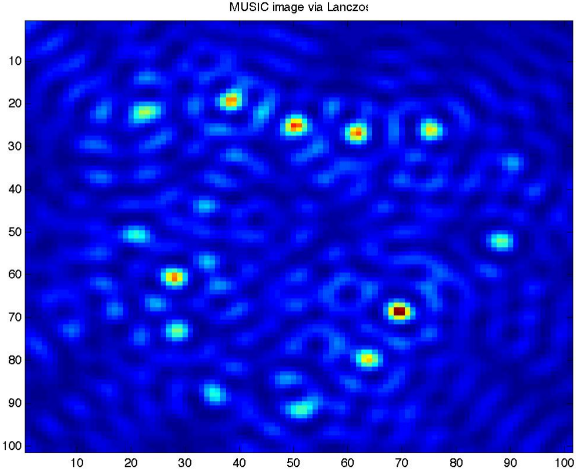
Iteration 3

Test 1: 16 scatterers clover pattern; $k = 9$



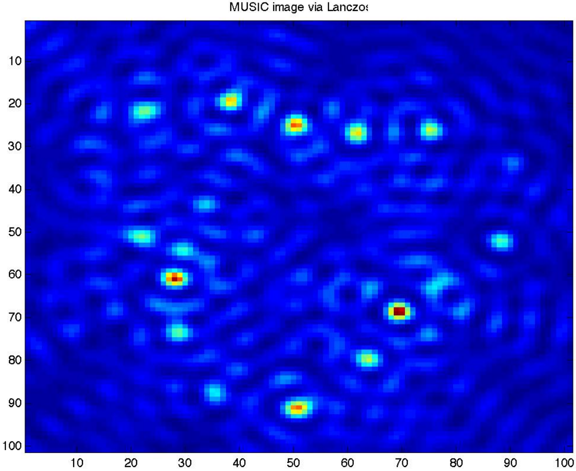
Iteration 4

Test 1: 16 scatterers clover pattern; $k = 9$



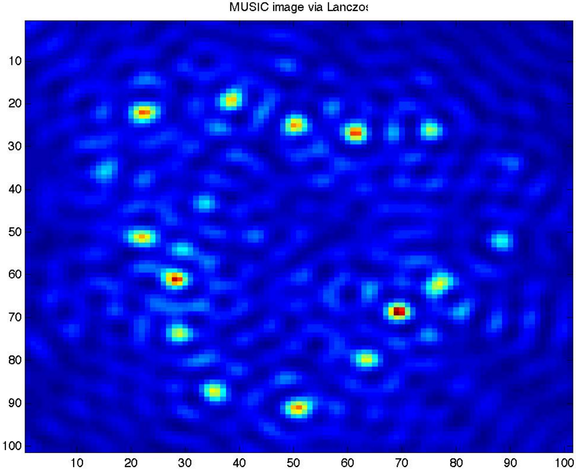
Iteration 5

Test 1: 16 scatterers clover pattern; $k = 9$



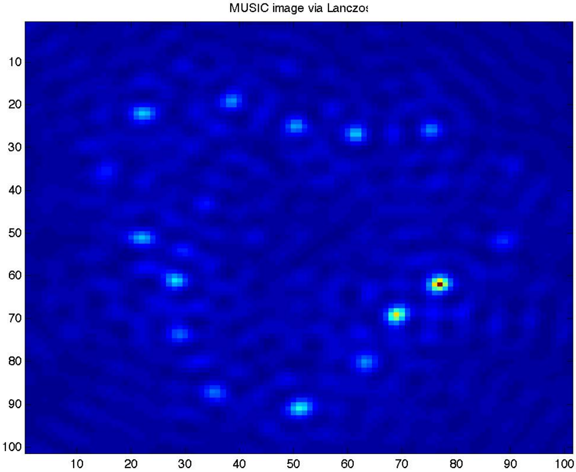
Iteration 6

Test 1: 16 scatterers clover pattern; $k = 9$



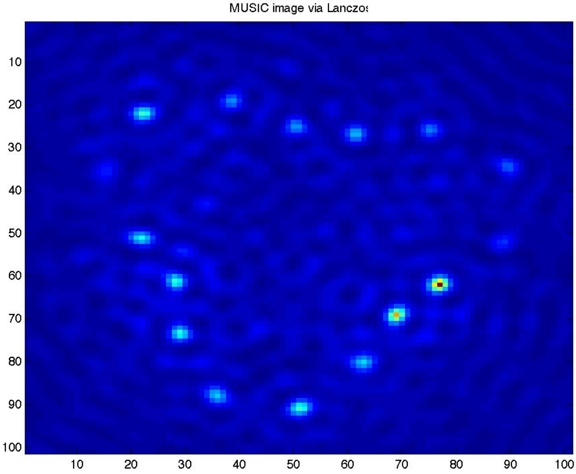
Iteration 7

Test 1: 16 scatterers clover pattern; $k = 9$



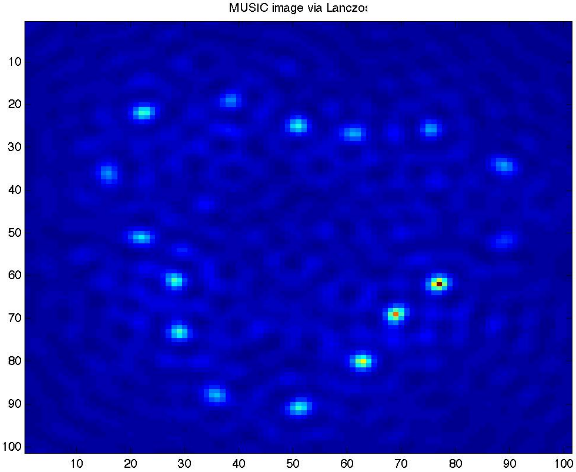
Iteration 8

Test 1: 16 scatterers clover pattern; $k = 9$



Iteration 9

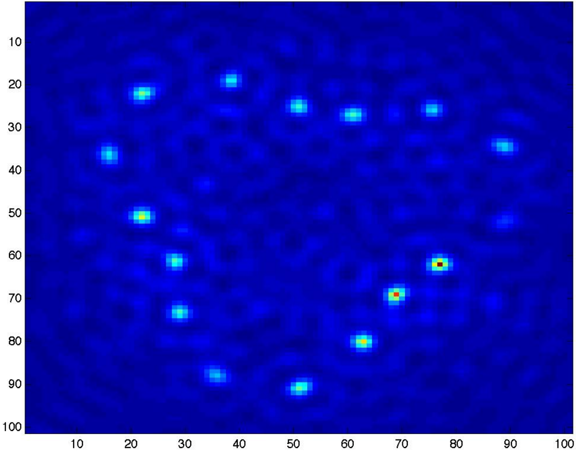
Test 1: 16 scatterers clover pattern; $k = 9$



Iteration 10

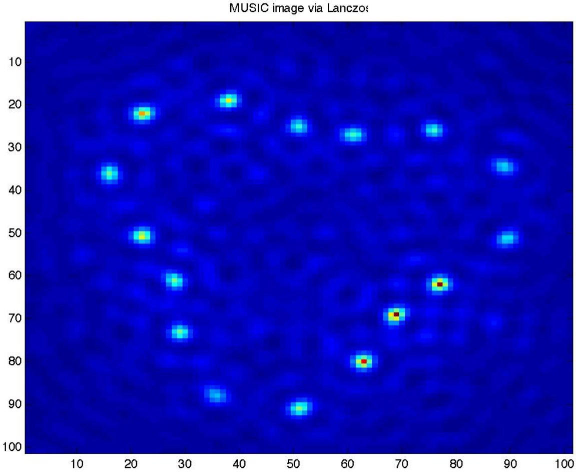
Test 1: 16 scatterers clover pattern; $k = 9$

MUSIC image via Lanczos



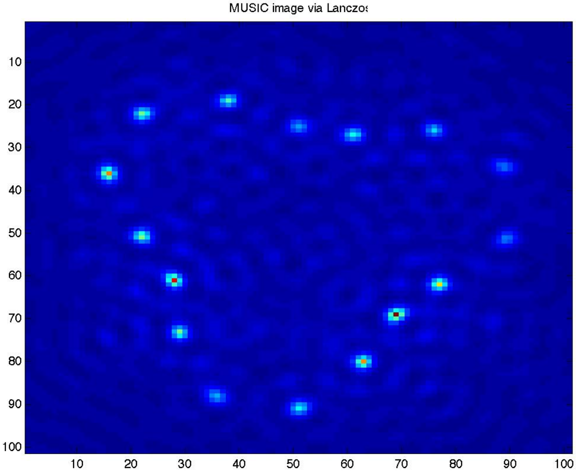
Iteration 11

Test 1: 16 scatterers clover pattern; $k = 9$



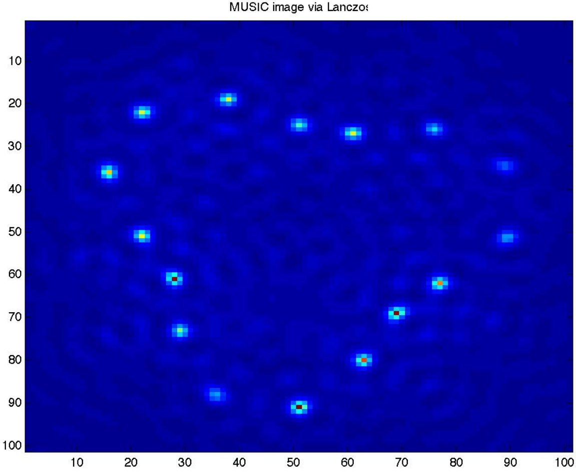
Iteration 12

Test 1: 16 scatterers clover pattern; $k = 9$



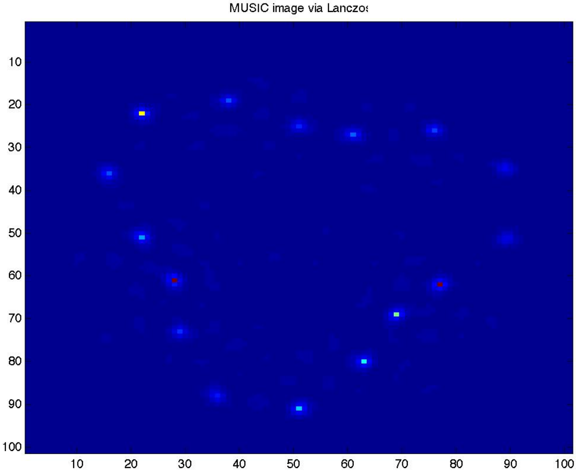
Iteration 13

Test 1: 16 scatterers clover pattern; $k = 9$



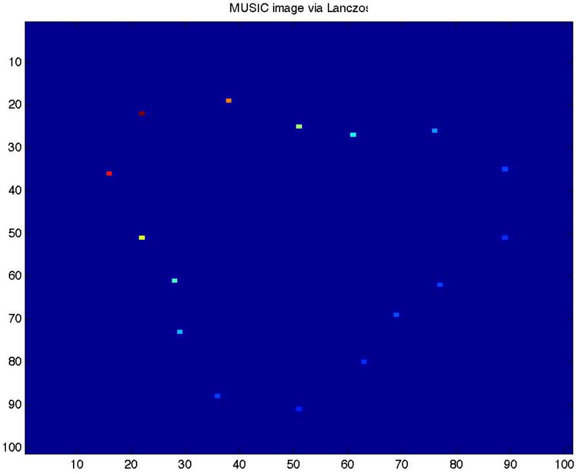
Iteration 14

Test 1: 16 scatterers clover pattern; $k = 9$



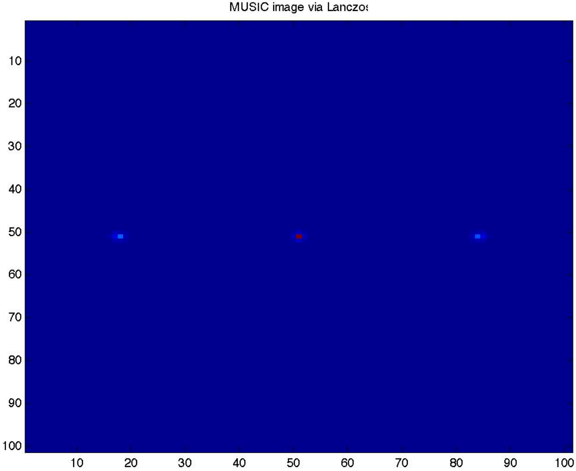
Iteration 15

Test 1: 16 scatterers clover pattern; $k = 9$



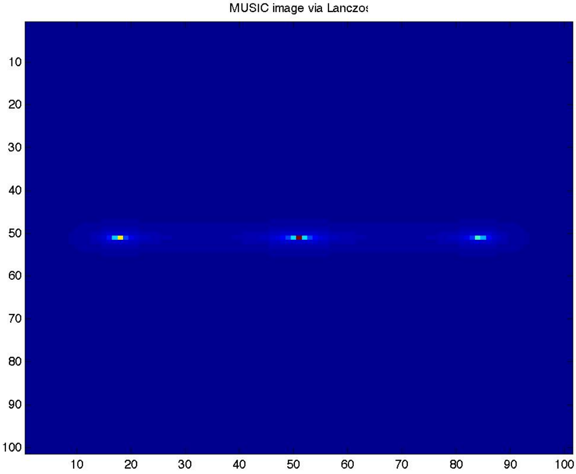
Iteration 16

Test 2: 3 scatterers in line



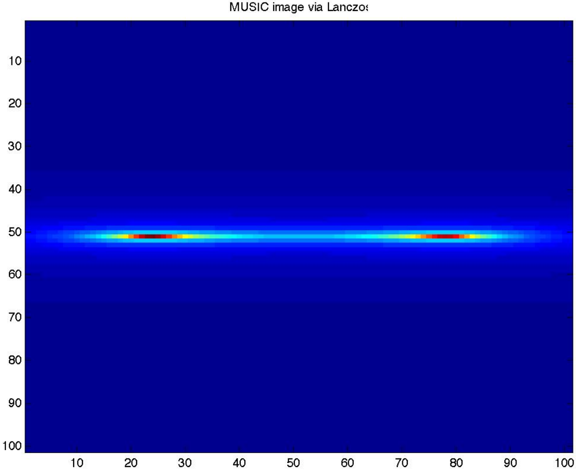
$k = 10$

Test 2: 3 scatterers in line



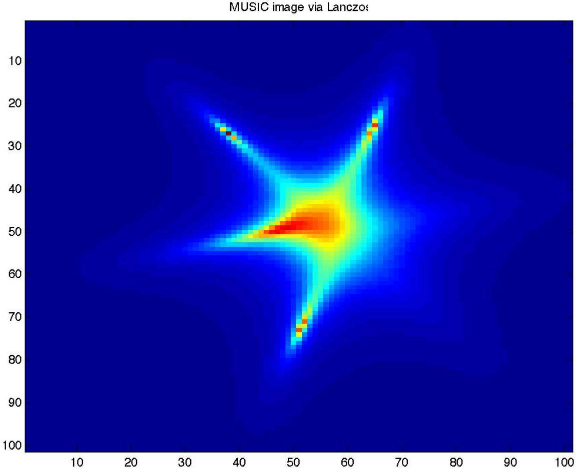
$k = 1$

Test 2: 3 scatterers in line



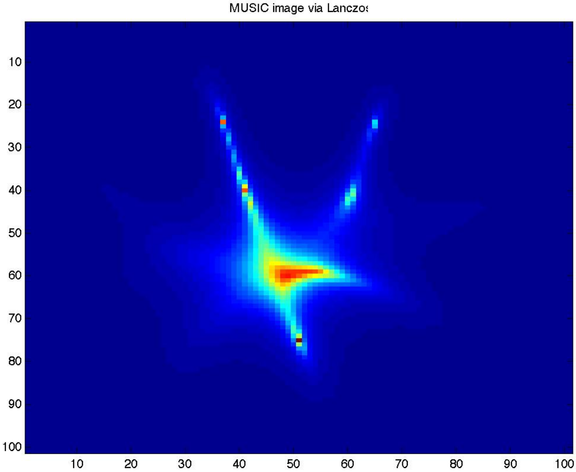
$k = 0.1$

Test 3: V-pattern of scatterers



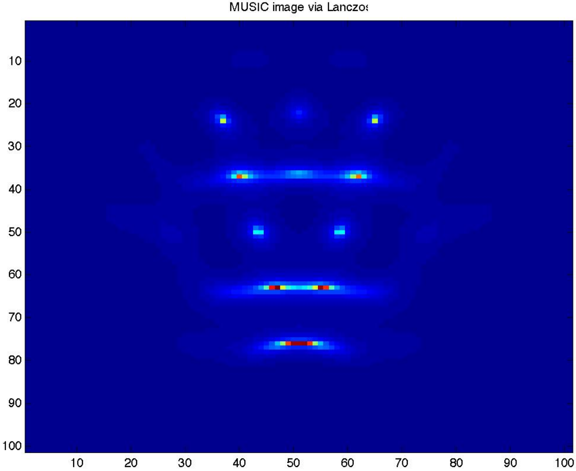
$k = 1$

Test 3: V-pattern of scatterers



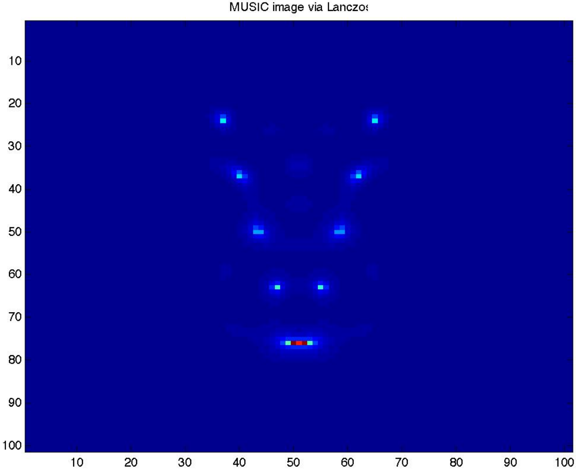
$k = 2$

Test 3: V-pattern of scatterers



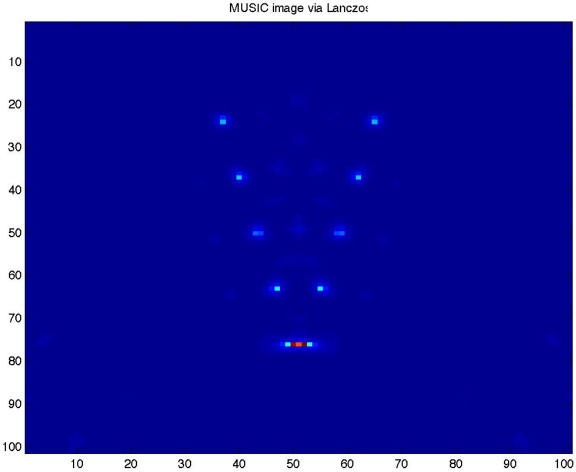
$k = 4$

Test 3: V-pattern of scatterers



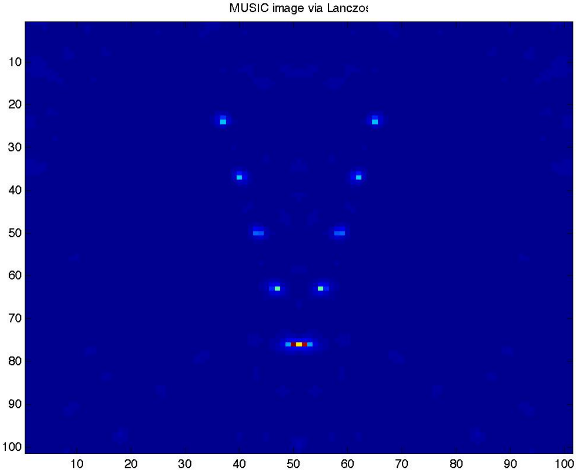
$k = 6$

Test 3: V-pattern of scatterers



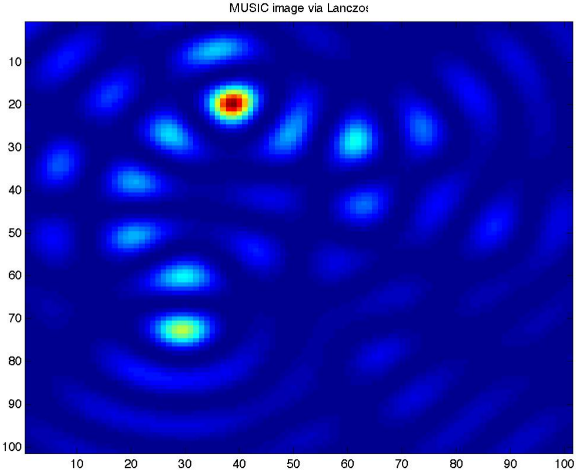
$k = 8$

Test 3: V-pattern of scatterers



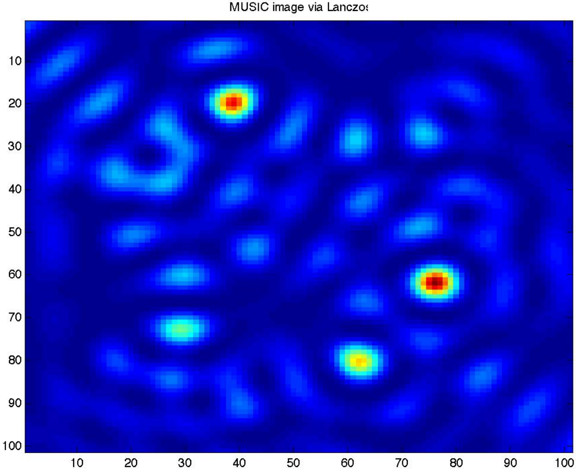
$k = 10$

Test 4: 16 scatterers clover pattern; $k = 5$



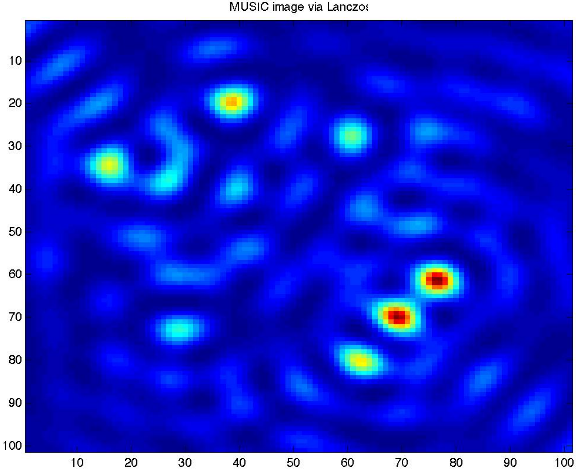
Iteration 1

Test 4: 16 scatterers clover pattern; $k = 5$



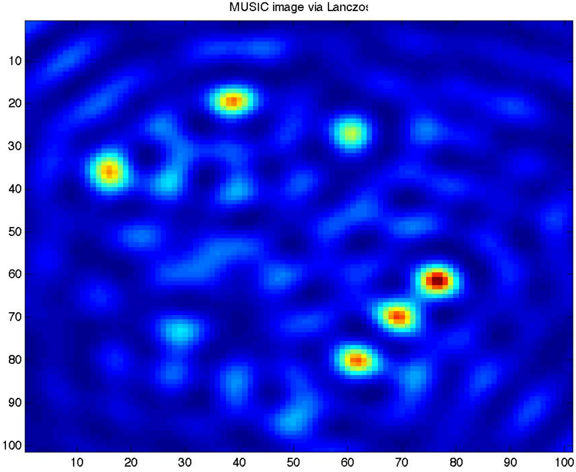
Iteration 2

Test 4: 16 scatterers clover pattern; $k = 5$



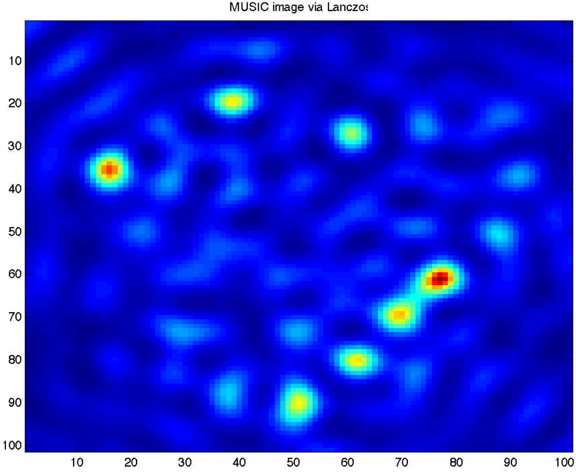
Iteration 3

Test 4: 16 scatterers clover pattern; $k = 5$



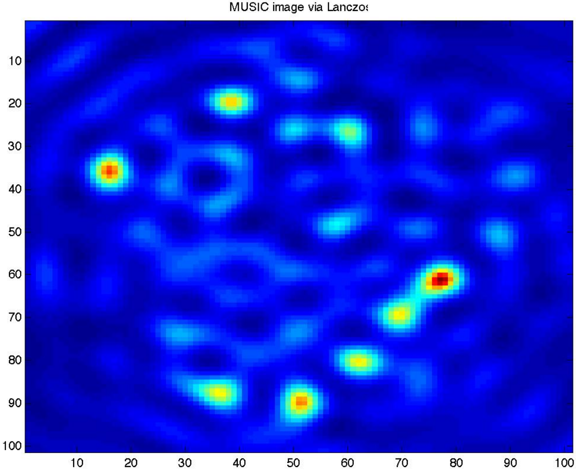
Iteration 4

Test 4: 16 scatterers clover pattern; $k = 5$



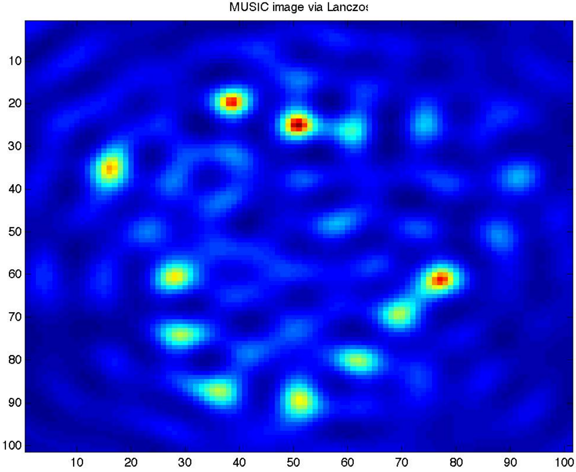
Iteration 5

Test 4: 16 scatterers clover pattern; $k = 5$



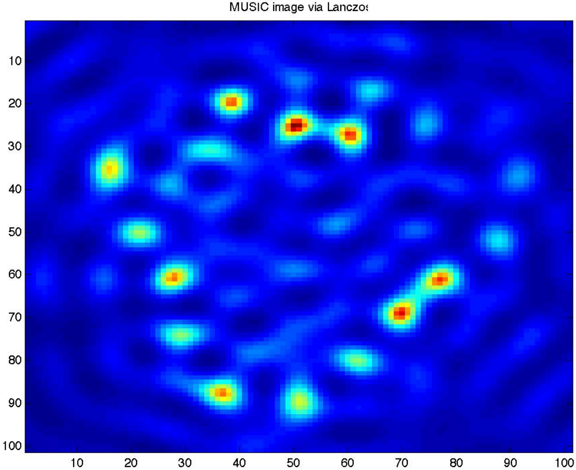
Iteration 6

Test 4: 16 scatterers clover pattern; $k = 5$



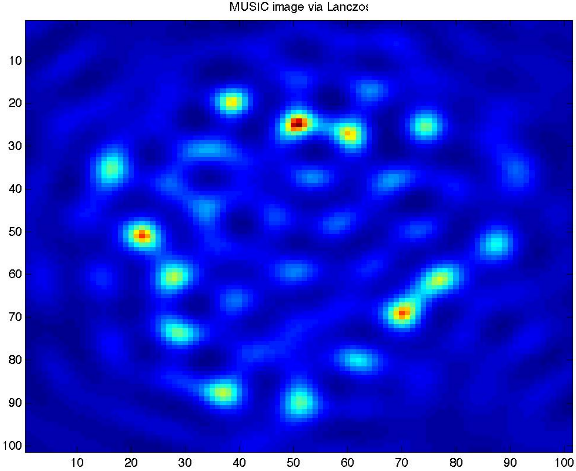
Iteration 7

Test 4: 16 scatterers clover pattern; $k = 5$



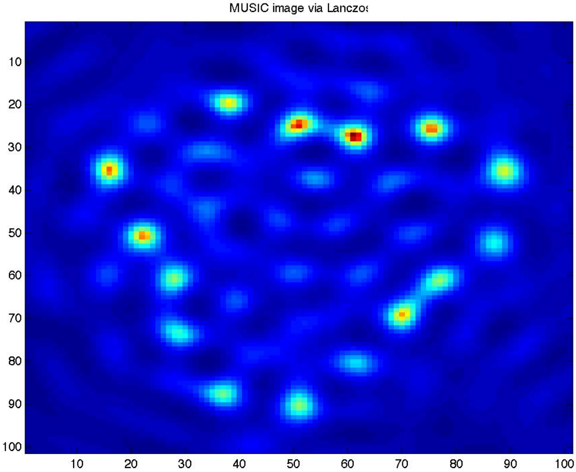
Iteration 8

Test 4: 16 scatterers clover pattern; $k = 5$



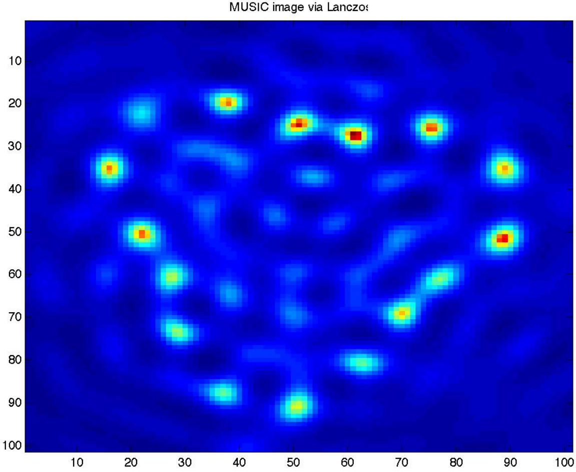
Iteration 9

Test 4: 16 scatterers clover pattern; $k = 5$



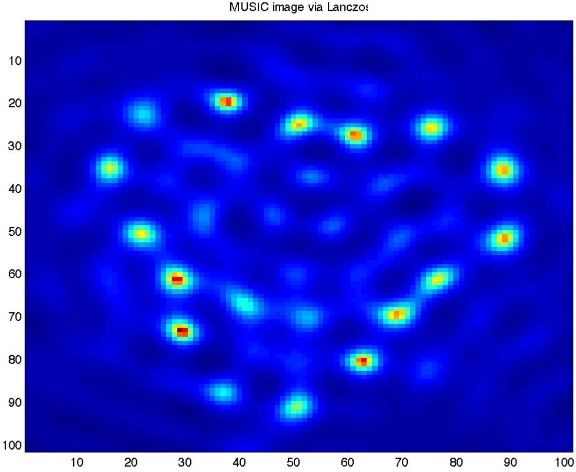
Iteration 10

Test 4: 16 scatterers clover pattern; $k = 5$



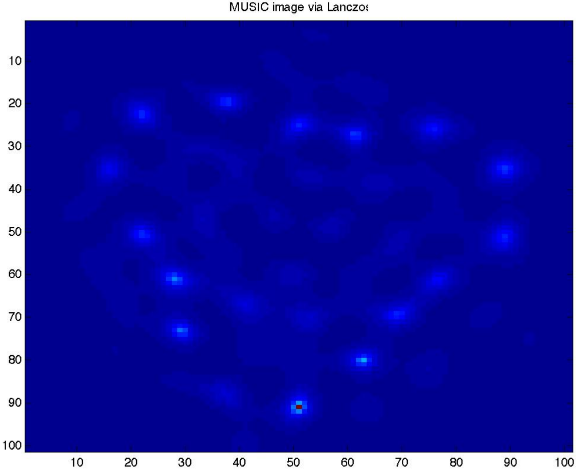
Iteration 11

Test 4: 16 scatterers clover pattern; $k = 5$



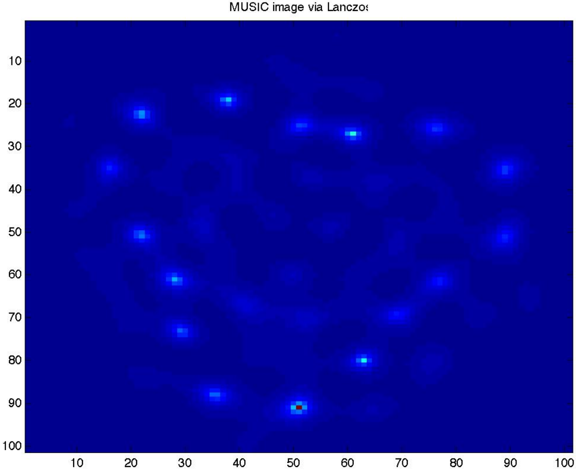
Iteration 12

Test 4: 16 scatterers clover pattern; $k = 5$



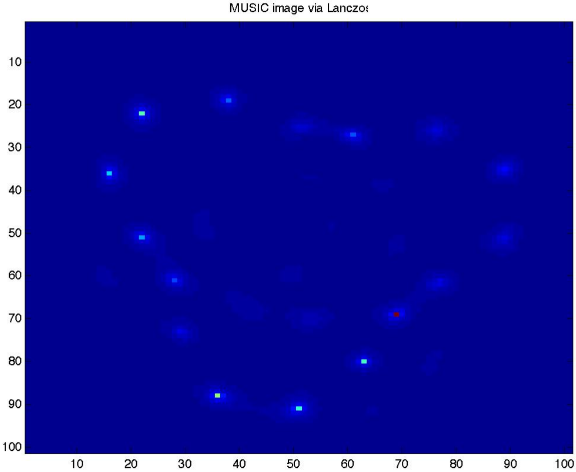
Iteration 13

Test 4: 16 scatterers clover pattern; $k = 5$



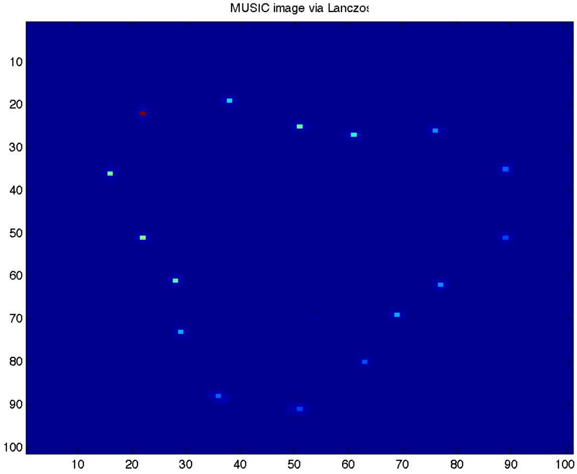
Iteration 14

Test 4: 16 scatterers clover pattern; $k = 5$



Iteration 15

Test 4: 16 scatterers clover pattern; $k = 5$



Iteration 16

Concluding Remarks

- Standard time reversal iterations = power method.
- Power iterations suboptimal.
- Lanczos iterations converge with far fewer transmissions.
- Few iterations $\Longleftrightarrow$ few acoustics transmissions.
- Lanczos provides v. direct method for implementing MUSIC.
- Breakdown of Lanczos: Standard reorthogonalization needed.

Looking ahead

- Key idea: Array is analog computer for linear operator.
- Just two examples of iterative linear operator methods in signal processing and imaging.
- Opportunity: Use temporal information.
- New Applications!