Linking Time-reversal to Krylov Methods in Acoustic Focusing and Imaging

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Outline

1. Introduction: Time-reversal focusing
2. Lanczos Iterated Time-reversal focusing
   - Numerical tests
3. MUSIC Imaging via Lanczos Time-reversal
   - Numerical tests
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Motivation

Time-reversal focusing & imaging

- NDE: Locating scatterers, flaws & voids (Johnson, et al. 2008(R))
- Medical applications: lithotripsy, HIFU radiation (Fink, et al. 2003(R))
- Underwater communication (1960’s; 1990’s - present)
- Decomposition of Time Reversal Operator (DORT) (Prada & Fink, 1990’s - present)
- TR Imaging (2000’s - present)
Motivation

Time-reversal focusing & imaging

- Underwater communication:
  - Parvulescu & Clay (1965 - matched-signal processing)
  - Dowling (phase conjugation) (early 1990’s).
  - TR Demonstration in ocean (Edelmann, et al. 2002)

- Decomposition of Time Reversal Operator (DORT):
  - Iterative focusing on isolated scatterer: Prada & Fink (1994)
  - Multiple scatterers (2004)
  - Demonstration of TRM in ocean waveguide (Gaumond et al. 2006, Prada et al. 2007)

- TR Imaging:
  - MULTiple SIgnal Classification - MUSIC (Devaney, mid 2000).
  - Extended to multiple scattering (Devany 2005).
  - TR = adjoint fields.
Motivation

Time-reversal focusing & imaging

- Drawbacks of iterative time-reversal:
  - Localization of multiple scatterers is tedious.
  - Poor convergence with scatterers of similar strengths.
  - Slow convergence limits application environments and frequency ranges.
  - Alternative: Measure entire multi-static response matrix, is worse.

- Goal: Efficiently identify & image multiple targets with few transmissions.
  - Quasi-stationary medium.
  - Large numbers of channels.
  - Signal strength issues when using small elements.
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Iterated Time-Reversal

Transmit Pulse

TRANSDUCER
Iterated Time-Reversal

Received Pulse
Iterated Time-Reversal

Re-Transmit Time Reversed Pulse
Iterated Time-Reversal

TRANSDUCER

Receive Pulse
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Iterated Time-Reversal

Re-Transmit Time Reversed Pulse
Iterated Time-Reversal

“Final” Received Pulse
Iterated Time-Reversal: Mathematics

Multistatic response matrix: $G$

$G_{ij} =$ measured signal on xdcr $j$ due to unit excitation on xdcr $i$.

Properties:

1. $G_{ij} = G_{ji}$ symmetric due to reciprocity.
2. $G \neq G^\dagger$ Not Hermitian.
3. $G^* = G^\dagger$ Conjugate = Hermitian transpose.
Iterated Time-Reversal: Mathematics

A single time-reversal iteration

1. Transmit $v^{(0)}(x, t)$
2. Receive $v^{(1/2)}(x, t) = G[v^{(0)}(x, t)]$
3. Transmit $v^{(1/2)}(x, T - t)$
4. Receive $v^{(1)}(x, t) = G[v^{(1/2)}(x, T - t)]$

Can write as $v^{(1)}(x, t) = H[v^{(0)}(x, t)]$, here $H = G^T G$ is the time-reversal operator

Time reversal iterations: $v^{(n)} = \overbrace{H \cdot H \cdots H}^{n}[v^{(0)}]$

As $n$ increases $v^{(n)}$ selectively focuses on the strongest scatterer. Why?
Iterated Time-Reversal: Mathematics

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Paul Barbone (BU)
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1. $v^{(n)} = H^n[v^{(0)}]$ are power iterations. As $n$ increases, $v^n \approx \phi^{(1)}$, eigenvector of the largest eigenvalue of $H$.

2. Let $\{s^{(i)}, \phi^{(i)}(x, t)\}$ be the $i$th eigenpair for $G$; then eigenpairs for $H$ are $\{\lambda^{(i)}, \phi^{(i)}(x, t)\}$, where $\lambda^{(i)} = |s^{(i)}|^2$.

3. $\phi^{(1)}$ focuses on strongest scatterer.

Criticism of Power iterations:

1. Tedious for multiple eigenvalues
2. Poor convergence for similar eigenvalues
3. Non-optimal convergence in general
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Time-reversal using Lanczos iterations

Can we do better? Yes, Lanczos iterations

Let $K^{n+1}(H, \nu^{(0)}) = \{ \nu^{(0)}, H\nu^{(0)}, \ldots, H^n\nu^{(0)} \}$ be the Krylov subspace

- For Power iterations $\lambda^{(1)}$ is approximated by the Rayleigh quotient of $H^n\nu^{(0)}$
- For Lanczos iterations $\lambda^{(1)}$ is approximated by the maximum of the Rayleigh quotient of all vectors in $K^{n+1}(H, \nu^{(0)})$
- That is, Lanczos is always better than power iterations.
Can we do better? Yes, Lanczos iterations

Let $\mathcal{K}^{n+1}(H, v^{(0)}) = \{ v^{(0)}, Hv^{(0)}, \ldots, H^n v^{(0)} \}$ be the Krylov subspace

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Time-reversal using Lanczos iterations

1. Begin with arbitrary signal $v^{(1)}$ such that $(v^{(1)}, v^{(1)}) = 1$
2. Let $w^{(1)} = H[v^{(1)}]$. (standard time-reversal operation.)
3. Iterate:
   1. $\alpha_1 = (w^{(1)}, v^{(1)})$
   2. $w^{(1)} = w^{(1)} - \alpha_1 v^{(1)}$ (orthogonalize)
   3. $\beta_2 = \sqrt{(w^{(1)}, w^{(1)})}$
   4. $v^{(2)} = w^{(1)}/\beta_2$ (normalize)
   5. $w^{(2)} = H[v^{(2)}] - \beta_2 v^{(1)}$

After $n$ iterations:
- Vectors $v^{(j)}$ represent orthonormal basis for $\mathcal{K}^n(H, v^{(1)})$.
- Create $n \times n$ matrix $T = \text{tridiag}(\alpha_i, \beta_i, \alpha_i)$
- Let $\{\omega^{(i)}, \psi^{(i)}\}$ be the $i$th eigenpair for $T$ . . . .
- . . . . . then $\lambda^{(i)} \approx \omega^{(i)}$ and $\phi^{(i)} \approx \sum_{j=1}^{n} \psi^{(i)}_{j} v^{(j)}$
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Lanczos Iterated Time-reversal focusing

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After $n$ iterations:
- Vectors $v^{(j)}$ represent orthonormal basis for $\mathcal{K}^n(H, v^{(1)})$.
- Create $n \times n$ matrix $T = \text{tridiag}(\alpha_i, \beta_i, \alpha_i)$
- Let $\{\omega^{(i)}, \psi^{(i)}\}$ be the $i$th eigenpair for $T$ . . . .
- . . . . then $\lambda^{(i)} \approx \omega^{(i)}$ and $\phi^{(i)} \approx \sum_{j=1}^{n} \psi^{(i)}_j v^{(j)}$
Iterated Lanczos Time-Reversal

Transmit Pulse
Iterated Lanczos Time-Reversal
Iterated Lanczos Time-Reversal

Re-Transmit Time Reversed Pulse
Iterated Lanczos Time-Reversal

TRANSDUCER

Receive Pulse

Paul Barbone (BU)
Iterated Lanczos Time-Reversal

Transmit Orthogonalized Pulse
Iterated Lanczos Time-Reversal
Time-reversal using Lanczos iterations

Changes to (Power-iterations based) time-reversal experiment

1. Number of time reversal operations remain the same
2. Each time reversal step followed by $\approx 10N$ flops ($N = N_s \times N_t$)
3. Keep 3 vectors of length $N$ in memory
4. Store $n$ Lanczos vectors on hard drive
5. $O(Nn)$ flops to evaluate $\{\lambda^{(i)}, \phi^{(i)}\}$
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Numerical Tests

Preliminaries

1. Domain $\Omega$ with inhomogeneities (scatterers)
2. Surface $\Gamma$ embedded with transmitters/receivers
3. Transmitted or received signal is a function of $x \in \Gamma$ and $t \in (0, T)$
4. $G$ is the **scattering operator**; $s$ is transmitted signal; $r = G[s]$ is received signal
5. Sometimes convenient to discretize space $x$ or time $t$ or both
Numerical Test 1

- Time-harmonic case
- Transmitters & receivers at $x = D$
- $D \gg \delta$
- Neglect multiple scattering
- Ideally separated
Numerical Test 1

For the scattering operator $G$:

- Eigenvalues, $s^{(i)}$, proportional to scatterer strength
- Eigenvectors given by

$$
\phi^{(i)}(y, z) = g^{(i)}(D, y, z) / \|g^{(i)}(D, y, z)\|
$$

where

$$
g^{(i)}(x) = \frac{e^{ik|x-x_i|}}{4\pi|x-x_i|}.
$$

For the time reversal operator $H$:

- Eigenvalues, $\lambda^{(i)} = |s^{(i)}|^2$
- Eigenvectors equal to $\phi^{(i)}(y, z)$
Numerical Test 1

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Numerical Test 1

Case A: Two scatterers ($N = 2$)

- For Power iterations, error in the first eigenvector $\epsilon_1 \propto \left| \frac{s^{(2)}}{s^{(1)}} \right|^{2n}$
  - For different strengths, $s^{(2)} = 0.70s^{(1)}$, $\epsilon_1 = 0.08\%$ after 10 iterations
  - For similar strengths, $s^{(2)} = 0.98s^{(1)}$, $\epsilon_1 = 66\%$ after 10 iterations

- Lanczos iterations converge to the correct answer in two iterations
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Case B: Multiple scatterers ($N = 200$)

- Scattering strength selected randomly between 0 and 1
- Initial transmitted signal: point source
- Power iterations: 10 iterations to determine $\phi^{(1)}$ and then 10 to determine $\phi^{(2)}$
- Lanczos iterations: 20 iterations
- 10,000 realizations to quantify performance
- For each realization compute error in the 1st and 2nd eigenfunction
- Plot histogram of error
Numerical Test 1

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Numerical Test 1

Error in $\phi^{(1)}$

Power; median error = 60%

Lanczos; median error = $10^{-6}\%$
Numerical Test 1

Error in $\phi^{(2)}$

Power; median error = $100\%$

Lanczos; median error = $10^{-3}\%$
Numerical Test 2: Focusing

- Time-harmonic case; \( \lambda \approx 0.25 \)
- 40 Transmitters & receivers at \( y = 0 \);
  \( L_x = 6 \)
- 30 point scatterers; \( D = 6 \); 2 strong scatterers
- Solve Helmholtz equation
- Test ability to target
Numerical Test 2: Focusing

Case A: Strong contrast $= 3/2$; number of iterations $n = 4$

Power: strongest

Lanczos: strongest
Numerical Test 2: Focusing

Case A: Strong contrast $= 3/2$; number of iterations $n = 4$

**Power: next**

**Lanczos: next**
Numerical Test 2: Focusing

Case B: Weak contrast $= 2.1/2$; number of iterations $n = 8$

Power: strongest

Lanczos: strongest
Numerical Test 2: Focusing

Case C: Weak contrast $= 2.1/2$; number of iterations $n = 8$

Power: next

Lanczos: next
Numerical Test 2: Focusing

Case B: Strong contrast = $3/2$; ratio of $d_y = 2/3$; $n = 8$

Power: strongest

Lanczos: strongest
Numerical Test 2: Focusing

Case B: Strong contrast $= \frac{3}{2}$; ratio of $d_y = \frac{2}{3}$; $n = 8$
Outline

1. Introduction: Time-reversal focusing

2. Lanczos Iterated Time-reversal focusing
   - Numerical tests

3. MUSIC Imaging via Lanczos Time-reversal
   - Numerical tests
Imaging point scatterers by MUSIC:

- **Goal:** Identify locations of several point-like targets
- **Method** based on computing range of multi-static response matrix, $G$.
- **MUSIC** = MUltiple Signal Classification, designed to pick out dominant frequency content of a signal.
- Adapted to imaging by Devaney 2000.
- Recently recognized as related to Colton’s “Linear Sampling Method,” and Kirsch’s “Factorization Method.”
Consider a scatterer at location $x_s$. Characterize scattered field as:

$$u_{scat}(x) = u_{inc}(x_s) \tau_s g(x, x_s)$$  \hspace{1cm} (1)

Here,

- $u_{inc}(x_s) =$ is the total field incident on the scatterer (including multiply scattered contributions).
- $\tau_s =$ is the scattering strength of the scatterer.
- $g(x, x_s) =$ Green’s function = field at $x$ due to a point source at $x_s$. 
Measured field at locations $\mathbf{x}_e^j$, $j = 1, \ldots, N_e$ is:

$$u(\mathbf{x}_e^j) = \sum_{s=1}^{N_s} A_s g(\mathbf{x}_e^j, \mathbf{x}_s)$$

**Remarks:**

- Different incident fields give rise to different $A_s$.
- For any incident field, measured field is in $\text{Span} \{ g(\mathbf{x}_e, \mathbf{x}_s^1), g(\mathbf{x}_e, \mathbf{x}_s^2), \ldots, g(\mathbf{x}_e, \mathbf{x}_s^{N_s}) \}$. 
MUSIC Imaging

Let $G =$ multistatic response matrix. Then for $N_e \gg N_s$:

\[
\text{Range}G = \text{Span}\{g(x_e, x_s^1), g(x_e, x_s^2), \ldots, g(x_e, x_s^{N_s})\} \quad (3)
\]

\[
= \text{“signal space”} \quad (4)
\]

Test for scatterer location:

\[
x \in \{x_s^1, x_s^2, \ldots, x_s^{N_s}\} \iff g(x_e, x) \in \text{Range}G \quad (5)
\]
MUSIC Imaging function.

- Let $V = N_e \times N_s$ be orthonormal projector onto signal space.
- Let $V_{noise} = I - V$ is orthonormal projector onto "noise" space.

\[
\text{indicator}(x) = \frac{1}{g'(x_e, x)V_{noise}g(x_e, x)}
\]  \hspace{1cm} (6)

Remark:

- $V =$non-null eigenvectors of $GG^* = H$.
- $V_{noise} = \text{null}(GG^*) \equiv \text{null}(H)$. 
Standard MUSIC Imaging

1. Excite emitters $i = 1, \ldots, N_e \gg N_s$ in turn.
2. For each excitation, $i$, measure $G_{ij} = \text{signal on } N_e \text{ emitters, } j$.
3. Compute $H = GG^*$. 
5. Compute $V = [\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \ldots] = \text{non-null eigenvectors of } GG^* = H$.
6. Compute $V_{\text{noise}} = I - V$.
7. For each point $x$ in region of interest, compute:

$$\text{indicator}(x) = \frac{1}{g'(x_e, x)V_{\text{noise}}g(x_e, x)}$$ (7)
Lanczos MUSIC Imaging

1. Initialization: \( \nu^{(1)} = H[\nu^{(0)}] \) (Eliminate null component of \( \nu^{(0)} \).)

2. Perform \( N_s \ll N_e \) Lanczos iterations to construct
   \[ V = [\nu^{(1)}, \nu^{(2)}, \nu^{(3)}, \ldots] \]

3. Compute \( V_{\text{noise}} = I - V \).

4. For each point \( x \) in region of interest, compute:
   \[
   \text{indicator}(x) = \frac{1}{g'(x_e, x) V_{\text{noise}} g(x_e, x)}
   \]
   (8)

Remark:
- Lanczos procedure automatically stops at \( N_s \) iterations, when \( \beta = 0 \).
Lanczos vs. Standard MUSIC Imaging

- Lanczos requires *many* fewer acoustic excitations.
- Lanczos requires no eigenvalue computations.
- Lanczos procedure provides new image with each new measurement.
- But does it work?
Lanczos vs. Standard MUSIC Imaging

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- But does it work?
Numerical Tests: Imaging point scatterers

- Far-field array of 240 transducers (emitters).
- Time-harmonic excitation.
- Scattered field computed including multiple scattering.
- For each excitation, update $V$ and compute indicator $(x)$.
- Stop iterating when $\beta < \text{tolerance}$.
- Field of view $= 6 \times 6$, so $k =$ number of wavelengths across FOV.
Test 1: 16 scatterers clover pattern; $k = 9$
Test 1: 16 scatterers clover pattern; \( k = 9 \)

Iteration 2
Test 1: 16 scatterers clover pattern; \( k = 9 \)
Test 1: 16 scatterers clover pattern; $k = 9$
Test 1: 16 scatterers clover pattern; $k = 9$

Iteration 5
Test 1: 16 scatterers clover pattern; $k = 9$

Iteration 6
Test 1: 16 scatterers clover pattern; $k = 9$
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Iteration 8
Test 1: 16 scatterers clover pattern; $k = 9$

Iteration 9
Test 1: 16 scatterers clover pattern; $k = 9$
Test 1: 16 scatterers clover pattern; $k = 9$

Iteration 11
Test 1: 16 scatterers clover pattern; $k = 9$

Iteration 12
Test 1: 16 scatterers clover pattern; $k = 9$

Iteration 13
Test 1: 16 scatterers clover pattern; $k = 9$
Test 1: 16 scatterers clover pattern; $k = 9$
Test 1: 16 scatterers clover pattern; $k = 9$
Test 2: 3 scatterers in line

\[ k = 10 \]
Test 2: 3 scatterers in line

$k = 1$
Test 2: 3 scatterers in line

\[ k = 0.1 \]
Test 3: V-pattern of scatterers

\[ k = 1 \]
Test 3: V-pattern of scatterers

\( k = 2 \)
Test 3: V-pattern of scatterers

\[ k = 4 \]
Test 3: V-pattern of scatterers

\[ k = 6 \]
Test 3: V-pattern of scatterers

\[ k = 8 \]
Test 3: V-pattern of scatterers

$k = 10$
Test 4: 16 scatterers clover pattern; $k = 5$

Iteration 1
Test 4: 16 scatterers clover pattern; $k = 5$
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Iteration 3
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Iteration 12
Test 4: 16 scatterers clover pattern; $k = 5$

Iteration 13

MUSIC image via Lanczos
Test 4: 16 scatterers clover pattern; $k = 5$

Iteration 14
Test 4: 16 scatterers clover pattern; $k = 5$
Test 4: 16 scatterers clover pattern; $k = 5$
Concluding Remarks

- Standard time reversal iterations = power method.
- Power iterations suboptimal.
- Lanczos iterations converge with far fewer transmissions.
- Few iterations \(\iff\) few acoustics transmissions.
- Lanczos provides \textit{v.} direct method for implementing MUSIC.
- \textbf{Breakdown of Lanczos:} Standard reorthogonalization needed.

Looking ahead

- Key idea: Array is analog computer for linear operator.
- Just two examples of iterative linear operator methods in signal processing and imaging.
- Opportunity: Use temporal information.
- \textbf{New Applications!}