Some math and mechanics of biomechanical imaging: Current status and open questions

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2008 BIRS Workshop on Inverse Problems: Recent Progress and New Challenges

Acknowledgments

BMI Team:

- Assad Oberai: RPI
- Isaac Harari: Tel Aviv

Collaborators:

- Jeffrey C Bamber, Gearoid Berry (Medical Physics: ICR London)
- Timothy J Hall (Medical Physics: U Wisconsin)
- Elise Morgan (ME & Orthopedic Surgery, BU)
- Jonathan Rubin (Radiology, U Mich)
- Dave Mountain (BME, BU)

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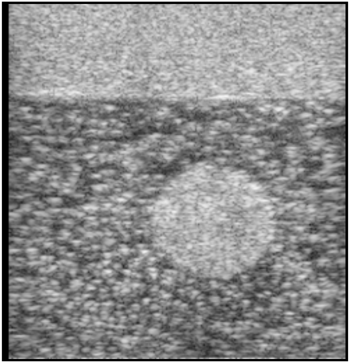
Outline

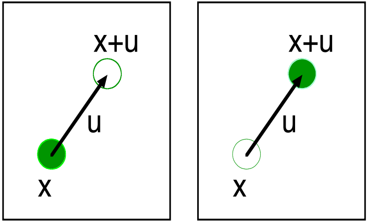
1. Background: Available data
   - Measuring interior displacement data
   - Going Forward: Math Model for Tissue Deformation
2. Inverse Problem Statement
3. Plane Stress
   - Forward Model
   - Inversion
4. Plane Strain
   - Forward Model
   - Inversion
5. Very few Examples: Images from lab and clinic.
6. Some open questions & challenges in elasticity imaging

Outline

Measuring Interior Displacement

Ultrasound shows *interior* of deforming medium.

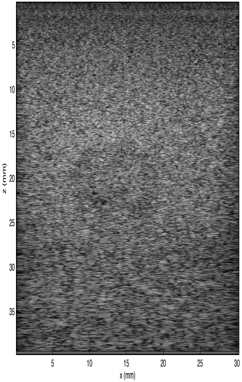




$$I_0(x) = I_1(x+u)$$

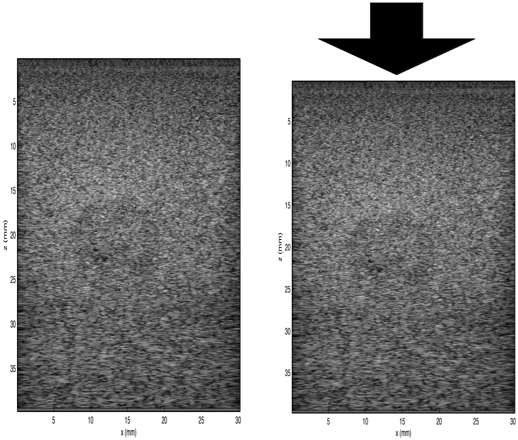
$$\Longrightarrow \boldsymbol{u}(\boldsymbol{x}) \tag{1}$$

Ultrasound Elastography: Strain Imaging[1]



Pre-compression Ultrasound Image

Ultrasound Elastography: Strain Imaging[1]



Pre-compression Ultrasound Image

Post-compression Ultrasound Image

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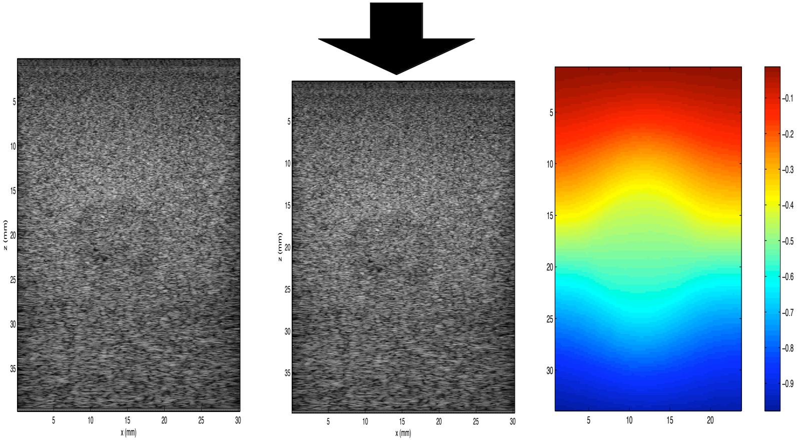
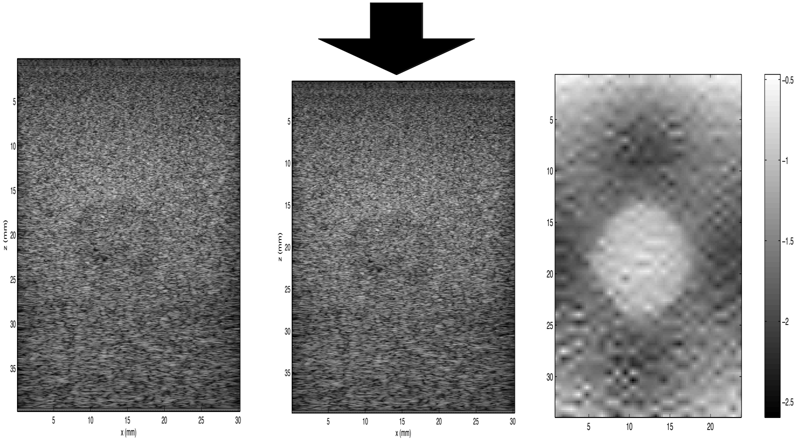


Pre-compression Ultrasound Image

Post-compression Ultrasound Image

Displacement Image

Strain images show "invisible" inclusions



Pre-compression Ultrasound Image

Post-compression Ultrasound Image

Strain Image

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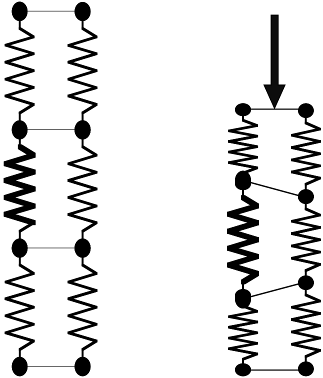
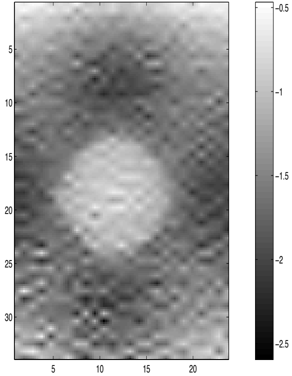


Interpretation



Strain Image

Potential Applications

Measuring tissue elastic property distribution *in vivo*

- Diseases: cancer, arterio-schlerosis, DVT, plaques, fibrosis, lymphedema, scirrhosis.
- Clinical: screening, differential diagnosis, treatment monitoring.
- Biomechanical function: muscles, lungs, cochlea, vascular tissue, bones, cartilage.
- Mechanobiology: cartilage, bone, cancer.

Necessary Ingredients

- Applied Deformation: Internal motion, external static, time-harmonic, mechanical, free-hand, via probe, radiation force.
- Imaging: Ultrasound, MR, microCT, OCT.
- Interpretation: Displacement, velocity, strain, reconstructed properties.

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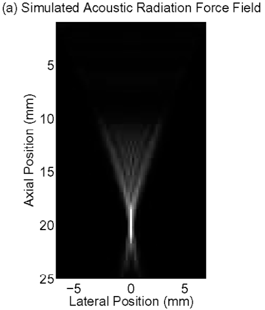
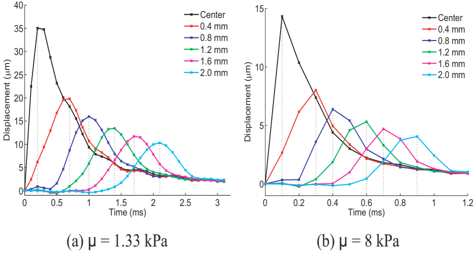
Acoustic Radiation Force[3, 4, 5]

$$F \approx 2\alpha I/c \qquad (1)$$



Distribution of ARF[2]



Transient Displacement

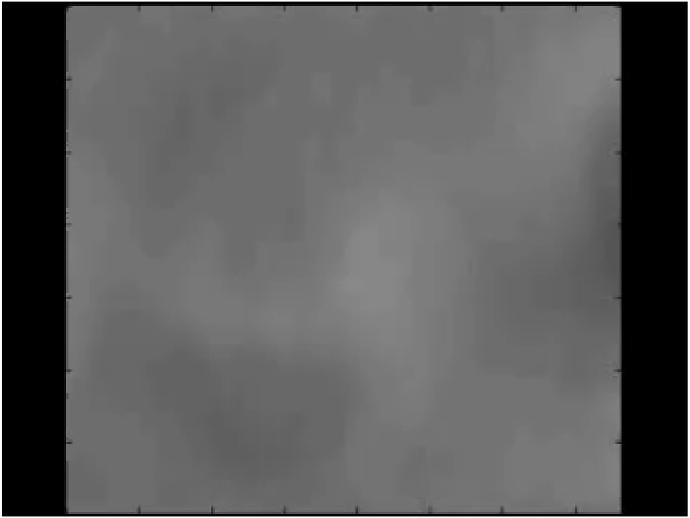
Acoustic Radiation Force Imaging

Interpretation:

- Algebraic inversion[1];
- Travel time[2];
- Time-to-peak[3];
- Vibro-acoustography[4];
- Static strain[5]

[1] Oliphant,et al. 2001
[2] McLaughlin, et al. 2004
[3] Nightingale, et al. 2008
[4] Greenleaf, et al. 1998
[5] Bamber, et al. 2007

ARFI Supersonic Imaging (www.supersonicimagine.fr)



Necessary Ingredients

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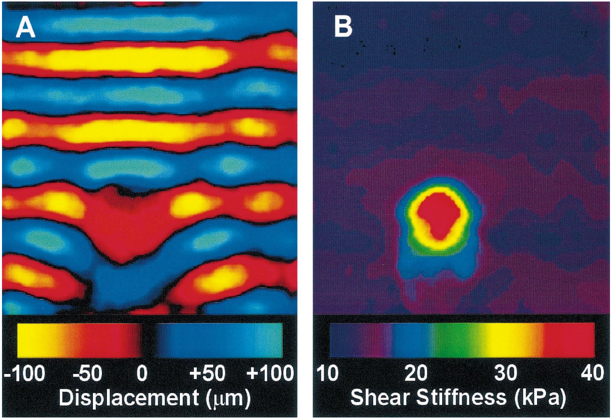
MR Elastography[7]



(Manduca & Oliphant [6])

MRE: Algebraic Inversion

Algebraic Inversion (Manduca & Oliphant[6]):

$$\mu \nabla^2 u = -\rho\omega^2 u \tag{2}$$

$$\Rightarrow \quad \mu = -\frac{\rho\omega^2 u}{\nabla^2 u} \tag{3}$$

Modified Algebraic Inversion [8]:

$$\mu \nabla^2 \nabla \times \boldsymbol{u} = -\rho\omega^2 \nabla \times \boldsymbol{u} \tag{4}$$

$$\Rightarrow \quad \mu = -\frac{\rho\omega^2 |\nabla \times \boldsymbol{u}|}{|\nabla^2 \nabla \times \boldsymbol{u}|} \tag{5}$$

Mathematical Modeling[9, 10]

Experimental Conditions

- Small Strains
- Excitation: 1 Hz – 1 kHz

Modeling Assumptions

- Single phase
- Elastic: $\boldsymbol{\sigma} = f(\epsilon)$
- Isotropic.

Mathematical Modeling[9, 10]

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- Isotropic.

Momentum and Constitutive Eqns:

$$
\begin{aligned}
\nabla(\lambda\nabla\cdot\boldsymbol{u}) + \nabla\cdot(\mu\nabla\boldsymbol{u}) + \nabla\cdot(\mu\nabla\boldsymbol{u})^T &= \rho\partial_{tt}\boldsymbol{u} \qquad (6)\\
+\text{boundary conditions} & \qquad (7)
\end{aligned}
$$

Some elastic parameter estimates[11]

- Relations between elastic constants:

$$\mu \equiv G = \frac{E}{2(1+\nu)} \quad ; \quad \lambda = \frac{2\mu\nu}{(1-2\nu)} \tag{8}$$

- Longitudinal wave speed:

$$c_L = \sqrt{(\lambda + 2\mu)/\rho} = 1540 m/s \pm 5\%$$

- Shear wave speed: $c_S = \sqrt{\mu/\rho} = 1 - 10 m/s$
- Poisson's ratio: $\nu \approx 1/2$.
- Density: $\rho = 1000\, kg/m^3 \pm 5\%$

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- Poisson's ratio: $\nu \approx 1/2$.
- Density: $\rho = 1000\, kg/m^3 \pm 5\%$
- $\Longrightarrow \quad \lambda \approx \text{constant} = \rho c_L^2 \pm 10\%$
- $\Longrightarrow \quad \lambda/\mu \approx 10^6$

Incompressible Elasticity Forward Model

Momentum and Constitutive Eqns:

$$-\nabla p + \nabla \cdot (\mu \nabla \boldsymbol{u}) + \nabla \cdot (\mu \nabla \boldsymbol{u})^T = \rho \partial_{tt} \boldsymbol{u} \tag{9}$$

$$\nabla \cdot \boldsymbol{u} = -p/\lambda \rightarrow 0 \tag{10}$$

$$+\text{boundary conditions} \tag{11}$$

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+\text{boundary conditions} & && (11)
\end{aligned}
$$

Remarks:

- Crude (but effective) model: $p = 0$; $\mu \approx$ const.

$$\mu \nabla^2 \boldsymbol{u} + \rho \omega^2 \partial_{tt} \boldsymbol{u} = 0 \qquad (12)$$

Incompressible Elasticity Forward Model

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-\nabla p + \nabla \cdot (\mu \nabla \boldsymbol{u}) + \nabla \cdot (\mu \nabla \boldsymbol{u})^T &= \rho \partial_{tt} \boldsymbol{u} && (9)\\
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Remarks:

- Crude (but effective) model: $p = 0$; $\mu \approx$ const.

$$\mu \nabla^2 \boldsymbol{u} + \rho \omega^2 \partial_{tt} \boldsymbol{u} = 0 \quad (12)$$

- “Worse” model:

$$\nabla \cdot (\mu \nabla \boldsymbol{u}) + \rho \omega^2 \partial_{tt} \boldsymbol{u} = 0 \quad (13)$$

Outline

Inverse Problem Statement

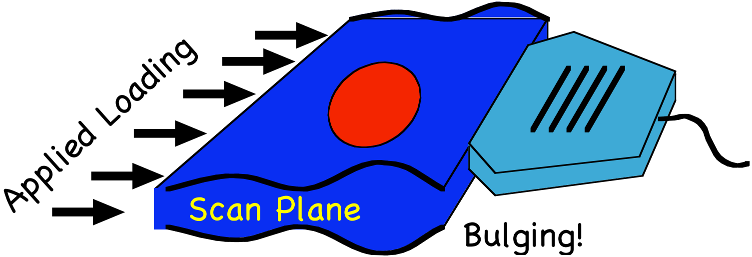
Given $\boldsymbol{u}(\boldsymbol{x}, t)$, ρ, for $\boldsymbol{x} \in \Omega$, determine $\mu(\boldsymbol{x})$ such that:

$$-\nabla p + 2\nabla \cdot (\mu\epsilon) = \rho\partial_{tt}\, bu \tag{14}$$

$$+\text{boundary conditions} \tag{15}$$

Outline

Plane Stress Approximation



No confining stress out of the plane.

Stress-strain relations

Plane stress assumption:

$$\begin{aligned} \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = \sigma_{zx} = \sigma_{zy} = & \quad 0, & (16) \\ \rightarrow \partial_z u_x = \partial_z u_y = & \quad 0. & (17) \end{aligned}$$

Solve for p using $\sigma_{zz} = 0$ and incompressibility:

$$p = 2\mu\epsilon_{zz} = -2\mu(\epsilon_{xx} + \epsilon_{yy}). \quad (18)$$

Then stress-strain relation reduces to:

$$\begin{aligned} \boldsymbol{\sigma} &= 2\mu(\boldsymbol{x})\boldsymbol{A} & (19) \\ \boldsymbol{A}(\boldsymbol{x}) &= \epsilon_{\alpha\alpha}\mathbf{1} + \boldsymbol{\epsilon}(\boldsymbol{x}) & (20) \\ &= 2(\epsilon_{xx} + \epsilon_{yy}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix}. & (21) \end{aligned}$$

Momentum Equation

Momentum eqn becomes:

$$\partial_x(2\mu(\epsilon_{xx}+\epsilon_{yy})) + 2\partial_x(\mu\epsilon_{xx}) + 2\partial_y(\mu\epsilon_{xy}) = \rho\,\partial_{tt} u_x \quad (22)$$

$$\partial_u(2\mu(\epsilon_{xx}+\epsilon_{yy})) + 2\partial_x(\mu\epsilon_{yx}) + 2\partial_y(\mu\epsilon_{yy}) = \rho\,\partial_{tt} u_y \quad (23)$$

Symbolically:

$$\nabla\cdot(\mu\boldsymbol{A}) = \rho\,\partial_{tt}\boldsymbol{u} \quad (24)$$

$$\boldsymbol{A}\nabla\mu + \mu(\nabla\cdot\boldsymbol{A}) = \rho\,\partial_{tt}\boldsymbol{u} \quad (25)$$

Inverse Problem Solution

Integrating (25) with $\partial_{tt}\boldsymbol{u} = 0$ gives:

$$\mu(\boldsymbol{x}) = \mu(\boldsymbol{x}_o)\exp\left\{-\int_{\boldsymbol{x}_o}^{\boldsymbol{x}} \boldsymbol{A}^{-1}\nabla\boldsymbol{A}\cdot d\boldsymbol{x}'\right\} \tag{26}$$

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Remarks:

- One unknown constant: solution is unique!
- See [12] for transient case.
- Solvability condition: $\nabla \times [\boldsymbol{A}^{-1}\nabla\cdot\boldsymbol{A}] = 0$.
  - Solution may not exist!!!
- $\nabla\cdot\boldsymbol{A} \Rightarrow \boldsymbol{u}$ is twice differentiable.
- “Worse" model exact solution similarly available.

Summary: Plane Stress

- Inverse solution is unique (when given u_x and u_y) up to calibration constant.
- Inverse problem is well-posed subject to solvability condition and calibration constant.
- u_x and u_y are constrained by nonlinear pde in the form of integrability condition.
- Transient problem = forced static problem parameterized by time.
- Exact solution exists, but requires $\boldsymbol{u}$ to be twice differentiable and yields continuous μ.
- Structure nearly identical to LFCI/MREIT.

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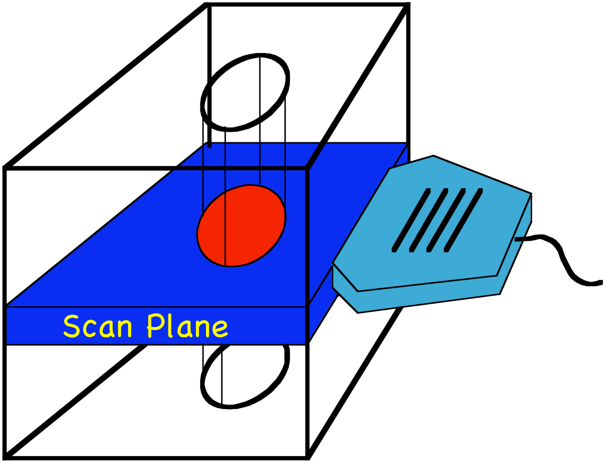

Confinement out of the plane prevents expansion.

Stress-strain relations

Plane strain assumption:

$$\epsilon_{xz} = \epsilon_{yz} = \epsilon_{zz} = \epsilon_{zx} = \epsilon_{zy} = 0, \tag{27}$$

$$\rightarrow \partial_z u_x = \partial_z u_y = 0. \tag{28}$$

Stress-strain relation reduces to:

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\mu\epsilon \tag{29}$$

$$\epsilon(\boldsymbol{x}) = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix}. \tag{30}$$

Remark: Pressure p is completely undetermined, unlike Plane Stress.

Momentum Equation

Momentum eqn becomes:

$$-\nabla p + 2\nabla \cdot (\mu \boldsymbol{\epsilon}) = \rho\, \partial_{tt} \boldsymbol{u} \tag{31}$$

In detail:

$$\begin{aligned} -\partial_x p + 2\partial_x(\mu\epsilon_{xx}) + 2\partial_y(\mu\epsilon_{xy}) &= \rho\,\partial_{tt} u_x && (32)\\ -\partial_y p + 2\partial_x(\mu\epsilon_{yx}) + 2\partial_y(\mu\epsilon_{yy}) &= \rho\,\partial_{tt} u_y. && (33) \end{aligned}$$

Plane Strain Inversion Equation

Eliminate p by taking curl(31):

$$(\partial_{yy} - \partial_{xx})(\epsilon_{xy}\,\mu) + 2\partial_{xy}(\epsilon_{xx}\,\mu) = \rho\partial_{tt}\omega_{yx} \tag{34}$$

Remarks:

- Hyperbolic, linear PDE.
- Characteristics are principal directions of strain.
- Requires boundary data (e.g. Cauchy or Goursat) to make well-posed.
- For any μ that satisfies (34), $\exists p$ such that (32,33) are satisfied.
- See [13, 14] for details.

Example of Nonuniqueness: Uniform strain field[13]

Suppose we are given measurements:

$$\epsilon_{xx} = -\epsilon_{yy} = \epsilon_o = \text{const.} \tag{35}$$

Then (34) gives:

$$\epsilon_o \partial_{xy} \mu = 0; \qquad \Longrightarrow \mu(x, y) = f(x) + g(y). \tag{36}$$

- Solution is determined only up to two independent functions of a single variable.
- Need boundary data related to μ.

Choosing from among all possible solutions

There are infinitely many modulus reconstructions for any given ϵ. How do we choose?

Three possible strategies:

1. Use regularization
   - Choose smallest possible μ.
   - Choose smoothest possible μ.
2. Use traction BC's.
   - Approximate (guess) boundary conditions.
   - Measure boundary conditions.
3. Use additional measured deformations.

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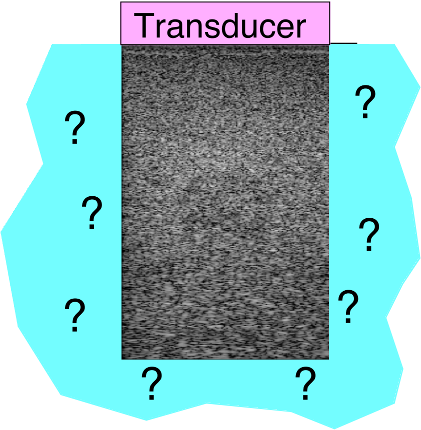
Available boundary data?



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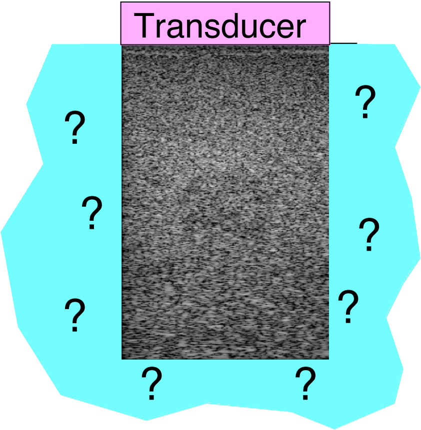

Hyperbolic eqn w/ Dirichlet data? Maybe it's just as well...

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Multiple strain fields: uniqueness[14]

Theorem:

Given *two* mutually compatible, linearly independent strain fields, $\epsilon^{(1)}$ and $\epsilon^{(2)}$, everywhere nonzero in Ω, with distinct eigendirections except at isolated points. Let $M^{(j)}$ be the set of all functions μ such that:

$$L(\epsilon^{(j)})\,\mu = 0. \tag{37}$$

Then:

$$M^{(1)} \bigcap M^{(2)} \leq \text{4 dimensional}. \tag{38}$$

Summary: Plane strain well-posedness

Remarks:

- Infinite number of solutions exist for single measured strain field.
- Transient problem = forced static problem parameterized by time.
- Single strain field with known traction BCs gives unique but unstable modulus distribution.
- With two measured strain fields, need four calibration constants to determine complete solution; solution is (probably) stable. Proof assumes $\mu \in C^4$.
- Not every pair of measured strain fields is mutually compatible: chance for averaging.

Direct Solution Strategies

Given $\boldsymbol{u}^m(\boldsymbol{x})$ = measured displacement field $\forall \boldsymbol{x} \in \Omega$.
Direct Inversion: Find $\mu(\boldsymbol{x})$ (and $p(\boldsymbol{x})$) s.t.

$$-\nabla p + \nabla \cdot (\mu \nabla \boldsymbol{u}^m) + \nabla \cdot (\mu \nabla \boldsymbol{u}^m)^T = 0 \qquad (39)$$

Remarks:

- Computationally efficient.
- No boundary conditions specified or needed for $\boldsymbol{u}^m$.
- Assumes $\boldsymbol{u}^m$ is a solution of the elasticity equation.
- Accuracy limited by least accurate displacement component.

Iterative/Optimization Solution Strategies

Given $\boldsymbol{u}^m(\boldsymbol{x})$ = measured displacement field $\forall \boldsymbol{x} \in \Omega$.
Iterative Inversion: Define $\boldsymbol{u}[\mu]$ s.t.

$$-\nabla p + \nabla \cdot (\mu \nabla \boldsymbol{u}) + \nabla \cdot (\mu \nabla \boldsymbol{u})^T = 0 \quad (40)$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad (41)$$

$$+\text{boundary conditions} \quad (42)$$

Find $\mu(\boldsymbol{x})$ to minimize:

$$\Pi[\mu] = \|\boldsymbol{u} - \boldsymbol{u}^m\| + \alpha R[\mu] \quad (43)$$

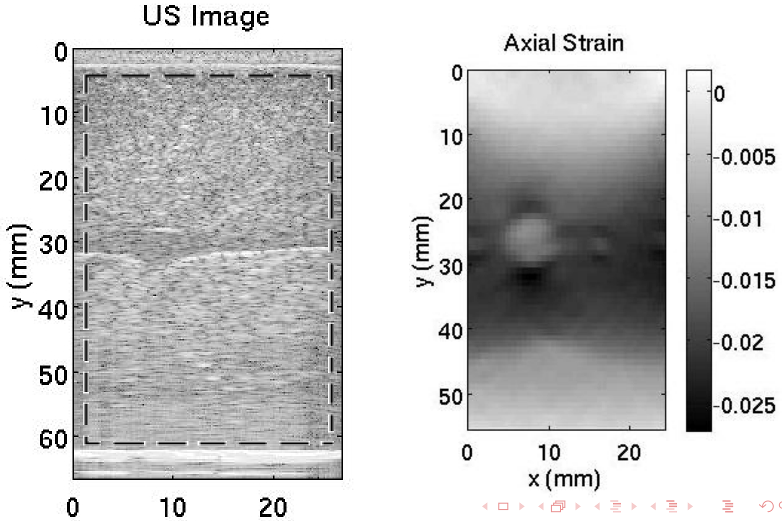
Remarks:

- Flexible but computationally intensive.
- Accommodates variety of corrupted data.
- Accuracy limited by accuracy of boundary conditions.

Outline

Example: Phantom Images

Example: 5 mm Inclusion



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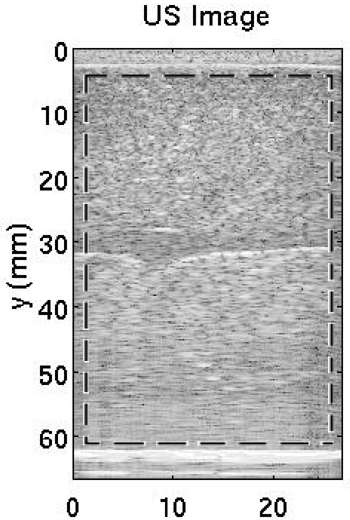
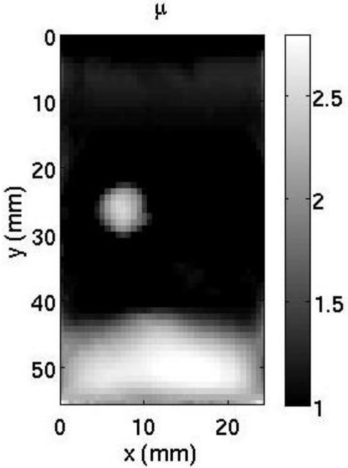
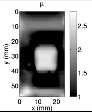
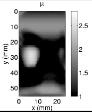
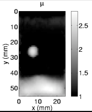
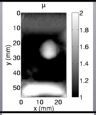
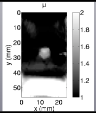
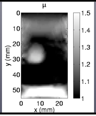
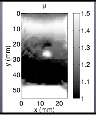




Resolution Experiment

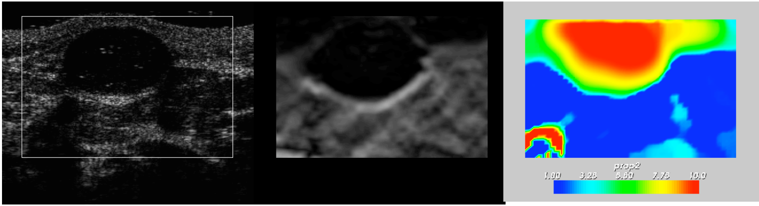
Size: **Stiffness:**	Large: 13 mm	Medium: 8 mm	Small: 5 mm
Large: 3:1	Visible!		
Medium: 2:1			
Small: 1+:1			Invisible?

Resolution Experiment

Size: **Stiffness:**	Large: 13 mm	Medium: 8 mm	Small: 5 mm
Large: 3:1			
Medium: 2:1			
Small: 1+:1			

Results from Clinical Images[6]

(KU0031) Fibroadenoma (benign tumor):



[6]w/ TJ Hall, U Wisc; AA Oberai, RPI

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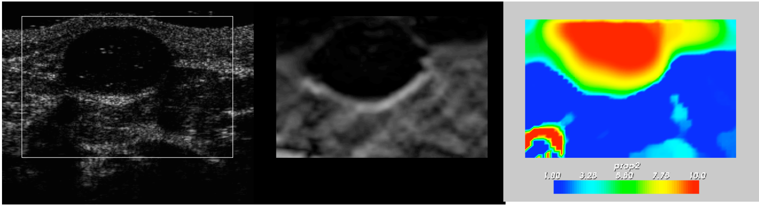
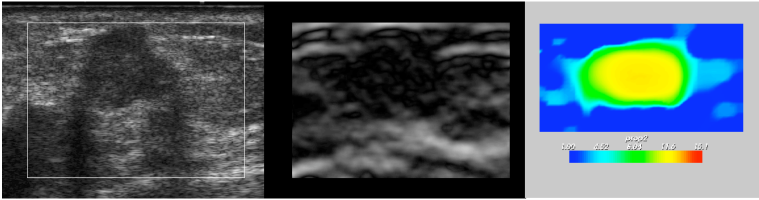


(CC011) Invasive Ductal Carinoma (malignant tumor)



[6] w/ TJ Hall, U Wisc; AA Oberai, RPI

Results from Clinical Images: IDC (CC193)

Inverse Elasticity

Barbone

Background

Measuring interior displacement data

Fwd Model

Inv Problem

Plane Stress

Forward Model

Inversion

Plane Strain

Forward Model

Inversion

Examples

Wish list

Summary

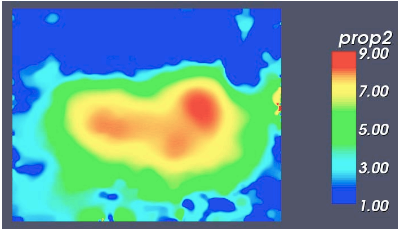
Further Reading

Ultrasound Image

Results from Clinical Images: IDC (CC193)

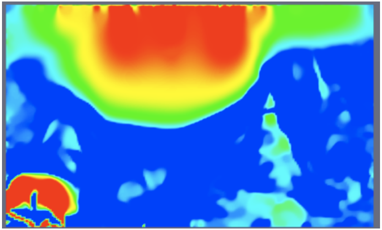
Ultrasound Image

Shear Modulus Reconstruction



Tumor heterogeneity hallmark of malignancy

Fibroadenoma (benign tumor)



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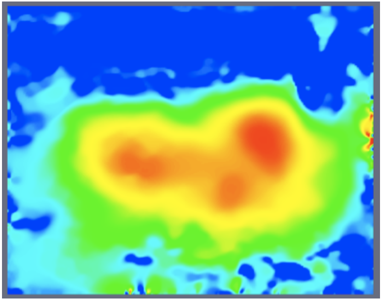


Outline

Challenge 1: Discontinuous Material Properties

Adjoint weighted variational equation (plane stress).
Given $\boldsymbol{A}$. Let $\mathcal{V} = \{w \in H^1(\Omega) | \int w d\Omega = 0\}$. Assume:

- $\nabla \cdot \boldsymbol{A} \in L_2(\Omega)$
- $\exists C_1$ and C_2 $s.t.$

$$(w, (\nabla \cdot \boldsymbol{A})^2 w) \leq C_1(\nabla w, \boldsymbol{A}^2 \nabla w) \leq C_2 \|w\|_1^2 \qquad (44)$$

- Define: $b(w, \mu) = \left(\boldsymbol{A} \nabla w, \nabla \cdot (\boldsymbol{A}\mu)\right)$
- $\mu = \tilde{\mu} + \mu_o$.

AWE: Find $\tilde{\mu} \in \mathcal{V}$ s.t.

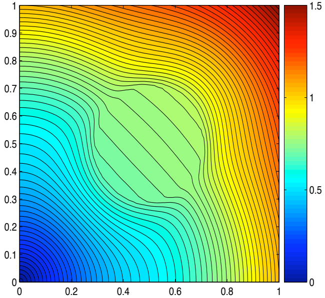
$$b(w, \mu) = 0 \qquad \forall w \in \mathcal{V} \qquad (45)$$

Challenge 1: Discontinuous Material Properties

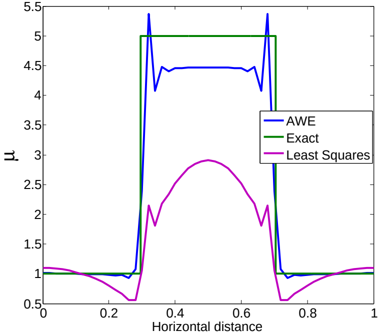
Remarks:

- AWE (45) is well-posed.
- $b(\cdot,\cdot)$ is coercive on $\mathcal{V}$.
- From coercivity, comes:
  1. Uniqueness & existence (Lax-Milgram).
  2. Equivalence to strong form.
  3. Convergence with Galerkin discretization.

Challenge 1: Discontinuous Material Properties



Input Displacement



Line plot through inclusion

Challenge 1: Discontinuous Material Properties

Remarks:

1. Strong (exact) solution, AWE formulation, and least squares all require $\boldsymbol{u} \in H^2$, or smoother.
2. All methods give $\mu \in H^1$ or smoother.
3. Can we obtain a direct formulation that allows $\boldsymbol{u} \in H^1$, $\mu \in L_1$, consistent with the forward problem?

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Challenge 2: Discontinuous Material Properties

Optimization formulation of inverse problem.

$$\mathcal{L}[\boldsymbol{u}, \boldsymbol{\lambda}, \mu] = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}^m\|_N^2 + a_1(\boldsymbol{\lambda}, \boldsymbol{u}; \mu) \tag{46}$$

$$a_1(\boldsymbol{\lambda}, \boldsymbol{u}; \mu) = \Big(\boldsymbol{\lambda}, \nabla \cdot (\mu \boldsymbol{A})\Big) \tag{47}$$

$$\mathcal{S} = \{\boldsymbol{u} \in H^s(\Omega) \mid \boldsymbol{u} = \boldsymbol{u}^m \text{ on } \Gamma\} \tag{48}$$

$$\mathcal{V} = \{\boldsymbol{v} \in H^s(\Omega) \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma\} \tag{49}$$

$$\mathcal{P} = \{\boldsymbol{\lambda} \in H^l(\Omega) \mid \boldsymbol{\lambda} = \boldsymbol{0} \text{ on } \Gamma\} \tag{50}$$

$$\mathcal{A} = \{\mu \in H^m(\Omega) \mid \int_\Omega \mu \, d\Omega = \overline{\mu}\} \tag{51}$$

$$\mathcal{B} = \{\gamma \in H^m(\Omega) \mid \int_\Omega \gamma \, d\Omega = 0\} \tag{52}$$

Challenge 2: Discontinuous Material Properties

Remarks:

1. Forward problem well defined for $\mu \in L_1$, $\boldsymbol{u} \in H^1$.
2. Can prove optimization problem well posed for $\boldsymbol{u} \in H^2$ and $\mu \in H^1$.
3. Can we prove the optimization formulation is well-posed for $\boldsymbol{u} \in H^1$, $\mu \in L_1$, consistent with the forward problem?
4. What are the weakest spaces where we can pose this problem?

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Challenge 3: New *Forward* FEM formulations

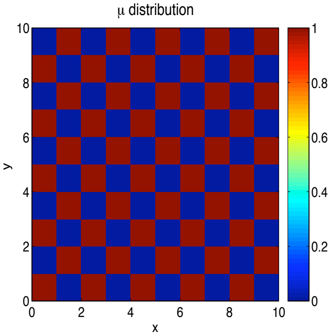
Consider optimization formulation of inverse problem.

$$\mathcal{L}[\boldsymbol{u}, \boldsymbol{\lambda}, \mu] = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}^m\|_1^2 + a_1(\boldsymbol{\lambda}, \boldsymbol{u}; \mu) \tag{53}$$

$$a_1(\boldsymbol{\lambda}, \boldsymbol{u}; \mu) = \Big(\boldsymbol{\lambda}, \nabla \cdot (\mu \boldsymbol{A})\Big) \tag{54}$$

- Discretize by standard FEM
  - u^h bilinear interpolation over element.
  - μ^h constant over each element.
- “Measure" $\boldsymbol{u}_x^m = y$ consistent with $\mu = \text{const}$.
- Solve by Newton iterations.

Challenge 3: New *Forward* FEM formulations



Reconstructed μ

- Noiseless data.
- Both data and modulus solution are exactly representable on mesh.
- Elasticity equations reduce to:

$$\partial_x \mu = 0 \; ; \; \partial_y \mu = 0 \quad (55)$$

Challenge 3: New *Forward* FEM formulations

Remarks:

1. Exact (strong) solution is $\mu =$ constant.
2. Checkerboard exactly satisfies FEM equations at $u = u^m$.
3. Checkerboard violates strong elasticity equations.
4. Strong elasticity eqn has enough information to determine μ.
5. Discrete (weak) eqn does not!
6. New discretization of forward problem needed to adequately enforce physical constraints.

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Challenge 4: Multiple Data Sets

Multiple datasets stabilizes inverse problem, reduces ambiguity. Given $\boldsymbol{u}_j^m$, $j = 1, \ldots, N_{meas}$, measured displacement fields.

AWE or LS formulation:

$$b(w, \mu) = \sum_{j=1}^{N_{meas}} b_j(w, \mu) = 0 \qquad \forall w \in \mathcal{V} \tag{56}$$

Optimization formulation:

$$\Pi[\mu] = \sum_{j=1}^{N_{meas}} \|\boldsymbol{u} - \boldsymbol{u}_j^m\|_N^2 \tag{57}$$

Challenge 4: Multiple Data Sets

Open questions:

1. For plane strain and $3D$, multiple deformations required for uniqueness. How “different" must they be?
2. Uniqueness for plane strain assumes distinct characteristics. What if they are the same?
3. Appropriate scaling and “orthogonalization” of data?

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Challenge 4: Multiple Data Sets

Linearity of forward problem:
If $\boldsymbol{u}_1^m$ and $\boldsymbol{u}_2^m$ are solutions of the forward problem, then

$$\boldsymbol{u}_\alpha^m = \alpha \boldsymbol{u}_1^m + (1-\alpha)\boldsymbol{u}_2^m \quad (58)$$

$$\boldsymbol{u}_\beta^m = \beta \boldsymbol{u}_1^m + (1-\beta)\boldsymbol{u}_2^m \quad (59)$$

are valid data sets.

- Conditioning, and hence solution, of the inverse problem depends upon choice of α and β.
- α and β probably ought to be selected so that $\boldsymbol{u}_\alpha^m$ and $\boldsymbol{u}_\beta^m$ are orthonormal in some sense.
- What sense?

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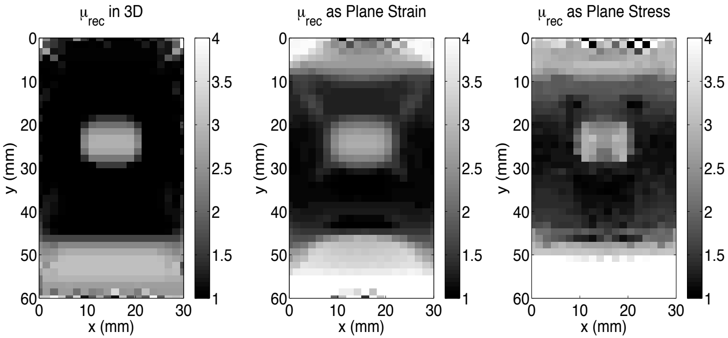
Challenge 5: Single (accurate) displacement component

Ultrasound (currently) provides accurate measurement of “axial” component of $\boldsymbol{u}$, but relatively noisy estimate of orthogonal components.

Questions & Opportunities:

1. What are the implications for uniqueness?
2. Plane stress: Solvability condition gives nonlinear pde coupling u_x to u_y:
$$\nabla \times [\boldsymbol{A}^{-1}\nabla \cdot \boldsymbol{A}] = 0. \tag{60}$$
3. Plane strain: Incompressibility condition gives linear pde coupling u_x to u_y:
$$\nabla \cdot \boldsymbol{u} = 0. \tag{61}$$

Challenge 6: 3*D* effects in 2*D* reconstructions



Challenge 7: Three-dimensional uniqueness

Eliminate pressure from momentum equation:

$$2\nabla \times [\nabla \cdot (\mu\epsilon)] = \rho\partial_{tt}\nabla \times \boldsymbol{u} \quad (62)$$

Questions:

1. Eqn (62) is “hyper-hyperbolic". What are its solution properties? What data does it require?
2. Given two sufficiently smooth measurements (i.e. equation coefficients $\epsilon(\boldsymbol{x})$), what is the dimension of the solution space?

Challenge 7: Three-dimensional uniqueness

Example - Uniform Strain:

$$u_x = \epsilon_1 x \tag{63}$$
$$u_y = \epsilon_2 y \tag{64}$$
$$u_z = \epsilon_3 z \tag{65}$$

with:

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \tag{66}$$
$$\epsilon_1 \neq \epsilon_2 \neq \epsilon_3 \neq \epsilon_1. \tag{67}$$

Then:

$$\mu(x, y, z) = f(x) + g(y) + h(z). \tag{68}$$

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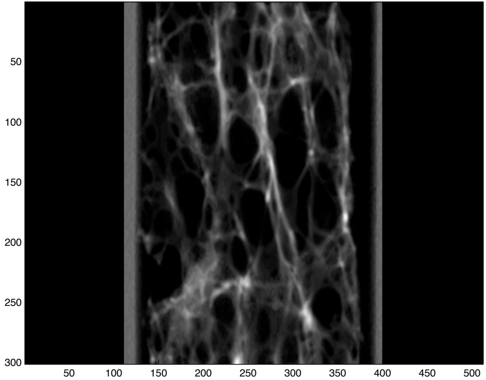
Structure similar to plane strain

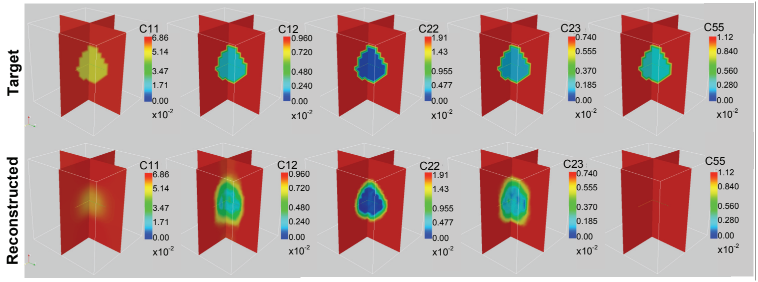
Challenge 8: Anisotropy[7]



- Improve understanding of osteoporosis.
- Image with μCT.
- Reconstruct distribution of aniostropic mat'l props.

[7] E.F. Morgan, unpub 2007

Challenge 8: Anisotropy[8]



Loading along 3 orthogonal axes are insensitive to C_{55}.
What loadings give unique reconstruction?

[8] A.A. Oberai & E.F. Morgan, unpub 2007

Biomechanical Imaging Summary

- Nice inverse problems with lots of applications.
- Pressure field due to incompressibility introduces ambiguity. *Not to be neglected!*
- Strain imaging and plane stress: enough "extra information" in the assumed model.
- Plane strain and 3D: need additional information.
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  - Forward solutions!
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Broader outlook:

- Linear shear stiffness.
- Nonlinear stiffness.
- Anisotropy.
- Viscosity.
- Hysteresis.
- Compressibility (for $f < 0.1Hz$).
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There remains lots of math and engineering to be done.

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