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A Hierarchical Linear Model for Explaining Variation in Math Achievement

The 1980 High School and Beyond (HS&B) survey data set offers a wealth of information about students and the possible factors affecting their achievements. Students in the samples are not chosen totally at random, but are grouped by schools. On account of this grouping, student-level outcomes such as the standardized math achievement score cannot be explained using standard ordinary least squares (OLS) regression techniques. Specifically the nesting of level-1 units (the students) in level-2 groups (the schools) leads to the possibility of unequal variances and correlated error terms, both of which violate the OLS regression assumptions. These violations of OLS lead to misestimation of standard errors and significance levels.

Previous approaches for dealing with multi-level data include disaggregation and aggregation. Under disaggregation, the investigator recodes the level-2 variables as though they were level-1 variables, and proceeds with OLS. However, we cannot assume that level-1 cases are independent, and thus we could introduce correlated errors and heteroskedasticity into the model. Further, we should not assume that the effect of the level-2 variables is consistent across level-2 units (e.g. heterogeneity in regression slopes is common). Finally, under disaggregation, we are subject to the atomistic fallacy – incorrectly interpreting a relationship at the wrong level of analysis. All of these biases can lead to incorrect conclusions. Under the aggregation approach, all level-1 variables are recoded to level-2 (e.g. average effect and average outcome at the level-2 unit). However, in the process we discard a lot of useful information about the level-1 units. Further, we discard the ability to analyze the variation in outcomes as being within or between level-2 units, since only between-unit variation remains. Finally, we are subject to the ecological effect, and thus we cannot assume that the relationship between two aggregated variables is the same as the relationship between the "equivalent" variables at their original level of analysis.

Hierarchical linear modeling (HLM) corrects for the unequal and non-independent variances found in multi-level data by allowing the model to estimate separately the regression coefficients and random error terms for each level-2 unit (e.g. the schools), and by having a more complex error term which controls for heteroskedasticity and correlated errors.

Descriptive Statistics

To familiarize ourselves with the data set, exploratory descriptive statistics were calculated for the relevant level-1 (student) and level-2 (school) variables. These results are presented in tables 1 and 2.

Table 1: Student-level predictor variables take from the 1980 High School and Beyond data set.

Variable Name	Description	Mean	Standard Deviation	Min	Max	Valid N
zmath	A standardized math achievement score.	0.00	1.00	-2.5202	2.0903	23936
zbbsesra	A standardized SES composite scale score.	-0.0894	0.7589	-3.9247	3.0191	26808
wrkhrs	The number of hours worked per week.	19.5688	12.1204	0	40	27278
tvhrs	The number of hours watching tv per day.	2.9457	1.6819	0	6	27176
cutclass	A dummy variable for whether the student cuts class. (1 = yes, 0 = no)	0.4305	0.4952	0	1	26703
female	A dummy variable for the student's sex. (female = 1, male = 0)	0.5205	0.4996	0	1	26427
racew	A dummy variable for race (1 = while, 0 = not white)	0.7753	0.4174	0	1	27131

Table 2: School-level predictor variables taken from the 1980 High School and Beyond data set.

Variable Name	Description	Mean	Standard Deviation	Min	Max	Valid N
zmeanses	The standardized school mean SES, aggregate variable based on the student data.	0.00	1.00	-2.3875	3.31 47	965
iqvf	The faculty index of qualitative variation, a measure of the racial diversity of the faculty.	0.1962	0.2066	0	0.82 5	988
iqvs	The student index of qualitative variation, a measure of the racial diversity of the students.	0.3068	0.2628	0	1	988
relig	A dummy variable indicating whether the school is religious.	0.1022	0.3031	0	1	988
private	A dummy variable indicating whether the school is private.	0.0182	0.1338	0	1	988

One-way ANOVA with Random Effects Model

In the first analysis of the standardized math scores from the 1980 High School and Beyond data[1], we rejected the null hypothesis that none of the variation in scores can be explained by differences between schools. For standardized math achievement scores, the variation between schools is almost

[1] SC705 Assignment 1, the fully unconditional model.

18.6% of total variation (intra-class correlation), compared with 82.4% of total variation being within schools. The significance of this between-school variation is born out by the χ^2-test value of 4930, which allows us to soundly reject the null hypothesis (H_0: $\tau(math) = 0$) in favor of the research hypothesis (H_A: $\tau(math) > 0$). Thus, math achievement scores do vary significantly across schools. Finally, school-level sample means are about 79% reliable in predicting the true school means.

A multi-level regression analysis is advised to further analyze the variation between schools. The remainder of this paper addresses this research hypothesis (e.g. explaining the variation between schools), by considering a one-way analysis of covariance (ANCOVA) with random effects model (e.g. fixed slopes for level-1 variables); a random coefficient regression model (e.g. allowing random variation in the level-1 slope for race effects); and an intercepts and slopes as outcomes model (e.g. allowing level-2 factors to explain variation in the slopes and intercepts across schools.

One-way ANCOVA with Random Effects Model

The initial model for the standardized math achievement score under the one-way ANCOVA model is expressed by

$$\begin{aligned} Y_{ij} = \beta_{0j} + \beta_{1j}(CUTCLASS) + \beta_{2j}(FEMALE) + \beta_{3j}(RACEW) + \beta_{4j}(ZBBSESRA) + \\ \beta_{5j}(WRKHRS) + \beta_{6j}(TVHRS) + r_{ij} \end{aligned}$$

$$\beta_{0j} = \gamma_{00} + u_{0j},$$

and all other $\beta_{ij} = \gamma_{ij}$

where Y_{ij} is the student-level outcome; γ_{00} is the grand-mean across all schools; β_{0j} is the mean for school j; u_{0j} is the residual variation in the intercept for school j after controlling for explanatory variables; all other $\beta_{ij} = \gamma_{ij}$ are fixed effect slopes for the level-1 independent variables; and r_{ij} is the remaining (unexplained) variation. The level-1 independent variables were defined in table 1.

Several of the student-level independent variables significantly affect the math achievement score, including the propensity to cut class ($p < 0.001$), sex ($p < 0.001$), race ($p < 0.001$), socioeconomic status ($p < 0.001$), work hours ($p < 0.001$), and TV hours ($p < 0.001$). The effect of these independent variables is given by their respective coefficients (see table 3).

Table 3. Effects of student-level predictor variables on standardized math achievement score: HLM One-way ANCOVA with random effects model.

outcome: ZMATH	coefficient	std err	t-ratio	df	p-value
beta00 = gamma00	0.066399	0.020005	3.319	934	0.001
beta01 (CUTCLASS)	-0.219616	0.013244	-16.582	18506	0.000
beta02 (FEMALE)	-0.266977	0.013126	-20.340	18506	0.000
beta03 (RACEW)	0.318936	0.018678	17.075	18506	0.000
beta04 (ZBBSESRA)	0.298335	0.007245	41.178	18506	0.000
beta05 (WRKHRS)	-0.004155	0.000540	-7.694	18506	0.000
beta06 (TVHRS)	-0.070604	0.003865	-18.268	18506	0.000

The effect of race is fairly strong, such that a white student is expected to score 0.319 standard units better than a non-white student. Socioeconomic status is also a fairly strong effect, whereby a 1 standard unit increase in SES predicts a 0.298 standard unit increase in math achievement score. The student's sex is also a strong predictor, such that a female student is expected to have a math achievement score 0.267 standard units lower than a comparable male student. Finally, the propensity to cut class has a strong negative effect on math achievement score, with students who cut class tending to have a math achievement score about 0.22 standard units lower than students who do not cut class. While TV hours and work hours are statistically significant predictors, their effects (e.g. coefficients) are relatively small ($\beta_{06}(WRKHRS)$ = -0.07, $\beta_{07}(TVHRS)$ = 0.004). Given the low absolute value of these coefficients, it is possible that the statistical significance is more a result of the large sample size than the effects being strong.

After allowing the one-way ANCOVA model to control for the student-level independent variables, we investigate whether math achievement still varies across schools. The remaining variation across schools, given by $\tau_{00} = 0.05524$ is still significant, as evidenced by the χ^2-test value of 2392.75 ($p < 0.001$). Therefore, we reject the null hypothesis (H_0: τ_{00} = 0) in favor of the research hypothesis (H_A: $\tau_{00} > 0$). This supports the assertion that schools do matter, e.g. have a significant effect on a student's math achievement score, even after controlling for the student-level independent variables.

The introduction of independent variables to the student-level model explains approximately 12.64% of the variation in math achievement within schools, and approximately 69.7% of the variation in math achievement between schools, compared to the baseline fully unconditional model. Given the significant reduction in between-school variation in math achievement scores, there is some evidence of composition effects relating to these level-1 independent variables. That is to say, the composition of the school groupings based on race, socioeconomic status, and so forth appear to have an effect on the average math achievement score for each school.

Random Coefficient Regression Model

The random coefficient regression model addresses the question of whether the effects of race vary across schools. The model estimated for the standardized math achievement score under the random coefficient regression model (RCRM) is expressed by

$$Y_{ij} = \beta_{0j} + \beta_{1j}(CUTCLASS) + \beta_{2j}(FEMALE) + \beta_{3j}(RACEW) + \beta_{4j}(ZBBSESRA)$$
$$+ \beta_{5j}(WRKHRS) + \beta_{6j}(TVHRS) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + u_{0j},$$

$$\beta_{3j} = \gamma_{03} + u_{3j},$$

and all other $\beta_{ij} = \gamma_{ij}$,

where Y_{ij} is the student-level outcome; γ_{00} is the grand-mean intercept across all schools; β_{0j} and β_{3j} are the mean intercept and race slope, respectively, for school j; u_{0j} and u_{3j} are residual variations in the intercept and race slope, respectively, for school j after controlling for explanatory variables; all other $\beta_{ij} = \gamma_{ij}$ are fixed effect slopes for the level-1 independent variables; and r_{ij} is the remaining (unexplained) variation. The level-1 independent variables were previously defined in table 1, and the level-1 independent variable RACEW is group mean centered.

This model allows the slope for race (β_{3j}) to vary for each school, and thus enables us to evaluate the variation of this coefficients across schools. The effects of race vary significantly across schools, as evidenced by the χ^2-test value of 782.8 ($p < 0.001$). This provides very strong evidence which allows us to reject the null hypothesis (H_0: $u_{3j} = 0$), in favor of the research hypothesis (H_A: $u_{3j} > 0$). We can interpret this as very strong evidence that the effect of race varies across schools, e.g., schools differ in the degree to which they help equalize the difference in race as it affects math achievement results.

The relationship between the intercept and slope coefficient for race provide some insight into the effects of student-level predictor variables after controlling for the grouping of students in schools. The positive average intercept (0.07029) is negatively correlated with the race dummy variable (τ_{03}= -0.160), suggesting that in schools with higher average math achievement scores, the effects of race in explaining differences between students' scores diminishes. Based on the results of this RCRM, further research into the school-level factors which could explain the varying effects of race across schools would be appropriate; this leads us to consider the intercepts and slopes as outcomes model.

Intercepts and Slopes as Outcomes Model

The intercepts and slopes as outcomes model addresses the question of the effect of level-2 independent variables (e.g. school-specific factors) on the estimated level-1 slopes and intercepts. The model estimated for the standardized math achievement score under the slopes and intercepts as outcomes model is expressed by

$$Y_{ij} = \beta_{0j} + \beta_{1j}(CUTCLASS) + \beta_{2j}(FEMALE) + \beta_{3j}(RACEW) + \beta_{4j}(ZBBSESRA) + \beta_{5j}(WRKHRS) + \beta_{6j}(TVHRS) + r_{ij}$$

$$\beta_{0j} = \gamma_{00} + \gamma_{1j}(RELIG) + \gamma_{2j}(PRIVATE) + \gamma_{3j}(ZMEANSES) + u_{0j} \text{ ,}$$

$$\beta_{1j} = \gamma_{10} \text{ ,}$$

$$\beta_{2j} = \gamma_{20} \text{ ,}$$

$$\beta_{3j} = \gamma_{30} + \gamma_{31}(IQVF) + \gamma_{32}(IQVS) + \gamma_{33}(RELIG) + \gamma_{34}(PRIVATE) + u_{3j} \text{ ,}$$

$$\beta_{4j} = \gamma_{40} \text{ ,}$$

$$\beta_{5j} = \gamma_{50} \text{ ,}$$

$$\beta_{6j} = \gamma_{60} \text{ ,}$$

where Y_{ij} is the student-level outcome; γ_{00} is the grand-mean intercept across all schools; γ_{1j}, γ_{2j}, and γ_{3j} are school-level slopes for the RELIG, PRIVATE, and ZMEANSES school-level independent variables for school *j*, respectively; u_{0j} is the residual variation in the intercept for school *j*; γ_{30} is the average race effect across all schools; γ_{31}, γ_{32}, γ_{33}, and γ_{34} are the school-level slopes for the IQVF, IQVS, RELIG, and PRIVATE school-level independent variables, respectively; u_{3j} is the residual variation in the race slope, respectively, for school *j*; all other $\beta_{ij} = \gamma_{ij}$ are fixed effect slopes for the level-1 independent variables; and r_{ij} is the remaining (unexplained) variation. The level-1 independent variable RACEW is centered around its group mean, and level-1 independent variables ZBBSESRA, WRKHRS, and TVHRS have been centered around their respective grand means. The level-2 independent variable ZMEANSES has been centered around its grand mean, and all other level-2 independent variables are uncentered.

This model allows us to consider the effects of school type and mean social-economic status (SES) on the average math achievement intercept, and the effects of school-type and faculty and student diversity on the average race effect slope (see table 4).

Table 4: Effects of school-level and student-level predictor variables on standardized math achievement score: HLM Intercepts and Slopes as Outcomes model.

outcome: ZMATH	coefficient	std err	t-ratio	df	p-value
beta00(intercept1)					
INTRCPT2, G00	0.316474	0.013643	23.197	931	0.000
RELIG, G01	0.017103	0.030624	0.558	931	0.577
PRIVATE, G02	0.119476	0.07753	1.541	931	0.124
ZMEANSES, G03	0.200268	0.011188	17.900	931	0.000
beta01(cutclass)					
INTRCPT2, G10	-0.227388	0.013215	-17.207	18501	0.000
beta02(female)					
INTRCPT2, G20	-0.268397	0.013037	-20.587	18501	0.000
beta03(racew)					
INTRCPT2, G30	0.201728	0.054689	3.689	930	0.000
IQVF, G31	-0.414369	0.135858	-3.050	930	0.002
IQVS, G32	0.470068	0.111597	4.212	930	0.000
RELIG, G33	-0.295818	0.075326	-3.927	930	0.000
PRIVATE, G34	-0.031787	0.321443	-0.099	930	0.921
beta04(zbbsesra)					
INTRCPT2, G40	0.255345	0.007749	32.952	18501	0.000
beta05(wrkhrs)					
INTRCPT2, G50	-0.004181	0.000537	-7.786	18501	0.000
beta06(tvhrs)					
INTRCPT2, G60	-0.067414	0.003852	-17.501	18501	0.000

The average intercept in this model (γ_{00}) after controlling for school-level predictor variables is 0.31647, and a 95% confidence interval for this intercept is the range [0.289734, 0.343214]. Of the level-2 variables, only the effect of mean SES is statistically significant, with ($p < 0.001$). The effect of the mean SES level for a given school is positive, with a coefficient of 0.200268. This can be interpreted as meaning that students in a school with an average SES one standard unit above the mean of school mean SES is expected to have a math achievement score about 0.2 standard units higher, which would be about 0.516. The effect of private schools appears to be somewhat strong (0.12), but it is not statistically significant ($p = 0.124$).

With regard to the effect of level-2 explanatory variables on the race slope, faculty diversity, student diversity, and religious school type appear to have statistically significant effects (p-values < 0.002 for each). Further, the effect of each of these explanatory variables appears to be strong. The of faculty diversity effect accounts for a -0.414369 change from the average race slope, meaning that a hypothetical school with a totally diverse faculty (IVFQ = 1) would reduce the effect of race on standardized math achievement by 0.41 standard units. However, a diverse student body appears to have an opposite effect with a coefficient of 0.47. Thus, a hypothetical school with total diversity (IQVS = 1) would have a difference of 0.47 in the race effect, compared with a school with no diversity. Finally, the effect of a religious school is to reduce the race effect on student-level math achievement outcomes, such that a religious school on average reduces the race gap between students by -0.295818 standard units. Compared to the RCRM model above, the introduction of these school-level independent variables explains about 15.9% of the variation in the race slope. While this percentage changes in variation does not follow a well-known distribution (e.g. the χ^2 distribution), given the size of the sample set one could reasonably argue that this is significant in explaining the race gap in math achievement scores.

There appears to be a composition effect with respect to socioeconomic status. In the slopes and intercepts as outcomes model, we are taking into consideration the average SES for a school in estimating the intercept for that school. Even after controlling for the effect of student-level SES which is strong (γ_{40}=0.255345) and statistically significant ($p < 0.001$), the effect of aggregate school-level SES is strong (γ_{03}=0.200268) and statistically significant ($p < 0.001$). Thus, we can conclude that the grouping of students in schools with higher average SES does have a significant effect on individual student-level math achievement. Stated another way, students perform better on math achievement in schools with higher average SES, even if the student has a low individual SES; perhaps these schools used the average SES to their advantage by hiring better teachers or employing better materials.

Overall, the effect of introducing these level-1 and level-2 explanatory variables accounts for about 13.7% of the variation within school, and about 77.1% of variation between schools (as measured by the relative improvement in variation compared to the fully unconditional model).

Concluding Remarks

In the one-way ANCOVA model, we established that the effects of race and socioeconomic status were significant predictors of student-level math achievement scores. The analysis given by the random coefficient regression model provides some evidence that the effects of these differences also varies across schools, meaning that some schools are doing a better job of neutralizing the effects of these variables (or equalizing the results). The slopes and intercepts as outcomes model led to the identification of some factors affecting the average level math achievement score (aggregate socioeconomic status at the school level) and the level of the race effect (faculty diversity, student diversity, and religious school type). Empowered with these statistics, education advocates can develop programs to address educational inequities resulting from race and socioeconomic status. Further research to address policy or practice to help effect these changes would be advised.