Modeling II
Linear Stability Analysis
and
Wave Equations
Nondimensional Equations

From previous lecture, we have a system of nondimensional PDEs:

\[
\frac{\partial c}{\partial T} = \frac{\partial}{\partial X} \left( D_c \frac{\partial c}{\partial X} \right) - \frac{\partial}{\partial X} \left( \chi_u c \frac{\partial u}{\partial X} \right) - \frac{\partial}{\partial X} \left( \chi_v c \frac{\partial v}{\partial X} \right) + \mu_c c(1 - c) \quad (21.1)
\]

\[
\frac{\partial v}{\partial T} = -\gamma u v + \mu_v v(1 - v) \quad (21.2)
\]

\[
\frac{\partial u}{\partial T} = \frac{\partial}{\partial X} \left( D_u \frac{\partial u}{\partial X} \right) + k_c c - k_u u \quad (21.3)
\]

where here the “*” sign has been dropped for convenience.
Parameters

The parameter values:

\[ D_c = 10^{-4} \quad \mu_c = 0.25 \quad k_c = 0.05 \]
\[ D_u = 10^{-2} \quad \mu_u = 0.15 \quad k_u = 0.3 \]
\[ \chi_u = 0.05 \quad \chi_v = 0.035 \quad \gamma = 8.15 \]

Initial conditions:

\[ c(X, 0) = \exp(-|X|^2/0.01) \]
\[ v(X, 0) = 1 - \exp(-|X|^2/0.01) \]
\[ u(X, 0) = \frac{1}{2} \exp(-|X|^2/0.01) \]

Boundary conditions: zero-flux on all boundaries.
Homogeneous Steady States

A homogeneous steady state of a PDE model is a solution that is constant in space in time. For equations (21.1), (21.2), and (21.3) we take

\[
\frac{\partial c}{\partial T} = \frac{\partial v}{\partial T} = \frac{\partial u}{\partial T} = 0
\]

\[
\frac{\partial c}{\partial X} = \frac{\partial v}{\partial X} = \frac{\partial u}{\partial X} = 0
\]

leaving

\[
0 = \mu_c c_{ss}(1 - c_{ss}) \tag{21.4}
\]

\[
0 = -\gamma u_{ss} v_{ss} + \mu_v v_{ss}(1 - v_{ss}) \tag{21.5}
\]

\[
0 = k_c c_{ss} - k_u u_{ss} \tag{21.6}
\]
Homogeneous Steady States

Possibilities of steady state points:

\[ c_{ss} = 0 \quad \rightarrow \quad u_{ss} = 0 \quad \text{and} \quad v_{ss} = 0 \quad \text{or} \quad v_{ss} = 1 \]

\[ c_{ss} = 1 \quad \rightarrow \quad u_{ss} = \frac{k_c}{k_u} \quad \text{and} \quad v_{ss} = 0 \quad \text{or} \quad v_{ss} = -\frac{\gamma}{\mu_v} u_{ss} + 1 \]

Hence, there are four possible steady state points:

\[ (c_{ss}, v_{ss}, u_{ss}) = (0, 0, 0) \]
\[ (c_{ss}, v_{ss}, u_{ss}) = (0, 1, 0) \]
\[ (c_{ss}, v_{ss}, u_{ss}) = \left(1, 0, \frac{k_c}{k_u}\right) \]
\[ (c_{ss}, v_{ss}, u_{ss}) = \left(1, 1 - \frac{\gamma k_c}{\mu_v k_u}, \frac{k_c}{k_u}\right) \]
Inhomogeneous Perturbations

• It is of interest to determine whether or not the steady states are stable.
• We analyze stability properties by considering the effect of small perturbations.
• To do this, we must look at spatially non-uniform (also called inhomogeneous) perturbations and explore whether they are amplified or attenuated.
• If an amplification occurs, then a situation close to the spatially uniform steady state will destabilize, leading to some new state in which spatial variations predominate.
Inhomogeneous Perturbations

We take the distributions of the variables

\[
    u(X, T) = u_{ss} + \tilde{u}(X, T)
\]

\[
    v(X, T) = v_{ss} + \tilde{v}(X, T)
\]

\[
    c(X, T) = c_{ss} + \tilde{c}(X, T)
\]

where \( \tilde{c}, \tilde{v}, \) and \( \tilde{u} \) are small.

Using the facts that \( c_{ss}, v_{ss}, \) and \( u_{ss} \) are constants and uniform, the temporal and spatial derivatives give

\[
    \frac{\partial c}{\partial X} = \frac{\partial (c_{ss} + \tilde{c})}{\partial X} = \frac{\partial \tilde{c}}{\partial X}
\]

\[
    \frac{\partial c}{\partial T} = \frac{\partial (c_{ss} + \tilde{c})}{\partial T} = \frac{\partial \tilde{c}}{\partial T}
\]
Inhomogeneous Perturbations

The second-order spatial derivatives:

\[
\frac{\partial}{\partial X} \left( D_c \frac{\partial c}{\partial X} \right) = \frac{\partial}{\partial X} \left( D_c \frac{\partial (c_{ss} + \tilde{c})}{\partial X} \right) = D_c \frac{\partial^2 \tilde{c}}{\partial X^2}
\]  
(21.8)

For taxis terms:

\[
\frac{\partial}{\partial X} \left( \chi_u c \frac{\partial u}{\partial X} \right) = \frac{\partial}{\partial X} \left( \chi_u (c_{ss} + \tilde{c}) \frac{\partial (u_{ss} + \tilde{u})}{\partial X} \right)
\]

\[
= \frac{\partial}{\partial X} \left( \chi_u c_{ss} \frac{\partial \tilde{u}}{\partial X} + \chi_u \tilde{c} \frac{\partial \tilde{u}}{\partial X} \right)
\]

\[
= \chi_u c_{ss} \frac{\partial^2 \tilde{u}}{\partial X^2} + \chi_u \frac{\partial \tilde{c}}{\partial X} \frac{\partial \tilde{u}}{\partial X} + \chi_u \tilde{c} \frac{\partial^2 \tilde{u}}{\partial X^2}
\]
Inhomogeneous Perturbations

The terms
\[
\frac{\partial \tilde{c}}{\partial X} \frac{\partial \tilde{u}}{\partial X} \quad \text{and} \quad \tilde{c} \frac{\partial^2 \tilde{u}}{\partial X^2}
\]
are quadratic in the perturbations or their derivatives and consequently are of smaller magnitude than other terms, thus they can be omitted, leaving

\[
\frac{\partial}{\partial X} \left( \chi_u c \frac{\partial u}{\partial X} \right) = \chi_u c_{ss} \frac{\partial^2 \tilde{u}}{\partial X^2} \tag{21.9}
\]
and, similarly

\[
\frac{\partial}{\partial X} \left( \chi_v c \frac{\partial v}{\partial X} \right) = \chi_v c_{ss} \frac{\partial^2 \tilde{v}}{\partial X^2} \tag{21.10}
\]
Inhomogeneous Perturbations

And for the reaction terms:

\[\mu_c c (1 - c) = \mu_c (c_{ss} + \tilde{c})(1 - c_{ss} - \tilde{c})\]

\[= \mu_c c_{ss} (1 - c_{ss} - \tilde{c}) + \mu_c \tilde{c} (1 - c_{ss} - \tilde{c})\]

\[= \mu_c c_{ss} (1 - c_{ss}) + \mu_c \tilde{c} - 2 \mu_c c_{ss} \tilde{c} - \mu_c \tilde{c}^2\]

\[= \mu_c \tilde{c} (1 - 2c_{ss})\]

\[-\gamma u v + \mu_v v (1 - v) = -\gamma (u_{ss} \tilde{v} + v_{ss} \tilde{u}) + \mu_v \tilde{v} (1 - 2v_{ss})\]

\[k_c c - k_u u = k_c \tilde{c} - k_u \tilde{u}\]
Inhomogeneous Perturbations

Combining all together we rewrite the approximate linearized equations of (21.1) – (21.3) as

\[
\frac{\partial \tilde{c}}{\partial T} = D_c \frac{\partial^2 \tilde{c}}{\partial X^2} - \chi_u c_s \frac{\partial^2 \tilde{u}}{\partial X^2} - \chi_v c_s \frac{\partial^2 \tilde{v}}{\partial X^2} + \mu_c \tilde{c} (1 - 2c_s) \tag{21.11}
\]

\[
\frac{\partial \tilde{v}}{\partial T} = -\gamma (u_s \tilde{v} + v_s \tilde{u}) + \mu_v \tilde{v} (1 - 2v_s) \tag{21.12}
\]

\[
\frac{\partial \tilde{u}}{\partial T} = D_u \frac{\partial^2 \tilde{u}}{\partial X^2} + k_c \tilde{c} - k_u \tilde{u} \tag{21.13}
\]

which are linear in the quantities \( \tilde{c} \), \( \tilde{v} \), and \( \tilde{u} \).
We find eigenvalues by setting the equations (21.11), (21.12), and (21.13) to have no spatial variations, or

\[
\frac{d\tilde{c}}{dT} = \mu_c \tilde{c} (1 - 2c_{ss}) \tag{21.14}
\]

\[
\frac{d\tilde{v}}{dT} = -\gamma u_{ss} \tilde{v} - \gamma v_{ss} \tilde{u} + \mu_v \tilde{v} (1 - 2v_{ss}) \tag{21.15}
\]

\[
\frac{d\tilde{u}}{dT} = k_c \tilde{c} - k_u \tilde{u} \tag{21.16}
\]

Then let

\[
\frac{d\tilde{c}}{dT} = F(\tilde{c}, \tilde{v}, \tilde{u}) \quad \frac{d\tilde{v}}{dT} = G(\tilde{c}, \tilde{v}, \tilde{u}) \quad \frac{d\tilde{u}}{dT} = H(\tilde{c}, \tilde{v}, \tilde{u})
\]
Finding Eigenvalues

Differentiating \( F(\tilde{c}, \tilde{v}, \tilde{u}), G(\tilde{c}, \tilde{v}, \tilde{u}), \) and \( H(\tilde{c}, \tilde{v}, \tilde{u}) \) with respect to \( \tilde{c}, \tilde{v}, \) and \( \tilde{u} \) gives us a Jacobian matrix of the reaction terms:

\[
J_R = \begin{bmatrix}
\mu_c(1 - 2c_{ss}) & 0 & 0 \\
0 & -\gamma u_{ss} + \mu_v(1 - 2v_{ss}) & -\gamma v_{ss} \\
k_c & 0 & -k_u
\end{bmatrix}
\]

(21.17)

Eigenvalues are obtained by taking

\[
|J_R - \lambda I| = 0
\]

(21.18)
Stability of the Steady States

Steady states in (21.7) are linearly stable if \( \text{Re} \lambda < 0 \) since in this case the perturbations go to zero as time goes to infinity.

Using the parameter values given and substituting into (21.18), we obtain the stability of the steady state:

- \((0, 0, 0)\) gives \(2 \lambda_s > 0\) and \(1 \lambda < 0\): unstable
- \((0, 1, 0)\) gives \(1 \lambda > 0\) and \(2 \lambda_s < 0\): unstable
- \((1, 0, \frac{k_c}{k_u})\) gives \(3 \lambda < 0\): stable
- \(\left(1, 1 - \frac{\gamma k_c}{\mu_v k_u}, \frac{k_c}{k_u}\right)\) gives \(1 \lambda > 0\) and \(2 \lambda_s < 0\): unstable
Dispersion Relation

Now we consider the full equations (21.11) – (21.13) and differentiate with respect to the second-order spatial derivatives

\[ \frac{\partial^2 \tilde{c}}{\partial X^2}, \quad \frac{\partial^2 \tilde{v}}{\partial X^2}, \quad \text{and} \quad \frac{\partial^2 \tilde{u}}{\partial X^2} \]

To get the transport Jacobian

\[ J_T = \begin{bmatrix} D_c & -\chi_v c_{ss} & -\chi_u c_{ss} \\ 0 & 0 & 0 \\ 0 & 0 & D_u \end{bmatrix} \quad \text{(21.19)} \]
Dispersion Relation

The linearized system of equations (21.11) – (21.13) can now be represented in a compact form

\[ \overline{w}_t = J_R \overline{w} + J_T \nabla^2 \overline{w} \]  \hfill (21.20)

where

\[ \overline{w} = \begin{bmatrix} \tilde{c} \\ \tilde{v} \\ \tilde{u} \end{bmatrix} \]

to be solved in a domain with zero-flux boundary conditions

\[ (n \cdot \nabla) \overline{w} = 0 \]  \hfill (21.21)
Dispersion Relation

To solve the system of equations in (21.20) subject to the boundary conditions, we first define $W(r)$ to be the time-independent solution of the spatial eigenvalue problem, defined by

$$\nabla^2 W + k^2 W = 0$$

$$(n \cdot \nabla)W = 0 \quad \text{for} \quad r \text{ on } \partial B$$

where $k$ is the eigenvalue. For example, if the domain is 1D, say $0 \leq x \leq L$, then

$$W \propto \cos\left(\frac{n\pi x}{L}\right)$$

where $n$ is an integer. This satisfies zero-flux boundary conditions at $x = 0$ and $x = L$. 
Dispersion Relation

The eigenvalue in this case is

\[ k = \frac{n\pi}{L} \]

So

\[ \frac{1}{k} = \frac{L}{n\pi} \]

is a measure of the wavelike pattern: the eigenvalue \( k \) is called the \textit{wavenumber} and \( 1/k \) is proportional to the wavelength \( \omega \):

\[ \omega = \frac{2\pi}{k} = \frac{2L}{n} \]

We shall refer to \( k \) in this context as the wavenumber.
Dispersion Relation

With finite domains there is a discrete set of possible wavenumbers since \( n \) is an integer.

We now look for solutions of (21.20) in the form

\[
\mathbf{w} = \sum_{k} a_k e^{\lambda T} \mathbf{W}_k
\]  \hspace{1cm} (21.23)

Substituting (21.23) into (21.20) with (21.21) and canceling \( e^{\lambda T} \) we get, for each \( k \),

\[
\lambda \mathbf{W}_k = J_R \mathbf{W}_k + J_T \nabla^2 \mathbf{W}_k
\]

\[
= J_R \mathbf{W}_k - J_T k^2 \mathbf{W}_k
\]
Dispersion Relation

We require nontrivial solutions for $W_k$ so the now the $\lambda$ are determined by the roots of

$$|\lambda I - J_R + J_T k^2| = 0$$  \hspace{1cm} (21.24)

Evaluating the determinant with $J_T$ and $J_R$ we get the eigenvalues $\lambda(k)$ as functions of the wavenumber $k$. 
Dispersion Relation

Dispersion relation from steady state (0,0,0)

- Max real part (blue line) of eigenvalues is 0.25
- It means that the perturbation grows with time.
- Imaginary part (red line) is zero.
- With the max real part, there are a range of $k$ where the eigenvalues are positive.
Dispersion relation from steady state (0,1,0)

- Max real part (blue line) of eigenvalues is 0.25.
- The perturbation grows with time.
- Imaginary part (red line) is zero.
- There are a range of $k$ (between $k=0$ and $k\approx30$) where the eigenvalues are positive.
Dispersion relation from steady state \( \left( 1, 0, \frac{k_c}{k_u} \right) \)

- Max real part (blue line) of eigenvalues is -0.0477.
- The perturbation is damped away.
- Imaginary part (red line) is zero.
Dispersion Relation

Dispersion relation from steady state \( \left( 1, 1 - \frac{\gamma k_c}{\mu_v k_u}, \frac{k_c}{k_u} \right) \)

- Max real part (blue line) of eigenvalues is 3.725.
- Max imaginary part (red line) is 2.1374.
- The perturbations grow with time.
- Imaginary part creates oscillating solutions.
Simulation Results
1D Wave Equations

An example of a hyperbolic PDE is a 1D wave equation for the amplitude function $u(x,t)$ as

$$\frac{\partial^2 u}{\partial t^2} = D \frac{\partial^2 u}{\partial x^2} \quad (21.25)$$

for $\text{xmin} \leq x \leq \text{xmax}, \ 0 \leq t \leq \text{tend}$

In order for this equation to be solvable, the following should be provided:

- boundary conditions at $x = \text{xmin}$ and at $x = \text{xmax}$
- initial condition $u(x,0) = f_0$
- initial velocity $u_t(x,0) = v_0$
**1D Wave Equations**

We replace the second derivatives on both sides by their three-point finite central difference approximation as

\[
\frac{U_{i,k-1} - 2U_{i,k} + U_{i,k+1}}{\Delta t^2} = D \left( \frac{U_{i-1,k} - 2U_{i,k} + U_{i+1,k}}{\Delta x^2} \right)
\]

Multiplying both sides by $\Delta t^2$ gives

\[
U_{i,k-1} - 2U_{i,k} + U_{i,k+1} = \frac{D \Delta t^2}{\Delta x^2} (U_{i-1,k} - 2U_{i,k} + U_{i+1,k})
\]

From which we solve for $U_{i,k+1}$:

\[
U_{i,k+1} = rU_{i-1,k} + 2(1 - r)U_{i,k} + rU_{i+1,k} - U_{i,k-1}
\]  

(21.26)
1D Wave Equations

where

\[ r = \frac{D \Delta t^2}{\Delta x^2} \]

From equation (21.26), if \( k = 0 \):

\[ U_{i,1} = r U_{i-1,0} + 2(1 - r)U_{i,0} + r U_{i+1,0} - U_{i,-1} \]

where \( U_{i,-1} \) is not given. Therefore, we approximate the initial condition on the derivative (initial velocity) by the central difference as

\[ \frac{\partial u}{\partial t} \approx \frac{U_{i,k+1} - U_{i,k-1}}{2\Delta t} = v_0 \quad (21.27) \]
1D Wave Equations

or

\[ U_{i,-1} = U_{i,1} - 2\Delta tv_0 \]

and make use of this to remove \( U_{i,-1} \) from equation (21.26)

\[ U_{i,1} = rU_{i-1,0} + 2(1 - r)U_{i,0} + rU_{i+1,0} - (U_{i,1} - 2\Delta tv_0) \]

Equating the terms with \( U_{i,1} \) yields

\[ 2U_{i,1} = rU_{i-1,0} + 2(1 - r)U_{i,0} + rU_{i+1,0} + 2\Delta tv_0 \]

or rewrite

\[ U_{i,1} = \frac{1}{2}rU_{i-1,0} + (1 - r)U_{i,0} + \frac{1}{2}rU_{i+1,0} + \Delta tv_0 \]  
(21.28)
1D Wave Equations

We use equation (21.28) together with the initial conditions to get $U_{i,1}$ and then go on with equation (21.26) for $k = 1, 2, \ldots$

The following must be taken into account:
- To guarantee stability, $r \leq 1$
- The accuracy of the solution gets better as $r$ becomes larger so that $\Delta x$ decreases.

The stability condition can be obtained by substituting

$$U_{i,k} = \lambda^k e^{ji\pi/P}, \quad P \text{ is any nonzero integer}$$

into equation (21.26) and applying the Jury test:
1D Wave Equations

to get

$$\lambda = 2r \cos(\pi/P) + 2(1 - r) - \lambda^{-1}$$

or

$$\lambda^2 + 2(r(1 - \cos(\pi/P)) - 1)\lambda + 1 = 0$$

We need the solution of this equation to be inside the unit circle for stability, which requires:

$$r \leq \frac{1}{1 - \cos(\pi/P)}$$

$$r = D \frac{\Delta t^2}{\Delta x^2} \leq 1$$
Exercise 1

Solve the following 1D wave equation:

\[
\frac{\partial^2 u}{\partial t^2} = D \frac{\partial^2 u}{\partial x^2}, \quad D = 0.1
\]

over the spatial domain \(0 \leq x \leq 1\) and within time interval \(0 \leq t \leq 8\), with

- at \(x = 0\): \(u(0, t) = 0\)
- at \(x = 1\): \(u(1, t) = 0\)
- at \(t = 0\): \(u(x, 0) = x(1 - x)\)
- at \(t = 0\): \(\frac{\partial u(x, 0)}{\partial t} = 0\)
2D Wave Equations

For 2D wave equations such as

\[
\frac{\partial^2 u}{\partial t^2} = D \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (21.29)
\]

In the same fashion as 1D equations, we apply three-point central difference approximation for the 2D equation:

\[
\frac{U_{i,j}^{k-1} - 2U_{i,j}^k + U_{i,j}^{k+1}}{\Delta t^2} =
\delta_{t^2} D \left( \frac{U_{i-1,j}^k - 2U_{i,j}^k + U_{i+1,j}^k}{\Delta x^2} + \frac{U_{i,j-1}^k - 2U_{i,j}^k + U_{i,j+1}^k}{\Delta y^2} \right)
\]
2D Wave Equations

which leads to the explicit central difference method:

\[ U_{i,j}^{k+1} = r_x (U_{i-1,j}^k + U_{i+1,j}^k) + 2(1 - r_x - r_y)U_{i,j}^k \]
\[ + r_y (U_{i,j-1}^k + U_{i,j+1}^k) - U_{i,j}^{k-1} \]

(21.30)

with

\[ r_x = D \frac{\Delta t^2}{\Delta x^2} \]
\[ r_y = D \frac{\Delta t^2}{\Delta y^2} \]

Since \( U_{i,j}^{-1} \) is not given when \( k = 0 \), we approximate the initial condition on the derivative (initial velocity) by the central difference, giving:

\[ U_{i,j}^{-1} = U_{i,j}^1 - 2\Delta tv_0(x_i, y_j) \]
and make use of this to remove $U_{i,j}^{-1}$ from equation (21.30) to get

$$U_{i,j}^1 = r_x (U_{i-1,j}^0 + U_{i+1,j}^0) + 2(1 - r_x - r_y)U_{i,j}^0 + r_y (U_{i,j-1}^0 + U_{i,j+1}^0) - (U_{i,j}^1 - 2\Delta t v_0(x_i, y_j))$$

or, rearrange:

$$U_{i,j}^1 = \frac{1}{2} r_x (U_{i-1,j}^0 + U_{i+1,j}^0) + (1 - r_x - r_y)U_{i,j}^0 + \frac{1}{2} r_y (U_{i,j-1}^0 + U_{i,j+1}^0) + \Delta t v_0(x_i, y_j)$$

(21.31)
Stability for approximation equation (21.31) is guaranteed if and only if:

\[ r = \frac{4D \Delta t^2}{\Delta x^2 + \Delta y^2} \leq 1 \]
Exercise 2

Solve the following 2D wave equation:

\[ \frac{\partial^2 u}{\partial t^2} = D \nabla^2 u, \quad D = 0.25 \]

over the spatial domain \( 0 \leq x \leq 2, 0 \leq y \leq 2 \), and within time interval \( 0 \leq t \leq 8 \), with

- at \( x = 0 \) and \( y = 0 \): \( u(0, y, t) = 0 \) and \( u(x, 0, t) = 0 \)
- at \( x = 2 \) and \( y = 2 \): \( u(2, y, t) = 0 \) and \( u(x, 2, t) = 0 \)
- at \( t = 0 \): \( u(x, y, 0) = 0.1 \sin(\pi x) \sin(\pi y/2) \)
- at \( t = 0 \): \( \frac{\partial u(x, 0)}{\partial t} = 0 \)
References

(2) Applied Numerical Methods Using MATLAB, Yang Chao Chung and Morris.