Linearization of Nonlinear Reaction Terms

Consider a system of two equations

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} - \alpha \frac{\partial}{\partial x} \left( \frac{u}{(1 + v)^2} \frac{\partial v}{\partial x} \right) + g(u, v)
\]

\[
\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + h(u, v)
\]

where

\[
g(u, v) = \rho u \left( \delta \frac{w}{1 + w} - u \right)
\]

\[
h(u, v) = \beta w \left( \frac{u^2}{\mu + u^2} - uv \right)
\]
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Steady states are obtained by setting spatial and temporal variations to zero, which gives:

\[ g(u, v) = 0 \quad \rightarrow \quad \rho u_{ss} \left( \delta \frac{w}{1 + w} - u_{ss} \right) = 0 \]

\[ h(u, v) = 0 \quad \rightarrow \quad \beta w \frac{u_{ss}^2}{\mu + u_{ss}^2} - u_{ss} v_{ss} = 0 \]

from which, there is one non-zero steady state, given by

\[ (u_{ss}, v_{ss}) = \left( \delta \frac{w}{1 + w}, \beta w \frac{u_{ss}}{\mu + u_{ss}^2} \right) \]
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We linearize the system about the steady state by setting

\[ u = u_{ss} + \tilde{u}, \quad \tilde{u} \ll 1 \]
\[ v = v_{ss} + \tilde{v}, \quad \tilde{v} \ll 1 \]

(A.2)

We substitute (A.2) into (A.1) and expand for small \( \tilde{u} \) and \( \tilde{v} \) using a Taylor expansion up to the first order. The expansions are given by

\[ f(u_{ss} + \tilde{u}) \approx f(u_{ss}) + \tilde{u} f'(u_{ss}) + O(\tilde{u}^2) \]

\[ f(v_{ss} + \tilde{v}) \approx f(v_{ss}) + \tilde{v} f'(v_{ss}) + O(\tilde{v}^2) \]
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For multivariable functions, the expansions are given by

\[ g(u_{ss} + \tilde{u}, v_{ss} + \tilde{v}) \approx g(u_{ss}, v_{ss}) + \tilde{u} \frac{\partial g}{\partial u_{ss}} + \tilde{v} \frac{\partial g}{\partial v_{ss}} \]

\[ h(u_{ss} + \tilde{u}, v_{ss} + \tilde{v}) \approx h(u_{ss}, v_{ss}) + \tilde{u} \frac{\partial h}{\partial u_{ss}} + \tilde{v} \frac{\partial h}{\partial v_{ss}} \]

Since

\[ g(u_{ss}, v_{ss}) = 0 \]

\[ h(u_{ss}, v_{ss}) = 0 \]
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and from the partial derivatives

\[
\frac{\partial g}{\partial u_{ss}} = \rho \left( \delta \frac{w}{1+w} \right) - 2\rho u_{ss}
\]

\[
\frac{\partial g}{\partial v_{ss}} = 0
\]

\[
\frac{\partial h}{\partial u_{ss}} = \beta w \left( \frac{2\mu u_{ss}}{(\mu + u_{ss}^2)^2} - v_{ss} \right)
\]

\[
\frac{\partial h}{\partial v_{ss}} = -\beta w u_{ss}
\]
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the expansions now become

\[ g(u_{ss} + \tilde{u}, v_{ss} + \tilde{v}) = \rho \left( \delta \frac{w}{1 + w} \right) \tilde{u} - 2\rho u_{ss} \tilde{u} \]

\[ h(u_{ss} + \tilde{u}, v_{ss} + \tilde{v}) = \beta w \left( \frac{2\mu u_{ss}}{(\mu + u_{ss}^2)^2} - v_{ss} \right) \tilde{u} - \beta w u_{ss} \tilde{v} \]

Hence, the linearized form of equations (A.1) is given by:

\[ \text{... see next slide ...} \]
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\[
\frac{\partial \tilde{u}}{\partial t} = D \frac{\partial^2 \tilde{u}}{\partial x^2} - \alpha u_{ss} \chi(u_{ss}, v_{ss}) \frac{\partial^2 \tilde{v}}{\partial x^2} + \rho \left( \delta \frac{w}{1 + w} \right) \tilde{u} - 2\rho u_{ss} \tilde{u}
\]

\[
\frac{\partial \tilde{v}}{\partial t} = D \frac{\partial^2 \tilde{v}}{\partial x^2} + \beta w \left( \frac{2\mu u_{ss}}{(\mu + u_{ss}^2)^2} - v_{ss} \right) \tilde{u} - \beta w u_{ss} \tilde{v}
\]