On Euler's Number e

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The number e, an irrational number whose first digits are 2.7182818284..., is usually presented to students in precalculus classes as the number toward which compounded interest approaches as the number of compounding intervals approaches infinity and the interval size approaches 0. This representation has the benefit of being navigable for the student who does not know any calculus, but as students' mathematical knowledge increases, the usual definition begins to seem awkward, and even arbitrary: what does compound interest have to do with derivatives and integrals?

This short monograph details a method for deriving e that is both succinct and comprehensive. I first saw it presented in an MIT lecture by David Jerison, and I believe it is the best method for making this elusive number clear in the context of single variable calculus.

Suppose we want to know the derivative of some exponential function a^x, $a \in \mathbb{R}_+$. Well, by definition the derivative is

$$\frac{d}{dx}a^x = \lim_{\Delta x \to 0} \frac{a^{x+\Delta x} - a^x}{\Delta x}$$

We are used to thinking of x as the "moving" variable, but here it is Δx that is changing. We can think of x as fixed. Thus, the above derivative simplifies by the laws of exponents:

$$\frac{d}{dx}a^x = \lim_{\Delta x \to 0} a^x \frac{a^{\Delta x} - 1}{\Delta x} = a^x \left\{ \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x} \right\}$$

This means that the derivative of a^x is proportional to itself, by some constant: the quantity inside the limit. Call that quantity M(a), a function of the base of the exponential function. Note that this quantity is *not dependent on* x. Then we have that

$$\frac{d}{dx}a^x = a^x M(a)$$

Let's pretend we have never heard of the number e. Define e as that number such that M(e) = 1. Therefore we have that, by the definition of this new e number,

$$\frac{d}{dx}e^x = e^x M(e) = e^x \ , \quad \frac{d}{dx}e^x|_{x=0} = 1$$

All of that is well and good, but the original task, that of finding the derivative of a^x, is still not solved. To accomplish this, I introduce the logarithm with base e. Define $w = \ln(x)$. So if $y =$ e^x, then $\ln(y) = x$. This function obeys all the usual laws of a logarithm of any base, i.e. $\ln(1) = 0$, $\ln(x) + \ln(y) = \ln(xy)$, and so on. Now I solve the derivative of this function using implicit differentiation.

$$w := \ln(x) \rightarrow \ e^w = e^{\ln(x)} = x,$$

and from the chain rule:

$$\frac{d}{dx}e^w \equiv \frac{d}{dx}x \quad \rightarrow \quad \frac{d}{dx}e^w = e^w \times \frac{dw}{dx} = 1$$

This last quantity implies that

$$\frac{dw}{dx} = \frac{1}{e^w}$$

But what did we define e^w to be? It's x. Thus, the derivative of $\ln(x)$ is $1/x$. Now we have the machinery to work out the derivative of a^x.

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln(a)})^x = \frac{d}{dx}e^{x\ln(a)}$$

and again from the chain rule, the result is

$$\frac{d}{dx}e^{x\ln(a)} = \ln(a) \times e^{x\ln(a)} = \ln(a) \times a^x$$

For some given a, say $a = 3$, the derivative of 3^x is $\ln(3)$ times 3^x. This is one of the reasons that the exponential function with base e is called the "natural" exponential function, and it is why the logarithm base e is called the natural logarithm. It naturally appears in the formula for the derivative of an exponential, and if we choose the base e for our calculations, we never need actually compute one of these clunky logarithm functions.

Alright. The correct, "natural'" base has been derived. But what is this number actually? To find that out, I invoke the common approximation to the exponential base (which is in fact inferior in almost every way to the power series definition, but that's another story). This formula is usually introduced in the context of compounding interest:

$$\lim_{n\to\infty} \big(1 + \frac{1}{n}\big)^n$$

Now take the logarithm of this function, which is equal by the properties of logarithms to $n \times \ln(1 + 1/n)$. Now define $\Delta x = 1/n$. Note that as n grows large, Δx will tend toward 0. Then we can rewrite the logarithm of the compounding interest formula as

$$\lim_{\Delta x\to 0} \frac{1}{\Delta x} \ln(1 + \Delta x)$$

Now subtract $0 = \ln(1)$ from the above quantity to yield

$$\lim_{\Delta x\to 0} \frac{1}{\Delta x}(\ln(1 + \Delta x) - \ln(1))$$

But look what this is equal to:

$$\lim_{\Delta x\to 0} \frac{1}{\Delta x}(\ln(1 + \Delta x) - \ln(1)) = \frac{d}{dx}\ln(x)|_{x=1}$$

But we know what this derivative is. It's $1/x$. So the limit is 1. Now recall that this entire formula was the **logarithm** of the compound interest formula. To recover its actual value, we need to perform the inverse of the logarithm base e, which is the natural exponentiation:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e^{\left\{\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n\right\}} = e^1 = e$$

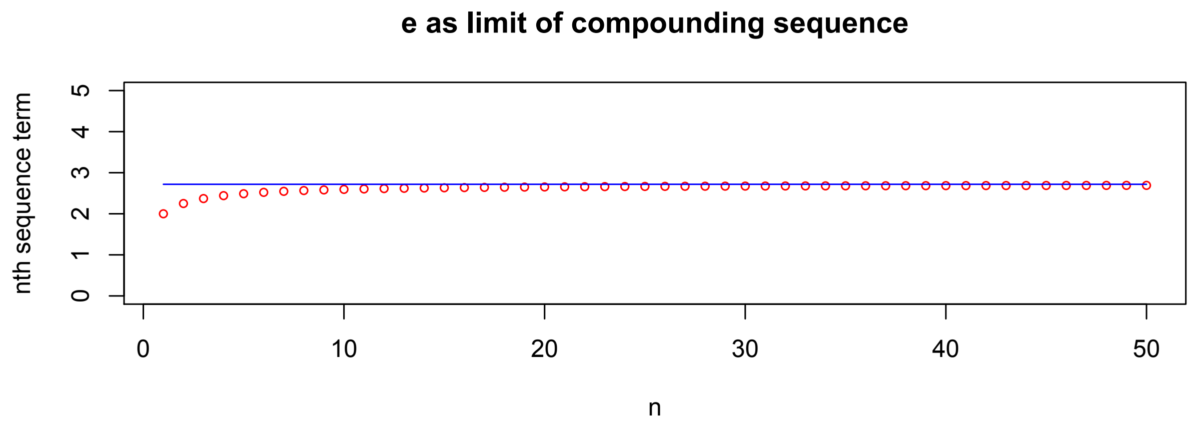
Now we know roughly what e is. If I take $n = 1000$, I get that $e \approx 2.71$.

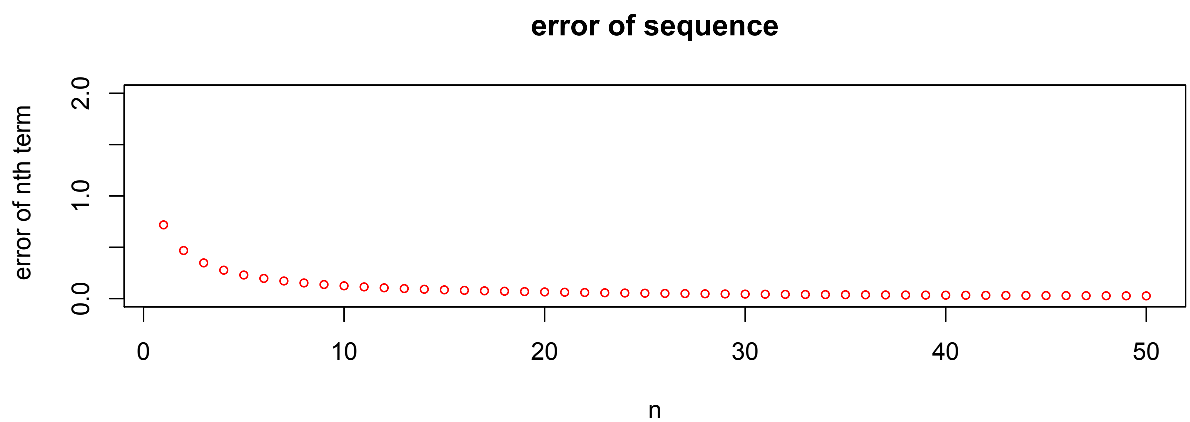
To recap, we decided to define e as that number such that the derivative of that number raised to some power is identical to the original number to some power. In more formal language, the derivative is idempotent and the function is of differentiability class C^∞, infinitely differentiable, or "smooth."

With that definition in hand, we found that any derivative of an exponential base to a power is just that function times a constant multiple involving the logarithm. Thus, e is the natural base to use in applications, as it eliminates the need to calculate a logarithm every time a derivative is taken.

Finally, I showed that this number e is between 2 and 3, and that it also happens to be the limit of compounding interest as the length of compounding intervals tends toward 0.

The figure below shows the relatively quick convergence of the sequence formula for e. Notice that after only 35 iterations the error is less than one tenth: about 0.03.


e as limit of compounding sequence
nth sequence term
0
1
2
3
4
5
0
10
20
30
40
50
n


error of sequence
error of nth term
0.0
1.0
2.0
0
10
20
30
40
50
n